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OSCILLATORY PROPERTIES OF SOLUTIONS OF HIGHER ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Oscillatory properties of solutions of the functional differential equation

$$u^{(n)}(t) = f(u)(t)$$

and its particular cases

$$u^{(n)}(t) = g\big(t, u(\tau_1(t)), \dots, u(\tau_m(t))\big),$$

$$u^{(n)}(t) = \sum_{k=1}^{m} g_k(t) \ln\big(1 + |u(\tau_k(t))|\big) \operatorname{sgn}(u(\tau_k(t)))$$

are investigated. Here, f is an operator acting from the space $C([a,+\infty[)$ to the space $L_{\rm loc}(\mathbb{R}_+)$, $a \le 0$, $g : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$ is a function satisfying the local Carathéodory conditions,

$$g_k \in L_{\rm loc}(\mathbb{R}_+) \ \ (k = 1, \dots, m),$$

and $\tau_k : \mathbb{R}_+ \to \mathbb{R}$ $(k = 1, \dots, m)$ are continuous functions such that

$$\tau_k(t) \le t \ \text{ for } \ t \in \mathbb{R}_+, \quad \lim_{n\to+\infty} \tau_k(t) = +\infty \ \ (k = 1, \dots, m).$$

We investigate oscillatory properties of solutions of the functional differential equation

$$u^{(n)}(t) = f(u)(t) \tag{1}$$

and its particular cases

$$u^{(n)}(t) = g\big(t, u(\tau_1(t)), \dots, u(\tau_m(t))\big), \tag{2}$$

$$u^{(n)}(t) = \sum_{k=1}^{m} g_k(t) \ln\big(1 + |u(\tau_k(t))|\big) \operatorname{sgn}(u(\tau_k(t))). \tag{3}$$

Here, f is an operator acting from the space $C([a,+\infty[)$ to the space $L_{\rm loc}(\mathbb{R}_+)$, $a \le 0$, $g : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$ is a function satisfying the local Carathéodory conditions,

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and $\tau_k : \mathbb{R}_+ \to \mathbb{R}$ $(k = 1, \dots, m)$ are continuous functions such that

$$\tau_k(t) \le t \ \text{ for } \ t \in \mathbb{R}_+, \quad \lim_{n\to+\infty} \tau_k(t) = +\infty \ \ (k = 1, \dots, m). \tag{4}$$

We use the following notation and definitions.

$$p_0(t) \equiv 1, \quad p_k(t) \equiv t^k \ \ (k = 1, 2, \dots).$$

If k is a natural number, then N_k^0 is the set of those $i \in \{1, \dots, k\}$ for which $i + k$ is even.

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, +\infty[$.

$C([a,+\infty[)$ is the space of continuous functions $u : [a,+\infty[\to \mathbb{R}$.

If $u \in C([a,+\infty[)$ and $a_0 \ge a$, then

$$x(a_0; u)(t) = \begin{cases} u(t) & \text{for } \ t \ge a_0, \\ u(a_0) & \text{for } \ t < a_0, \end{cases}$$

$$v(u)(t) = \max\big\{|u(s)| : \ a \le s \le t\big\} \ \text{ for } \ t > a.$$

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$L_{\rm loc}(\mathbb{R}_+)$ is the space of functions $\nu : \mathbb{R}_+ \to \mathbb{R}$, Lebesgue integrable on every finite interval contained in $\mathbb{R}_+$.

An operator $f_0 : C([a,+\infty[) \to L_{\rm loc}(\mathbb{R}_+)$ is said to be **Volterra operator** if for any $t > a$ and $u_i \in C([a,+\infty[)$ $(k = 1, 2, \dots)$ satisfying the condition

$$u_1(s) = u_2(s) \ \text{ for } \ s \in [0,t],$$

we have

$$f_0(u_1)(s) = f_0(u_2)(s) \ \text{ for almost all } \ s \in [0,t].$$

An operator $f_0 : C([a,+\infty[) \to L_{\rm loc}(\mathbb{R}_+)$ is said to be **continuous** if for any $u \in C([a,+\infty[)$ and any sequence $u_k \in C([a,+\infty[)$ $(k = 1, 2, \dots)$, satisfying the condition

$$\lim_{k\to+\infty} v(u_k - u)(t) = 0 \ \text{ for } \ t \geq a,$$

the equality

$$\lim_{k\to+\infty} \int\limits_0^t \big|f_0(u_k)(s) - f_0(u)(s)\big|\, ds = 0 \ \text{ for } \ t > 0$$

holds.

Everywhere below, it is assumed that $f : C([a,+\infty[) \to L_{\rm loc}(\mathbb{R}_+)$ is a continuous Volterra operator.

