Transactions of A. Razmadze
Mathematical Institute
Vol. **176** (2022), issue 1, 159–162

ON RECONSTRUCTION OF COEFFICIENTS OF WALSH SERIES WITH GAPS

SHAKRO TETUNASHVILI[1,2] AND TENGIZ TETUNASHVILI[1,3]

Abstract. In the present paper, theorems concerning the Walsh series with gaps are formulated. Rademacher series are particular cases of considered Walsh series.

Formulas to calculate coefficients of such a Walsh series by means of values of the sum of this series at certain two points are presented. These two points vary depending on the index of the coefficient being calculated.

1. Notation, Definitions and Known Results

The Rademacher function $r_k(t)$, where $k = 0, 1, 2, \dots$, can be defined as follows (see [2]):

$$r_0(t) = \begin{cases} 1, & \text{for } \ t \in [0, \frac{1}{2}); \\ -1, & \text{for } \ t \in [\frac{1}{2}, 1). \end{cases}$$

We extend the function $r_0(t)$ over the whole real line under the condition that

$$r_0(t) = r_0(t+1),$$

for any real number t.

To define $r_k(t)$ for every integer $k \geq 1$ and any real number t, we set

$$r_k(t) = r_0\left(2^k t\right).$$

A system of functions $\{r_k(t)\}_{k=0}^{\infty}$, where $t \in [0, 1)$, is called the Rademacher system.

The Walsh system $\{W_n(t)\}_{n=0}^{\infty}$, where $t \in [0, 1)$ can be defined as follows (see [3]): $W_0(t) = 1$ for any $t \in [0, 1)$.

Let $\Omega_k = \left\{n : \ 2^k \leq n < 2^{k+1} \ \ \& \ n \in \mathbf{N}\right\}$ for every nonnegative integer k, where $\mathbf{N}$ stands for the set of all positive integers. So, for every positive integer n, there exists a number k such that $n \in \Omega_k$. Let

$$n = \sum_{i=0}^{k} \varepsilon_i 2^i$$

be a dyadic expansion of n, thus $\varepsilon_k = 1$ and $\varepsilon_i \in \{0, 1\}$ if $0 \leq i \leq k-1$. To define $W_n(t)$ for $n \geq 1$, we set

$$W_n(t) = \prod_{i=0}^{k} \left(r_i(t)\right)^{\varepsilon_i} = r_k(t) \prod_{i=0}^{k-1} \left(r_i(t)\right)^{\varepsilon_i}.$$

So, $W_{2^k}(t) = r_k(t)$ for every nonnegative integer k.

Bellow, μE stands for the Lebesgue linear measure of the set $E \subset [0, 1)$. The symbol $\dot{+}$ denotes the summation defined in [2] (see [2, Chs. I and II]).

Let σ be any rearrangment of nonnegative integers.

Consider the series

$$\sum_{k=0}^{\infty} a_k r_{\sigma(k)}(t) \tag{1}$$

with respect to the rearranged Rademacher system.

Theorems concerning the reconstruction of coefficients of the series (1) by values of the sum of this series are presented in [3], [1] and [4].

2020 *Mathematics Subject Classification.* 40A05, 40B05.

Key words and phrases. Walsh series; Rademacher series; Rearrangement of series.

According to S. Stechkin and P. Ul'yanov's theorem proved in [3], the following statement holds:

Theorem A. *Let the series* (1) *converge to some constant* C *on the set* $E \subset [0,1)$. *Then:*

1) *if* $\mu E > \frac{1}{2}$, *then* $a_k = 0$ *for every integer* $k \geq 0$ *and, as a consequence,* $C = 0$*;*

2) *if* $\mu E > 0$, *then there exists an integer* $k_0 \geq 0$ *such that* $a_k = 0$ *for every* $k \geq k_0$, *i.e., the series* (1) *is a polynomial.*

The following generalization of Theorem A is proved in [1]:

Theorem B. *Let* $E \subset [0,1)$. *Then:*

1) *if* $\mu E > \frac{1}{2}$, *then there exists a countable set* $P \subset E$ *such that the convergence of the series* (1) *on the set* P *to an arbitrary constant* C *implies that* $a_k = 0$ *for any integer* $k \geq 0$ *and* $C = 0$*;*

2) *if* $\mu E > 0$, *then there exists a countable set* $P_1 \subset E$ *such that the convergence of the series* (1) *to zero on the set* P_1 *implies that* $a_k = 0$ *for every integer* $k \geq k_0$, *for some* k_0. *So, the series* (1) *is a polynomial.*

Further generalization of the above mentioned results is presented in [4]. Namely, the following statement holds:

Theorem C. *Let*

$$\sum_{k=0}^{\infty} a_k r_{\sigma(k)}(t) = f(t), \quad t \in E.$$

Then:

1) *if* $\mu E > \frac{1}{2}$, *then for any integer* $k \geq 0$, *there exists a point* $t_k \in E$ *such that* $t_k \dot{+} \frac{1}{2^{\sigma(k)+1}} \in E$ *and*

$$a_k = \frac{1}{2} r_{\sigma(k)}\left(t_k\right) \left[f\left(t_k\right) - f\Big(t_k \dot{+} \frac{1}{2^{\sigma(k)+1}}\Big)\right]; \tag{2}$$

2) *if* $\mu E > 0$, *then there exists the number* k_0 *such that for any integer* $k \geq k_0$, *there exists the point* $t_k \in E$ *such that* $t_k \dot{+} \frac{1}{2^{\sigma(k)+1}} \in E$ *and equalities* (2) *are fulfilled.*

2. New Results

Let $\{n_k\}_{k=0}^{\infty}$ be a sequence of natural numbers such that $n_k \in \Omega_k$ for every integer $k \geq 0$.

