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ON BINARY SEMI-OPEN SETS AND BINARY SEMI-ω-OPEN SETS IN BINARY TOPOLOGICAL SPACES

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Abstract. As a generalization of binary open sets in binary topological spaces, we use the notions of semi-open sets in topological spaces to introduce and study the notions of binary semi-open sets and binary semi-ω-open sets in binary topological spaces. Furthermore, we develop some properties on binary semi-ω-compact spaces and binary semi-ω-connected spaces. Moreover, we define and discuss the concept of binary semi-continuous functions and binary semi-ω-continuous functions.

1. Introduction and Preliminaries

The notion of binary topological space was introduced by Nithyanantha and Thangavelu in 2011 [8]. A binary topology from X to Y is a binary structure which satisfies the conditions of Definition 1.1 that are analogous to the axioms of topology. This concept has been studied by many mathematicians in different fields of the general topology. In 2017, Mustafa [7] introduced the concept of a binary generalized closed set in binary topological spaces in which he showed some properties in the well-known open sets such as semi-open, pre-open, b-open and much more. Further, in 2018, Chacko and Susha [1] used the notion of a binary topological space to introduce the concept of binary linear topology in metric spaces. Then, in 2019, Mehmood et.al. [6] used the notions of soft semi-open sets to introduce the notion of a soft binary topological space in which they showed and proved some applications and characterizations in separation axioms. On the other hand, Hdeib [4] introduced the concept of ω-closed set as generalized of closed sets. A point $x \in X$ is said to be a condensation point of C if for each $V \in \tau$ with $x \in V$, the set $V \cap C$ is uncountable. C is said to be ω-closed [4], if it contains all its condensation points. The complement of a ω-closed set is called ω-open. The collection of all ω-open sets of (X, τ) is denoted by τ_ω in which it is finer than τ. This notion has been studied in different field of general topology (see [2,3]). In this paper, as a generalization of binary open sets in binary topological spaces, we use the notions of semi-open sets [5] to introduce and study the notions of binary semi-open sets and binary semi-ω-open sets. Besides, we show some of their properties and characterizations. Moreover, we introduce the notions of binary semi-continuous functions, binary semi-ω-continuous functions, strongly binary semi-ω-continuous functions and perfectly binary semi-ω-continuous functions.

Throughout this paper, $P(X)$ and $P(Y)$ are the power sets of X and Y, respectively. Now, we show some definitions which are useful for the developing of this paper.

Definition 1.1 ([8])**.** Let X, Y be any two empty sets and $A \subseteq X$ and $B \subseteq Y$. A binary topology from X to Y is a binary structure $M \subseteq P(X) \times P(Y)$ that satisfies the following conditions:
1) $(\emptyset, \emptyset)$ and $(X, Y) \in M$.
2) $(A_1 \cap A_2, B_1 \cap B_2) \in M$, for any $(A_1, B_1) \in M$ and $(A_2, B_2) \in M$.
3) If $\{(A_\delta, B_\delta) : \delta \in \Delta\}$ is a family of members of M, then $\big(\underset{\delta\in\Delta}{\cup} A_\delta, \underset{\delta\in\Delta}{\cup} B_\delta\big) \in M$.

Definition 1.2 ([8])**.** If M is a binary topology from X to Y, then the triplet (X, Y, M) is said to be a binary topological space and the members of M are called binary open sets of (X, Y, M). The elements of X$\times Y$ are said to be the binary points of the binary topological space (X, Y, M).

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Definition 1.3 ([8])**.** Let (X, Y, M) be a binary topological space and let $(x, y) \in (X, Y)$. The binary open set (A, B) is said to be a binary neighbourhood of (x, y) if $x \in A$ and $y \in B$.

