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AN ABSTRACT VERSION OF SUP-MEASURABILITY

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Abstract. A certain class Φ of functions of two variables is introduced in connection with the general superposition operator and with the extended notion of sup-measurability. Some cases are considered where there exist sup-measurable functions of two variables which do not belong to the class Φ.

Let E be an infinite ground set and let ϕ be a function which acts from $E \times E$ into E. In addition, let $\mathcal{F}_0$ be a family of functions, all of which act from E into E. Then, for any $f \in \mathcal{F}_0$, we have two associated superpositions – the left superposition $L_{\phi,f} : E \to E$ and the right superposition $R_{\phi,f} : E \to E$. More precisely, the function $L_{\phi,f}$ is defined by the formula

$$L_{\phi,f}(x) = \phi(f(x), x) \ (x \in E)$$

and the function $R_{\phi,f}$ is defined by the formula

$$R_{\phi,f}(x) = \phi(x, f(x)) \ (x \in E).$$

Actually, if ϕ is fixed, then we associate to ϕ the two superposition operators L_ϕ and R_ϕ on $\mathcal{F}_0$. Namely, for each $f \in \mathcal{F}_0$, the value $L_\phi(f)$ coincides with the function $x \to \phi(f(x), x)$ and the value $R_\phi(f)$ coincides with the function $x \to \phi(x, f(x))$.

A function $\phi : E \times E \to E$ will be called left sup-measurable (right sup-measurable) with respect to $\mathcal{F}_0$ if the range of L_ϕ (of R_ϕ) is contained in $\mathcal{F}_0$.

It is not difficult to show that there exist left sup-measurable functions which are not right sup-measurable and, conversely, there exist right sup-measurable functions which are not left sup-measurable.

A function ϕ will be called sup-measurable with respect to $\mathcal{F}_0$ if ϕ is simultaneously left sup-measurable and right sup-measurable with respect to $\mathcal{F}_0$.

Various aspects of the question concerning superposition operators and the sup-measurability of functions of two-variables were studied, e.g., in the works [1–10, 14, 16]. In those works the role of E is played by the real line $\mathbf{R}$. Here we would like to present some results connected with the introduced abstract version of sup-measurability.

Let Φ be a family of functions, all of which act from $E \times E$ into E. The main goal in this note is to consider, for Φ and $\mathcal{F}_0$, the above superposition operators and to show that, under some natural set-theoretical assumptions, there are many sup-measurable functions which do not belong to Φ.

For this purpose, we will assume below that an ideal $\mathcal{J}$ of subsets of $E \times E$ is given and satisfies the following conditions:

(a) $\mathcal{J}$ covers $E \times E$ and possesses a base of cardinality not exceeding card(E);

(b) $\mathcal{J}$ is card(E)-additive (i.e., for every family $\{Z_i : i \in I\}$ of members of $\mathcal{J}$ with card$(I) <$ card(E), the set $\cup\{Z_i : i \in I\}$ is also a member of $\mathcal{J}$);

(c) the quotient set $\Phi/\mathcal{J}$ has cardinality not exceeding card(E);

(d) if $Z \in \mathcal{J}$, then $Z^{-1} \in \mathcal{J}$ (where $Z^{-1} = \{(x, y) : (y, x) \in Z\}$).

(e) for any $f \in \mathcal{F}_0$, the graph of f belongs to $\mathcal{J}$;

Remark 1. Conditions (a) and (b) imply that all subsets of $E \times E$ whose cardinalities are strictly less than card(E), belong to $\mathcal{J}$. The same conditions also imply that card(E) is a regular cardinal number.

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Remark 2. In the family Φ we have the equivalence relation $S(\phi,\psi)$ canonically associated with $\mathcal{J}$ and defined by

$$S(\phi,\psi) \Leftrightarrow \{x \in E : \phi(x) \neq \psi(x)\} \in \mathcal{J}.$$

The above condition (c) says that the total number of S-equivalence classes is less than or equal to $\mathrm{card}(E)$.

Now, pick two distinct elements e_0 and e_1 in E and, for any subset X of E, define its characteristic function $\chi_X : E \to \{e_0, e_1\}$ by the formula:

$\chi_X(x) = e_1$ if $x \in X$, and $\chi_X(x) = e_0$ if $x \in E \setminus X$.

Further, suppose that the following condition holds:

$(*)$ if X is a subset of E with $\mathrm{card}(X) < \mathrm{card}(E)$, then the characteristic function χ_X belongs to $\mathcal{F}_0$.

We shall assume below that the conditions (a)–(e) and $(*)$ are fulfilled.

Lemma 1. *Under the above assumptions on E and $\mathcal{J}$, for any set $Z \subset E \times E$ not belonging to $\mathcal{J}$, there exists a set $Z' \subset Z$ such that:*

(1) $\mathrm{card}(Z') = \mathrm{card}(E)$*;*

(2) *if P is an arbitrary set from $\mathcal{J}$, then* $\mathrm{card}(P \cap Z') < \mathrm{card}(E)$.

In particular, Z' does not belong to $\mathcal{J}$.

The proof of this lemma is similar to the well-known transfinite constructions of Luzin sets and of generalized Luzin sets on the real line $\mathbf{R}$ (cf. [9, 11, 13, 15]).

Lemma 2. *Under the same assumptions on E and $\mathcal{J}$, let Z be as in Lemma* 1*. Then there exists a family $\{Z_t : t \in T\}$ of subsets of Z such that* $\mathrm{card}(T) > \mathrm{card}(E)$*, no set Z_t belongs to $\mathcal{J}$, and* $\mathrm{card}(Z_t \cap Z_r) < \mathrm{card}(E)$ *for any two distinct indices $t \in T$ and $r \in T$.*

The proof of Lemma 2 is based on Lemma 1 and uses Sierpiński's theorem on almost disjoint families of subsets of a given infinite set (see, e.g., [17]).

