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MAZURKIEWICZ SETS OF UNIVERSAL MEASURE ZERO

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Abstract. Assuming Martin's Axiom, it is shown that in the Euclidean plane $\mathbf{R}^2$ there exists a Mazurkiewicz set which is an absolute null subset of $\mathbf{R}^2$.

Consider the Euclidean plane $\mathbf{R}^2$ and the family $\mathcal{L}$ of all those straight lines in $\mathbf{R}^2$, each of which is parallel either to $\mathbf{R} \times \{0\}$ or to $\{0\} \times \mathbf{R}$. Clearly, for any point $z \in \mathbf{R}^2$, there are exactly two distinct lines from $\mathcal{L}$ passing through z. The dual (in the sense of projective geometry) version of this fact is highly nontrivial. Namely, there exists a set $Z \subset \mathbf{R}^2$ such that for every straight line $l \subset \mathbf{R}^2$, one has $\mathrm{card}(l \cap Z) = 2$. The first example of Z was given by Mazurkiewicz [9] who used some transfinite construction for obtaining a plane set with the indicated property (see also [10]).

We shall say that a subset Z of $\mathbf{R}^2$ is a Mazurkiewicz set if Z possesses the above property.

There are some works devoted to different aspects of Mazurkiewicz sets. For instance, in [1], certain Mazurkiewicz sets are constructed having additional purely geometric properties. In [2] it is shown that there is a zero-dimensional compact subset K of the unit circle $\mathbf{S}_1$ such that K cannot be included in a Mazurkiewicz set. Several geometric problems closely related to Mazurkiewicz sets are discussed in [7]. In [8], it is demonstrated that the original construction of Mazurkiewicz can be successfully extended to the case where $\mathbf{R}^2$ is replaced by a vector space over an infinite field.

In works [5] and [6], Mazurkiewicz sets are studied from the point of view of their measurability with respect to nonzero σ-finite translation invariant measures on $\mathbf{R}^2$. In this context, it should be mentioned that the behavior of Mazurkiewicz sets with respect to the standard Lebesgue measure λ_2 on $\mathbf{R}^2$ is not uniquely determined. For example, there exists a Mazurkiewicz set A such that $\lambda_2(A) = 0$ and there exists a Mazurkiewicz set B which meets any λ_2-measurable set of strictly positive measure (so, B is not λ_2-measurable).

In this note we consider the question of the existence of Mazurkiewicz sets which are very small from the view-point of topological measure theory. Below, we denote by $\mathbf{c}$ the cardinality of the continuum.

Lemma 1. *Let Z be a subset of the plane $\mathbf{R}^2$ such that for every straight line $l \subset \mathbf{R}^2$, the equality $\mathrm{card}(l \cap Z) = \mathbf{c}$ is valid.*

Then Z contains some Mazurkiewicz set.

This lemma is known and its proof can be obtained by a slight modification of the original Mazurkiewicz construction (cf. [3]).

Remark 1. Let C denote Cantor's set on the real line $\mathbf{R}$. In the plane $\mathbf{R}^2$, consider the set $Z = (C \times \mathbf{R}) \cup (\mathbf{R} \times C)$. Then Z satisfies the condition of Lemma 1, so there exists a Mazurkiewicz set $Z' \subset Z$. Obviously, this Z' is of first category in $\mathbf{R}^2$ and $\lambda_2(Z') = 0$ (see [3]). Lemma 1 also implies that any Bernstein subset of the plane contains some Mazurkiewicz set.

Recall that a Hausdorff topological space E is universal measure zero (or absolute null) if there exists no nonzero σ-finite Borel measure on E vanishing at all singletons in E. Accordingly, a subset X of a Hausdorff topological space E is called universal measure zero (or absolute null) if X turns out to be universal measure zero with respect to the induced topology.

Lemma 2. *Under Martin's Axiom, there exists a universal measure zero set $Z \subset \mathbf{R}^2$ such that $\mathrm{card}(Z \cap l) = \mathbf{c}$ for every straight line l in $\mathbf{R}^2$.*

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The transfinite construction of such a set Z is given in [4]. Notice that Lemma 2 was used in [4] for obtaining the following statement:

Under Martin's Axiom, there exists a universal measure zero subset T of $\mathbf{R}^2$ such that for each straight line $l \subset \mathbf{R}^2$, the orthogonal projection of T on l is absolutely nonmeasurable with respect to the class of the completions of all those nonzero σ-finite Borel measures on l which vanish at all singletons in l.