Proposition 1.1 ([8])**.** *Let $(A, B) \subseteq (C, D) \subseteq (X, Y)$ and (X, Y, M) be a binary topological space. Then the following statements hold:*

1) $\operatorname{Int}(A, B) \subseteq (A, B)$.
2) *If (A, B) is binary open, then* $\operatorname{Int}(A, B) = (A, B)$.
3) $\operatorname{Int}(A, B) \subseteq \operatorname{Int}(C, D)$.
4) $\operatorname{Int}(\operatorname{Int}(A, B)) = \operatorname{Int}(A, B)$.
5) $(A, B) \subseteq Cl(A, B)$.
6) *If (A, B) is binary closed, then $Cl(A, B) = (A, B)$.*
7) $Cl(A, B) \subseteq Cl(C, D)$.
8) $Cl(Cl(A, B)) = Cl(A, B)$.

Definition 1.4 ([8])**.** Let $f : Z \to X \times Y$ be a function. Let $A \subseteq X$ and $B \subseteq Y$. We define $f^{-1}(A, B) = \{z \in Z : f(z) = (x, y) \in (A, B)\}$.

Definition 1.5 ([8])**.** Let (X, Y, M) be a binary topological space and let (Z, τ) be a topological space. Now, let $f : (Z, \tau) \to X \times Y$ be a function, then f is said to be binary continuous if $f^{-1}(A, B)$ is open in (Z, τ) for every binary open set (A, B) in $X \times Y$.

2. Binary Semi-open Sets and Binary Semi-continuous Functions

In this section, we use the notion of a binary open set to introduce and study the notion of a binary semi-open set.

Definition 2.1. Let (A, B) be a subset of a binary topological space (X, Y, M). Then (A, B) is said to be binary semi-open if $(A, B) \subseteq Cl(\operatorname{Int}(A, B))$. The complement of a binary semi-open set is called binary semi-closed.

Remark 2.1. The collection of all binary semi-open sets and binary semi-closed sets are denoted by $BSO(X, Y, M)$ and $BSC(X, Y, M)$, respectively.

Proposition 2.1. *Every binary open set is binary semi-open.*

Proof. Let (A, B) be a binary open set in (X, Y, M), then $(A, B) \subseteq Cl(\operatorname{Int}(A, B))$, since (A, B) is a binary open set, $\operatorname{Int}(A, B) = (A, B)$, then $(A, B) \subseteq Cl(A, B)$, now either $Cl(A, B) = (X, Y)$ or $Cl(A, B) = (C, D)$, where $A \subseteq C$ and $B \subseteq D$, therefore (A, B) is binary semi-open. □

The converse of the above Proposition need not be true as can be seen in the following example.

Example 2.1. Let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$ and $M = \{(\emptyset, \emptyset), (\{a\}, \{1\}), (X, Y)\}$. Then $(\{a, b\}, \{1, 2\})$ is a binary semi-open set, but it is not a binary open set.

Theorem 2.1. *Let (X, τ) and (Y, σ) be two any topological spaces. If A and B are semi-open in X and Y, respectively, then (A, B) is binary semi-open in (X, Y, M).*

Proof. Let A and B be semi-open in X and Y, respectively. Suppose that (A, B) is not binary semi-open in (X, Y, M), this implies that $(A, B) \nsubseteq Cl(\operatorname{Int}(A, B))$. Indeed, we may assume that $\operatorname{Int}(A, B) = (\emptyset, \emptyset)$. Thus $\operatorname{Int}(A) = \emptyset$ and $\operatorname{Int}(B) = \emptyset$, therefore $A \nsubseteq Cl(\operatorname{Int}(A))$ and $B \nsubseteq Cl(\operatorname{Int}(B))$, and hence A and B are not semi-open sets and this is a contradiction. Therefore (A, B) is binary semi-open. □

Lemma 2.1. *Let $A \subseteq X$ and $B \subseteq Y$. If (A, B) is binary semi-open in (X, Y, M), then A is semi-open in (X, τ) and B is semi-open in (Y, σ).*

Proof. The proof is followed by the Theorem 2.1. □

Theorem 2.2. *Let (A, B) and (C, D) be binary open sets in a binary topological space (X, Y, M). Then the following statements hold:*

(1) $Cl((A, B) \cup (C, D)) \supseteq Cl(A, B) \cup Cl(C, D)$.
(2) $Cl((A, B) \cap (C, D)) \subseteq Cl(A, B) \cap Cl(C, D)$.