Consider a series

$$\sum_{k=0}^{\infty} a_k W_{n_{\sigma(k)}}(t) \tag{3}$$

with respect to the system $\left\{W_{n_{\sigma(k)}}(t)\right\}_{k=0}^{\infty}$.

The following statements are valid:

Theorem 1. *Let* $\{n_k\}_{k=0}^{\infty}$ *be a sequence such that* $n_k \in \Omega_k$ *for every integer* $k \geq 0$ *and* $E \subset [0,1)$. *Then:*

1) *if* $\mu E > \frac{1}{2}$, *then there exists a sequence* $\{t_m\}_{m=0}^{\infty}$ *such that* $t_m \in E$ *for every integer* $m \geq 0$ *and if*

$$\sum_{k=0}^{\infty} a_k W_{n_{\sigma(k)}}(t_m) = f(t_m)$$

for every integer $m \geq 0$, *then*

$$a_k = \frac{1}{2} W_{n_{\sigma(k)}}(t_{2k}) \left[f(t_{2k}) - f(t_{2k+1})\right]$$

for every integer $k \geq 0$*;*

2) *if* $\mu E > 0$, *then there exist an integer* $k_0 \geq 0$ *and a sequence* $\{t'_m\}_{m=0}^{\infty}$ *such that* $t'_m \in E$ *for every integer* $m \geq 0$ *and if*

$$\sum_{k=0}^{\infty} a_k W_{n_{\sigma(k)}}(t'_m) = f(t'_m)$$

for every integer $m \geq 0$*, then*

$$a_k = \frac{1}{2} W_{n_{\sigma(k)}}(t'_{2k}) \left[f(t'_{2k}) - f(t'_{2k+1}) \right]$$

for every integer $k \geq k_0$*.*

The following statement is a corollary of Theorem 1:

Theorem 2. *Let* $\{n_k\}_{k=0}^{\infty}$ *be a sequence such that* $n_k \in \Omega_k$ *for every integer* $k \geq 0$ *and* $E \subset [0, 1)$*. Then:*

1) *if* $\mu E > \frac{1}{2}$*, then there exists a sequence* $\{t_m\}_{k=0}^{\infty}$ *such that* $t_m \in E$ *for every integer* $m \geq 0$ *and if*

$$\sum_{k=0}^{\infty} a_k W_{n_{\sigma(k)}}(t_m) = f(t_m),$$

for every integer $m \geq 0$*, and* A *is a nonempty subset of the set of all nonnegative integer numbers, then*

$$\sum_{k \in A} |a_k| = 0$$

if and only if $f(t_{2k}) = f(t_{2k+1})$ *for every* $k \in A$*. In a particular case, if* $f(t_{2k}) = f(t_{2k+1})$ *for every integer* $k \geq 0$*, then*

$$\sum_{k=0}^{\infty} |a_k| = 0;$$

2) *if* $\mu E > 0$*, then there exist an integer* $k_0 \geq 0$ *and a sequence* $\{t'_m\}_{m=0}^{\infty}$ *such that* $t'_m \in E$ *for every* $m \geq 0$ *and if*

$$\sum_{k=0}^{\infty} a_k W_{n_{\sigma(k)}}(t'_m) = f(t'_m),$$

for every integer $m \geq 0$*, and* B *is a nonempty subset of the set* $\{k_0, k_0 + 1, k_0 + 2, \dots\}$*, then*

$$\sum_{k \in B} |a_k| = 0$$

if and only if $f(t'_{2k}) = f(t'_{2k+1})$ *for every* $k \in B$*. In a particular case, if* $f(t'_{2k}) = f(t'_{2k+1})$ *for every integer* $k \geq k_0$*, then*

$$\sum_{k=k_0}^{\infty} |a_k| = 0.$$

Note that if a sequence $\{n_k\}_{k=0}^{\infty}$ is such that $n_k = 2^k$ for every integer $k \geq 0$, then $W_{2^k}(t) = r_k(t)$ and the series (3) coincides with the series (1). So, Theorem 1 generalizes Theorem C and Theorem 2 generalizes both Theorem A and Theorem B.

Acknowledgement

This work was supported by the Shota Rustaveli National Scientific Foundation of Georgia through the Grant No. FR-18-2499.

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(Received 08.08.2021)

[1]Department of Mathematics, Georgian Technical University, 77 Kostava Str., Tbilisi 0171, Georgia

[2]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia

[3]Ilia Vekua Institute of Applied Mathematics, I. Javakhishvili Tbilisi State University, 2 University Str., Tbilisi 0186, Georgia

E-mail address: stetun@hotmail.com

E-mail address: tengiztetunashvili@gmail.com