Let $t_0 \geq 0$. An $(n-1)$-times continuously differentiable function $u : [t_0,+\infty[\to \mathbb{R}$ is said to be a **solution of equation** (1) if $u^{(n-1)}$ is absolutely continuous on every finite interval contained in $[t_0,+\infty[$, and there exists a continuous function $u_0 : [a,t_0] \to \mathbb{R}$ such that almost everywhere on $[t_0,+\infty[$ equality (1) is fulfilled, where

$$u(t) = u_0(t) \ \text{ for } \ a \leq t \leq t_0.$$

A solution u of equation (1) defined on some interval $[t_0,+\infty[\subset \mathbb{R}_+$ is said to be **proper** if it does not identically equal to zero in any neighbourhood of $+\infty$.

A proper solution $u : [t_0,+\infty[\to \mathbb{R}$ is said to be **oscillatory** if it changes sign in any neighbourhood of $+\infty$, and it is said to be a **Kneser solution** if on some interval $[t_1,+\infty[\subset [t_0,+\infty[$ it satisfies the inequalities

$$(-1)^i u^{(i)}(t)u(t) > 0 \ \ (i = 1, \dots, n-1).$$

Equation (1) has **Property** A_0 if every its proper solution for n even is oscillatory and for n odd either is oscillatory, or is a Kneser solution.

Equation (1) has **Property** B_0 if every its proper solution for n even is either oscillatory, or is a Kneser solution, or satisfies the condition

$$\lim_{t\to+\infty} |u^{(n-2)}(t)| = +\infty, \tag{5}$$

and for n odd is either oscillatory, or satisfies condition (5).

Unlike Properties A and B, whose notions in the oscillation theory have been introduced by V. A. Kondrat'ev [11] and I. Kiguradze [6], Properties A_0 and B_0 do not assume that Kneser solutions of the equation under consideration are vanishing at infinity, and the unbounded solutions satisfy the harder than (5) condition

$$\lim_{t\to+\infty} |u^{(n-1)}(t)| = +\infty.$$

We have found integral conditions under which equation (1) has, respectively, Properties A_0 and B_0. In contrast to the well-known earlier results (see, e.g., [1–5, 7–10, 12–14] and references therein), the proven by us general oscillation theorems yield the necessary and sufficient conditions for equation (3) to have Properties A_0 and B_0 in the case where g_k $(k = 1, \dots, m)$ are of the constant sign functions of the same sign.

We investigate oscillatory properties of equation (1) in the case where the operator f is monotone, or more precisely, when f satisfies one of the following two conditions:

(M_-) : $f(0)(t) \equiv 0$, and for any numbers $t_1 \geq 0$, $t_2 > t_1$, $t_0 \in [t_1, t_2]$ and functions $u_i \in C(\mathbb{R}_+)$ $(i = 1, 2)$, satisfying the conditions

$$u_1(t_0) \neq 0, \quad u_i(t)u_1(t_0) \geq 0 \ \ (i = 1, 2), \quad u_1(t) \leq u_2(t) \ \text{ for } \ t_1 \leq t \leq t_2 \tag{6}$$

almost everywhere on $[t_1, t_2]$, the inequality

$$f(\chi(t_1; u_1))(t) \geq f(\chi(t_1; u_2))(t)$$

is fulfilled.

(M_+) : $f(0)(t) \equiv 0$, and for any numbers $t_1 \geq 0$, $t_2 > t_1$, $t_0 \in [t_1, t_2]$ and functions $u_i \in C(\mathbb{R}_+)$ $(i = 1, 2)$, satisfying condition (6), almost everywhere on $[t_1, t_2]$, the inequality

$$f(\chi(t_1; u_1))(t) \leq f(\chi(t_1; u_2))(t)$$

is fulfilled.

Theorem 1. *If the operator f satisfies condition (M_-) and*

$$\int_0^{+\infty} t^{n-i-1} |f(xp_{i-1})(t)| \, dt = -\infty \ \ \textit{for} \ \ x \neq 0, \ \ i \in N^0_{n-1}, \tag{7}$$

then equation (1) *has Property A_0.*

Theorem 2. *If the operator f satisfies condition (M_+) and*

$$\int_0^{+\infty} t^{n-i-1} |f(xp_{i-1})(t)| \, dt = +\infty \ \ \textit{for} \ \ x \neq 0, \ \ i \in N^0_{n-1}, \tag{8}$$

then equation (1) *has Property B_0.*

Conditions (7) and (8) in Theorems 1 and 2 are unimprovable. In particular, the following theorems hold.