(3) $\text{Int}((A,B)\cup(C,D)) \supseteq \text{Int}(A,B)\cup\text{Int}(C,D)$.
(4) $\text{Int}(A,B)\cap\text{Int}(C,D) \supseteq \text{Int}((A,B)\cap(C,D))$.

Proof.

(1) Let $(A,B)\subseteq(A,B)\cup(C,D)$ and $(C,D)\subseteq(A,B)\cup(C,D)$. Then $Cl(A,B)\subseteq Cl((A,B)\cup(C,D))$ and $Cl(C,D)\subseteq Cl((A,B)\cup(C,D))$ and hence $Cl(A,B)\cup Cl(C,D)\subseteq Cl((A,B)\cup(C,D))$.
(2) Let $(A,B)\cap(C,D)\subseteq(A,B)$ and $(A,B)\cap(C,D)\subseteq(C,D)$. Then $Cl((A,B)\cap(C,D))\subseteq Cl(A,B)$ and $Cl((A,B)\cap(C,D))\subseteq Cl(C,D)$, and hence $Cl((A,B)\cap(C,D))\subseteq Cl(A,B)\cap Cl(C,D)$.
(3) Let $(A,B)\subseteq(A,B)\cup(C,D)$ and $(C,D)\subseteq(A,B)\cup(C,D)$. Then $\text{Int}(A,B)\subseteq\text{Int}((A,B)\cup(C,D))$ and $\text{Int}(C,D)\subseteq\text{Int}((A,B)\cup(C,D))$, and hence $\text{Int}(A,B)\cup\text{Int}(C,D)\subseteq\text{Int}((A,B)\cup(C,D))$.
(4) Let $(A,B)\cap(C,D)\subseteq(A,B)$ and $(A,B)\cap(C,D)\subseteq(C,D)$. Then $\text{Int}((A,B)\cap(C,D))\subseteq\text{Int}(A,B)$ and $\text{Int}((A,B)\cap(C,D))\subseteq\text{Int}(C,D)$, and hence $\text{Int}((A,B)\cap(C,D))\subseteq\text{Int}(A,B)\cap\text{Int}(C,D)$.

□

The equality in part (1), (2), (3) and (4) in the above Theorem need not be true as can be seen in the following example.

Example 2.2. Let $X=\{1,2,3\}$, $Y=\{4,5\}$ and $M=\{(\emptyset,\emptyset),(X,Y),(\{1\},\{5\}),(\{2\},Y),(\{1,2\},Y)\}$. Then $Cl(\{1\},\emptyset)\cup Cl(\{2\},\emptyset)=(\{1,3\},\emptyset)\cup(\{2,3\},\{4\})=(X,\{4\})$, but $Cl((\{1\},\emptyset)\cup(\{2\},\emptyset))=(X,Y)$. Besides, $Cl((\{1,2\},\{4\})\cap(\{3\},\{5\}))=(\{3\},\emptyset)$, but $Cl(\{1,2\},\{4\})\cap Cl(\{3\},\{5\})=(X,Y)$.
Now, $\text{Int}(\{1\},\{4\})\cup\text{Int}(\{2\},\{5\})=(\emptyset,\emptyset)$, but $\text{Int}((\{1\},\{4\})\cup(\{2\},\{4\}))=(\{1,2\},Y)$. Besides, $\text{Int}((\{1\},\{3\})\cap(\{2\},Y))=(\emptyset,\emptyset)$, but $\text{Int}(\{1\},\{3\})\cap\text{Int}(\{2\},Y)=(\emptyset,\{2\})$.