Theorem 1. *There exists a family Ψ of functions from $E \times E \to E$ such that:*

(1) $\mathrm{card}(\Psi) > \mathrm{card}(E)$*;*

(2) *each function $\psi \in \Psi$ is sup-measurable with respect to $\mathcal{F}_0$;*

(3) *no function from Ψ belongs to Φ.*

Remark 3. Consider the special case where $E = \mathbf{R}$. Denote by λ_1 the standard Lebesgue measure on $\mathbf{R}$ and denote by λ_2 the standard Lebesgue measure on the plane $\mathbf{R}^2$. Further, put:

$\mathcal{F}_0 =$ the family of all λ_1-measurable functions from $\mathbf{R}$ into $\mathbf{R}$;

$\mathcal{J} =$ the σ-ideal of all λ_2-measure zero subsets of $\mathbf{R}^2$;

$\Phi =$ the family of all λ_2-measurable functions from $\mathbf{R}^2$ into $\mathbf{R}$.

Assuming Martin's Axiom (**MA**), one can easily deduce from Theorem 1 that there exists a sup-measurable with respect to $\mathcal{F}_0$ function $\psi : \mathbf{R}^2 \to \mathbf{R}$ which does not belong to Φ.

Under the same **MA**, Theorem 1 also implies the analogous result for real-valued functions possessing the Baire property (in this case the role of $\mathcal{J}$ is played by the ideal of all sets of first category in $\mathbf{R}^2$).

Remark 4. If $\mathrm{card}(E) = \omega_{\alpha+1}$, then there is a family $\mathcal{F}_0$ of functions acting from E into E such that $\mathrm{card}(\mathcal{F}_0) = \omega_\alpha$ and there exists no ideal $\mathcal{J}$ of subsets of $E \times E$ which satisfies conditions (b), (d) and (e).

The above results have some natural discrete analogs. Here we would like to formulate the most simple discrete version of Theorem 1, which does not need the concept of an ideal of subsets of $E \times E$. Take $E = \mathbf{N}$, where $\mathbf{N}$ is the set of all natural numbers.

We shall say that a family $\mathcal{F}$ of functions acting from $\mathbf{N}$ into itself is admissible if these two conditions are fulfilled:

$(**)$ $\mathrm{card}(\mathcal{F}) = \mathrm{card}(\mathbf{N})$;

$(***)$ the characteristic functions of all finite subsets of $\mathbf{N}$ belong to $\mathcal{F}$.

Remark 5. The family $\mathcal{F} = \mathcal{F}_1$ of all recursive functions satisfies both conditions $(**)$, $(***)$. The family $\mathcal{F}_2$ of all arithmetical functions also satisfies $(**)$ and $(***)$. Recall that $\mathcal{F}_1$ is properly contained in $\mathcal{F}_2$ (see, e.g., [12]).

Theorem 2. *Let $\mathcal{F}$ be an admissible family of functions from $\mathbf{N}$ into itself.*

In **ZF** *theory there exists a family Ψ of functions, all of which act from $\mathbf{N}^2$ into $\{0,1\}$, and for which the following relations are valid:*

(1) $\mathrm{card}(\Psi) = \mathbf{c}$*;*

(2) *if $\phi \in \Psi$ and $\psi \in \Psi$ are any two distinct functions, then the sets $\phi^{-1}(1)$ and $\psi^{-1}(1)$ are infinite and their intersection is finite;*

(3) *for every function $f \in \mathcal{F}$ and for every function $\psi \in \Psi$, both superpositions*

$$n \to \psi(f(n), n), \quad n \to \psi(n, f(n)) \quad (n \in \mathbf{N})$$

belong to the family $\mathcal{F}$.

Remark 6. It directly follows from the above theorem that:

(1) there are continuum many functions $\phi : \mathbf{N}^2 \to \{0,1\}$ which are not recursive and for which all superpositions

$$n \to \phi(f(n), n), \quad n \to \phi(n, f(n)) \quad (n \in \mathbf{N}),$$

where $f \in \mathcal{F}_1$, are recursive functions;

(2) there are continuum many non-arithmetical functions $\psi : \mathbf{N}^2 \to \{0,1\}$ for which all superpositions

$$n \to \psi(f(n), n), \quad n \to \psi(n, f(n)) \quad (n \in \mathbf{N}),$$

where $f \in \mathcal{F}_2$, are arithmetical functions.

The analogous results hold true for partial recursive and partial arithmetical functions acting from $\mathbf{N}$ into $\mathbf{N}$.

Remark 7. In view of the existence of a canonical recursive isomorphism between $\mathbf{N}$ and $\mathbf{N}^2$, it is clear that:

(1) if for a given function $\phi : \mathbf{N}^2 \to \mathbf{N}$ all superpositions of the form

$$n \to \phi(f(n), g(n)) \quad (n \in \mathbf{N}),$$

where $\{f, g\} \subset \mathcal{F}_1$, belong to $\mathcal{F}_1$, then ϕ is a recursive function;

(2) if for a given function $\psi : \mathbf{N}^2 \to \mathbf{N}$ all superpositions of the form

$$n \to \psi(f(n), g(n)) \quad (n \in \mathbf{N}),$$

where $\{f, g\} \subset \mathcal{F}_2$, belong to $\mathcal{F}_2$, then ψ is an arithmetical function.

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