Theorem 3. *Let the operator f satisfy condition (M_-), and for any $x \neq 0$, there exist the numbers $t_x \geq 0$ and $\delta(x) > 0$ such that*

$$t^{n-i-2} |f(xp_{i-1})(t)| \geq \delta(x) |f(xp_{n-1})(t)| \ \ \textit{for} \ \ t \geq t_x, \ \ i \in N^0_{n-1}.$$

Then for equation (1) *to have Property A_0, it is necessary and sufficient that equalities* (7) *be satisfied.*

Theorem 4. *Let $n \geq 3$, the operator f satisfy condition (M_-), and for any $x \neq 0$, there exist the numbers $t_x \geq 0$ and $\delta(x) > 0$ such that*

$$t^{n-i-2} |f(xp_{i-1})(t)| \geq \delta(x) |f(xp_{n-2})(t)| \ \ \textit{for} \ \ t \geq t_x, \ \ i \in N^0_{n-2}.$$

Then for equation (1) *to have Property B_0, it is necessary and sufficient that equalities* (8) *be fulfilled.*

Everywhere below, when discussing equations (2) and (3), we assume that the functions τ_i $(i = 1, \dots, n)$ satisfy condition (4).

We investigate equation (2) in the case where the function g satisfies one of the following two conditions:

$$\begin{gathered} g(t, 0, \dots, 0) = 0, \\ g(t, x_1, \dots, x_m) \geq g(t, y_1, \dots, y_m) \ \text{ for } \ t > 0, \ \ x_i x_1 > 0, \ \ y_i x_1 > 0, \ \ x_i \leq y_i \ \ (i = 1, \dots, m) \end{gathered} \tag{9}$$

and

$$\begin{gathered} g(t, 0, \dots, 0) = 0, \\ g(t, x_1, \dots, x_m) \leq g(t, y_1, \dots, y_m) \ \text{ for } \ t > 0, \ \ x_i x_1 > 0, \ \ y_i x_1 > 0, \ \ x_i \leq y_i \ \ (i = 1, \dots, m). \end{gathered} \tag{10}$$

Theorems 1 and 3 imply the following corollaries.

Corollary 1. *If the function g satisfies condition* (9) *and*

$$\int_0^{+\infty} t^{n-i-1}\Big|g\big(t,x|\tau_1(t)|^{i-1},\dots,x|\tau_m(t)|^{i-1}\big)\Big|\,dt=+\infty \;\; for \;\; x\neq 0, \;\; i\in N^0_{n-1}, \tag{11}$$

then equation (2) *has Property A_0.*

Corollary 2. *Let the function g satisfy condition* (9), *and for any $x\neq 0$, there exist the numbers $t_x>0$ and $\delta(x)>0$ such that*

$$t^{n-i-1}\Big|g\big(t,x|\tau_1(t)|^{i-1},\dots,x|\tau_m(t)|^{i-1}\big)\Big|$$
$$\geq \delta((x)\Big|g\big(t,x|\tau_1(t)|^{n-1},\dots,x|\tau_m(t)|^{n-1}\big)\Big| \;\; for \;\; t>t_x, \;\; i\in N^0_{n-1}.$$

Then for equation (2) *to have Property A_0, it is necessary and sufficient that equalities* (11) *be fulfilled.*

Theorems 3 and 4 for equation (2) take the following forms.

Corollary 3. *If the function g satisfies condition* (10) *and*

$$\int_0^{+\infty} t^{n-i-1}\Big|g\big(t,x|\tau_1(t)|^{i-1},\dots,x|\tau_m(t)|^{i-1}\big)\Big|\,dt=+\infty \;\; for \;\; x\neq 0, \;\; i\in N^0_{n-2}, \tag{12}$$

then equation (2) *has Property B_0.*

Corollary 4. *Let $n\geq 3$, the function g satisfy condition* (10), *and for any $x\neq 0$, there exist the numbers $t_x>0$ and $\delta(x)\neq 0$ such that*

$$t^{n-i-2}\Big|g\big(t,x|\tau_1(t)|^{i-1},\dots,x|\tau_m(t)|^{i-1}\big)\Big|$$
$$\geq \delta((x)\Big|g\big(t,x|\tau_1(t)|^{n-2},\dots,x|\tau_m(t)|^{n-2}\big)\Big| \;\; for \;\; t>t_x, \;\; i\in N^0_{n-2}.$$

Then for equation (2) *to have Property B_0, it is necessary and sufficient that equalities* (12) *be fulfilled.*

Finally, let us consider equation (3). Corollaries 2 and 3 result in the following corollaries.

Corollary 5. *If $n>2$ and*

$$g_k(t)\leq 0 \;\; for \;\; t>0 \;\; (k=1,\dots,m),$$

then for equation (3) *to have Property A_0, it is necessary and sufficient that the equality*

$$\int_0^{+\infty}\bigg(\sum_{k=1}^m g_k(t)\ln\big(1+|\tau_k(t)|\big)\bigg)\,dt=-\infty$$

be fulfilled.

Corollary 6. *If $n>3$ and*

$$g_k(t)\geq 0 \;\; for \;\; t>0 \;\; (k=1,\dots,m),$$

then for equation (3) *to have Property B_0, it is necessary and sufficient that the equality*

$$\int_0^{+\infty} t\bigg(\sum_{k=1}^m g_k(t)\ln\big(1+|\tau_k(t)|\big)\bigg)\,dt=+\infty$$

be fulfilled.

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