Theorem 2.3. *An arbitrary union of binary semi-open sets is binary semi-open.*

Proof. Let $\{(A,B)_\delta:\delta\in\Delta\}$ be a collection of family of binary semi-open sets of (X,Y,M), then $(A,B)_\delta\subseteq Cl(\text{Int}((A,B_\delta)))$. Now, let $\underset{\delta\in\Delta}{\cup}(A,B)_\delta\subseteq\underset{\delta\in\Delta}{\cup}Cl(\text{Int}((A,\,B)_\delta)))$, by Theorem 2.2 parts (1) and (3), we have $\underset{\delta\in\Delta}{\cup}(A,B)_\delta\subseteq Cl(\text{Int}(x\underset{\delta\in\Delta}{\cup}(A,B)_\delta)))$. Therefore $\underset{\delta\in\Delta}{\cup}(A,B)_\delta$ is a binary semi-open set. □

The following example shows that an arbitrary intersection of binary semi-open sets need not be a binary semi-open set.

Example 2.3. Let $X=\{a,b,c\}$, $Y=\{1,2,3\}$ and $M=\{(\emptyset,\emptyset),(\{a\},\{1\}),(\{b\},\{2\}),(\{a,b\},\{1,2\}),(X,Y)\}$. Then $(\{b,c\},\{2,3\})$ and $(\{a,c\},\{1,3\})$ are binary semi-open sets, but $(\{b,c\},\{2,3\})\cap(\{a,c\},\{1,3\})=(\{c\},\{3\})$ is not a binary semi-open set.

Definition 2.2. Let (A,B) be a subset of a binary topological space (X,Y,M). Then (A,B) is said to be binary semi-closed if $(A,B)\supseteq Cl(\text{Int}(A,B))$.

Theorem 2.4. *An arbitrary intersection of binary semi-closed sets need not be a binary semi-closed set.*

Proof. Let $\{(A,B)_\delta:\delta\in\Delta\}$ be a collection of family of binary semi-closed sets of (X,Y,M), then $(A,B)_\delta\supseteq Cl(\text{Int}((A,B_\delta)))$. Now, let $\underset{\delta\in\Delta}{\cup}(A,B)_\delta\supseteq\underset{\delta\in\Delta}{\cap}Cl(\text{Int}((A,B)_\delta)))$, by Theorem 2.2 parts (2) and (4), we have $\underset{\delta\in\Delta}{\cap}(A,B)_\delta\supseteq\underset{\delta\in\Delta}{\cap}Cl(\text{Int}((A,B)_\delta)))\supseteq Cl(\text{Int}(\underset{\delta\in\Delta}{\cap}(A,B)_\delta)))$. Therefore $\underset{\delta\in\Delta}{\cap}(A,B)_\delta$ is a binary semi-closed set. □

The following example shows that an arbitrary union of binary semi-closed sets need not be a binary semi-closed set.

Example 2.4. By Example 2.3, $(\{a\},\{1\})$ and $(\{b\},\{2\})$ are the binary semi-closed sets, but $(\{a\},\{1\})\cup(\{b\},\{2\})=(\{a,b\},\{1,2\})$ is not a binary semi-closed set.

Definition 2.3. Let (X, Y, M) be a binary topological space and let (Z, τ) be a topological space. Now, let $f : (Z, \tau) \to X \times Y$ be a function, then f is said to be binary semi-continuous if $f^{-1}(A, B)$ is open in (Z, τ) for every binary semi-open set (A, B) in $X \times Y$.

Theorem 2.5. *Every binary continuous function is binary semi-continuous.*

Proof. The proof is followed by the fact that every binary open set is a binary semi-open set. □

The converse of the above Theorem need not be true as can be seen in the following

Example 2.5. Let $X = \{a, b, c\}$, $Y = \{0, 1\}$, $M = \{(\emptyset, \emptyset), (\{a\}, \{1\}), (\{b\}, \{0, 1\}), (X, Y)\}$, $Z = \{q, w, e\}$ and $\tau = \{\emptyset, Z, \{q\}, \{w\}, \{q, w\}\}$. Define the function $f : (Z, \tau) \to X \times Y$ by $f(q) = (a, 1)$, $f(w) = (b, 0)$ and $f(e) = (c, 1)$. Then f is a binary semi-continuous function, but it is not a binary continuous function, because $f^{-1}(\{b, c\}, \{2\}) = \{w, e\}$ is not an open set.