Lemmas 1 and 2 imply the following result.

Theorem 1. *Assuming Martin's Axiom, there exists a Mazurkiewicz set which is of universal measure zero.*

Remark 2. The existence of universal measure zero Mazurkiewicz subsets of $\mathbf{R}^2$ cannot be established within the framework of **ZFC** theory. Indeed, there exists a model M of **ZFC** such that in M we have $\omega_1 < \mathbf{c}$ and every universal measure zero set in $\mathbf{R}^2$ has cardinality not exceeding ω_1 (as was shown by Baumgartner and Laver, the role of M can be played by the random real model of **ZFC**). Since the cardinality of any Mazurkiewicz set is $\mathbf{c}$, one easily concludes that in the same M there are no universal measure zero Mazurkiewicz sets.

Every absolute null subset of $\mathbf{R}^2$ is totally imperfect, so it directly follows from Theorem 1 that under Martin's Axiom, there exists a totally imperfect Mazurkiewicz set in the plane. On the other hand, the existence of totally imperfect Mazurkiewicz sets does not need additional set-theoretical hypotheses. A certain modification of the original Mazurkiewicz construction gives

Theorem 2. *There exists a Mazurkiewicz set which is totally imperfect in* $\mathbf{R}^2$.

In this context, it should be noticed that if in **ZF** & **DC** theory there exists a totally imperfect Mazurkiewicz set, then there is a non-Lebesgue measurable subset of the real line $\mathbf{R}$. A more general statement holds true. To formulate it, we need one auxiliary notion.

Let l be a straight line in $\mathbf{R}^2$ and let Z be a subset of $\mathbf{R}^2$. We say that Z is finite in direction l if for every straight line $l' \subset \mathbf{R}^2$ parallel to l, the set $l' \cap Z$ is finite (see, e.g., [10]).

Theorem 3. *If in* **ZF** & **DC** *theory there exists a totally imperfect subset Z of $\mathbf{R}^2$, finite in direction $\{0\} \times \mathbf{R}$ and satisfying $\mathrm{pr}_1(Z) = \mathbf{R} \times \{0\}$, then there exists a subset of $\mathbf{R}$, nonmeasurable in the Lebesgue sense.*

Remark 3. Let l be a straight line in $\mathbf{R}^2$ and let Z be a subset of $\mathbf{R}^2$. We say that Z is closed in direction l if for every straight line $l' \subset \mathbf{R}^2$, parallel to l, the set $l' \cap Z$ is closed in l'. A somewhat strengthened version of Theorem 3 is as follows:

If in **ZF** & **DC** theory there exists a totally imperfect subset Z of $\mathbf{R}^2$ which is closed in direction $\{0\} \times \mathbf{R}$ and satisfies $\mathrm{pr}_1(Z) = \mathbf{R} \times \{0\}$, then there exists a subset of $\mathbf{R}$, nonmeasurable in the Lebesgue sense.

Remark 4. In classical point set theory some special subsets of $\mathbf{R}$ or of $\mathbf{R}^2$ are studied, which have nontrivial applications in real analysis, measure theory, and general topology (mostly, for constructing various kinds of counterexamples). Among such subsets, one may mention Hamel bases, Vitali sets, Bernstein sets, Luzin sets, Sierpiński sets, etc. Mazurkiewicz sets belong to this collection, too. Obviously, no Mazurkiewicz set can be a Bernstein subset of $\mathbf{R}^2$. Also, it is not difficult to show that no Mazurkiewicz set can coincide with a Luzin set in $\mathbf{R}^2$ or with a Sierpiński set in $\mathbf{R}^2$ (indeed, the graph of a function acting from $\mathbf{R}$ into $\mathbf{R}$ cannot be a Luzin set in $\mathbf{R}^2$ and cannot be a Sierpiński set in $\mathbf{R}^2$). At the same time, there exists a Mazurkiewicz set which simultaneously is a Hamel basis of $\mathbf{R}^2$ (see, e.g., [5,6]). In this context, we also would like to note that there exists a Mazurkiewicz set which contains the graph of some Sierpiński–Zygmund function acting from $\mathbf{R}$ into $\mathbf{R}$. As is known, the graph of any Sierpiński–Zygmund function is a totally imperfect subset of $\mathbf{R}^2$.

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