3. Binary Semi-ω-open Sets and Binary Semi-ω-continuous Functions

In this section, we use the notions of a binary open set, binary semi-open set and ω-closed set to introduce and study the notion of a binary semi-ω-open set.

Definition 3.1. Let (X, Y, M) be a binary topological space and $(A, B) \subseteq X \times Y$. Then (A, B) is said to be binary semi-ω-open if for each $(x, y) \in (A, B)$ there exits a binary semi-open set $(V, U)_x$ containing (x, y) such that $(V, U)_x - (A, B)$ is a countable set. The complement of a binary semi-ω-open set is called a binary semi-ω-closed set.

Remark 3.1. The collection of all binary semi-ω-open sets and binary semi-ω-closed sets are denoted by $BS\omega O(X, Y, M)$ and $BS\omega C(X, Y, M)$.

Lemma 3.1. *Every binary semi-open set is binary semi-ω-open.*

Proof. The proof is followed by Definition 3.1. □

The converse of the above Lemma need not be true as can be seen in the following example.

Example 3.1. Let $X = \{a, b, c, d\}$, $Y = \{q, w, e, r\}$ and $M = \{(\emptyset, \emptyset), (\{a\}, \{w\}), (\{c\}, \{r\}), (X, Y), (\{a, c\}, \{w, r\})\}$. Then $(\{c, d\}, \{e, r\})$ is a binary semi-ω-open set, but it is not a binary semi-open set.

Lemma 3.2. *Let (X, Y, M) and (X_1, Y_1, M) be two binary topological spaces such that $(A, B) \subseteq (X_1, Y_1)$ and $X_1 \subseteq X$ and $Y_1 \subseteq Y$. If (A, B) is a binary semi-ω-open set of (X, Y, M), then (A, B) is a binary semi-ω-open set of (X_1, Y_1, M).*

Proof. Let (A, B) be a binary semi-ω-open set of (X, Y, M). Then for every $(x, y) \in (A, B)$, there exits a semi-ω-open set (U, V) of (X, Y, M) containing (x, y) such that $(U, V) - (A, B)$ is, countable. As a consequence, we have that (U, V) is a semi-ω-open set of (X_1, Y_1, M) containing (x, y). This proves that (A, B) is a semi-ω-open set of (X_1, Y_1, M). □

Theorem 3.1. *Let (X, Y, M) be a binary topological space and $(A, B) \subseteq (X, Y)$. Then (A, B) is said to be binary semi-ω-open if and only if for every $(x, y) \in (A, B)$, there exits a binary semi-open set $(U, V)_x$ containing (x, y) and a countable subset (A, B) such that $(U, V)_x - (C, D) \subseteq (A, B)$.*

Proof. Necessity. Let (A, B) be a binary semi-ω-open set and $(x, y) \in (A, B)$, then there exits a binary semi-open set $(U, V)_x$ containing (x, y) such that $(U, V)_x - (A, B)$ is countable. Now, let $(C, D) = (U, V)_x - (A, B) = (U, V)_x \cap ((X - A), X - B))$. Then $(U, V)_x - (C, D) \subseteq (A, B)$.

Sufficiency. Let $(x, y) \in (A, B)$. Then there exits a binary semi-ω-open set $(U, V)_x$ containing (x, y) and a countable set (C, D) such that $(U, V)_x - (C, D) \subseteq (A, B)$. Therefore $(U, V)_x - (A, B) \subseteq (C, D)$ and $(U, V)_x - (A, B)$ is countable. □

Definition 3.2. Let $\{\psi_\delta : \delta \in \Delta\}$ be a collection of binary semi-open sets in a binary topological space (X, Y, M) is said to be a binary semi-open cover of a subset (A, B) of (X, Y) if $(A, B) \subseteq \bigcup_{\delta \in \Delta} \psi_\delta$.

Definition 3.3. Let (X, Y, M) be a binary topological space. Then (X, Y) is said to be binary-semi-Lindeloff if every binary semi-open cover of (X, Y) has a countable sub-cover.

Theorem 3.2. *Let (X, Y, M) be a binary topological space. Then the following statements are equivalent:*

(1) *(X, Y) is binary-semi-Lindeloff.*
(2) *Every countable cover of (X, Y) by binary semi-open sets has a countable sub-cover.*

Proof. (2) $\Rightarrow$ (1): Since every binary semi-open set is binary semi-ω-open set, the proof follows.

(1) $\Rightarrow$ (2) : Let $\{\psi_\delta : \delta \in \Delta\}$ be a cover of (X, Y) by binary semi-ω-open sets of (X, Y). Now, for each $(x, y) \in (X, Y)$, there exits a $\delta_{(x,y)} \in \Delta$ such that $(x, y) \in \psi_{\delta_{(x,y)}}$. Since $(U, V)_{\delta_{(x,y)}}$ is a binary semi-ω-open, there exits a binary semi-open set $(N, M)_{\delta_{(x,y)}}$ such that $(x, y) \in (N, M)_{\delta_{(x,y)}}$ and $(N, M)_{\delta_{(x,y)}} - (U, V)_{\delta_{(x,y)}}$ is countable. Then the family $\{(N, M)_\delta : \delta \in \Delta\}$ is a binary cover of (X, Y) and (X, Y) is binary-semi-Lindeloff. Therefore there exits a countable sub-cover $\delta_{(x,y)_i}$ with $i \in I$ such that $(X, Y) = \underset{i\in I}{\cup}(N, M)_{\delta_{(x,y)_i}}$. Since $(X, Y) = \underset{i\in I}{\cup}[(N, M)_{\delta_{(x,y)_i}} - (U, V)_{\delta_{(x,y)_i}}) \cup (U, V)_{\delta_{(x,y)_i}}] = \underset{i\in I}{\cup}\big[(N, M)_{\delta_{(x,y)_i}} - (U, V)_{\delta_{(x,y)_i}}) \cup \underset{i\in I}{\cup}(U, V)_{\delta_{(x,y)_i}}\big]$. Since $(N, M)_{\delta_{(x,y)_i}} - (U, V)_{\delta_{(x,y)_i}}$ is a countable set, for each $\delta((x, y)_i)$, there exits a countable subset $\Delta_{\delta((x,y)_i)}$ of Δ such that $(N, M)_{\delta_{(x,y)_i}} - (U, V)_{\delta_{(x,y)_i}} \subseteq \underset{\Delta_{\delta((x,y)_i)}}{\cup}(U, V)_\delta$ and therefore $(X, Y) \subseteq \underset{i\in I}{\cup}\big(\underset{\delta\in\Delta_{\delta((x,y)_i)}}{\cup}(U, V)_\delta\big) \cup \big(\underset{i\in I}{\cup}(U, V)_{\delta_{(x,y)_i}}\big)$. □

Theorem 3.3. *Let (X, Y, M) be a binary topological space and $(C, D) \subseteq (X, Y)$. If (A, B) is a binary semi-ω-closed set, then $(C, D) \subseteq (J, K) \cup (A, B)$ for some binary semi-ω-closed set (J, K) and a countable set (A, B).*

Proof. If (C, D) is a binary semi-ω-closed set, then $(X - C, X - D)$ is binary semi-ω-open, and hence by Theorem 3.1, for every $(x, y) \in (X - C, X - D)$, there exits a binary semi-ω-open set (U, V) containing (x, y) and a countable set (A, B) such that $(U - A, V - B) \subseteq (X - C, X - D)$. Thus $(C, D) \subseteq ((X - (U - A), (X - (V - B))) = X - ((U, V) \cap ((X - A), (X - B))) = ((X - U), (X - V)) \cup (A, B)$. Let $(J, K) = (X - U, X - V)$. Then (J, K) is a binary semi-ω-closed set such that $(C, D) \subseteq (J, K) \cup (A, B)$. □

Theorem 3.4. *The union of any family of binary semi-ω-open sets is a binary semi-ω-open set.*

Proof. Let $\{(A, B)_\delta : \delta \in \Delta\}$ be a collection of binary semi-ω-open subsets of (X, Y). Then for every $(x, y) \in \underset{\delta\in\Delta}{\cup}(A, B)_\delta$, $(x, y) \in (A, B)_\delta$, for some $\delta \in \Delta$. Hence there exits a binary semi-ω-open subset (U, V) containing (x, y) such that $(U - A, V - B)_\delta$ is countable. Now, as $((U - (\underset{\delta\in\Delta}{\cup} A_\delta), (V - (\underset{\delta\in\Delta}{\cup} B_\delta)) \subseteq (U - A, V - B)_\delta$, thus $((U - (\underset{\delta\in\Delta}{\cup} A_\delta)), (V - (\underset{\delta\in\Delta}{\cup} B_\delta)))$ is countable. Therefore $\underset{\delta\in\Delta}{\cup}(A, B)_\delta$ is a binary semi-ω-open set. □

Definition 3.4. The union of all binary semi-ω-open sets contained in $(A, B) \subseteq (X, Y)$ is called a binary semi-ω-interior of (A, B) and is denoted by $\mathrm{Int}_{s\omega}(A, B)$.

Definition 3.5. The intersection of all binary semi-ω-closed sets of (X, Y) containing (A, B) is called a binary semi-ω-closure of (A, B) and is denoted by $Cl_{s\omega}(A, B)$.

The $\mathrm{Int}_{s\omega}(A, B)$ is a binary semi-ω-open set and the $Cl_{s\omega}(A, B)$ is a binary semi-ω-closed set.

Theorem 3.5. *Let (X, Y, M) be a binary topological space and $(A, B), (C, D) \subseteq (X, Y)$. Then the following statements hold:*

(1) $\mathrm{Int}_{s\omega}(\mathrm{Int}_{s\omega}(A, B)) = \mathrm{Int}_{s\omega}(A, B)$.
(2) *if* $(A, B) \subset (C, D)$, *then* $\mathrm{Int}_{s\omega}(A, B) \subset \mathrm{Int}_{s\omega}(C, D)$.
(3) $\mathrm{Int}_{s\omega}((A, B) \cap (C, D)) \subset \mathrm{Int}_{s\omega}(A, B) \cap \mathrm{Int}_{s\omega}(C, D)$.
(4) $\mathrm{Int}_{s\omega}(A, B) \cup \mathrm{Int}_{s\omega}(C, D) \subset \mathrm{Int}_{s\omega}((A, B) \cup (C, D))$.
(5) $\mathrm{Int}_{s\omega}(A, B)$ *is the largest binary semi-ω-open subset of* (X, Y) *contained in* (A, B).
(6) (A, B) *is binary semi-ω-open if and only if* $(A, B) = \mathrm{Int}_{s\omega}(A, B)$.
(7) $Cl_{s\omega}(Cl_{s\omega}(A, B)) = Cl_{s\omega}(A, B)$.

(8) *If* $(A,B) \subset (C,D)$*, then* $Cl_{s\omega}(A,B) \subset Cl_{s\omega}(C,D)$.
(9) $Cl_{s\omega}(A,B) \cup Cl_{s\omega}(C,D) \subset Cl_{s\omega}((A,B) \cup (C,D))$.
(10) $Cl_{s\omega}((A,B) \cap (C,D)) \subset Cl_{s\omega}((A,B)) \cap Cl_{s\omega}(C,D)$.

Proof. (1), (2), (6), (7) and (8) follow directly from Definition 3.1. (3), (4) and (5) follow from part (2) of this Theorem. (9) and (10) follow by applying part (8) of this Theorem. □

Theorem 3.6. *Let* (X,Y,M) *be a binary topological space and* $(A,B) \subset (X,Y)$*. Then the following statements hold:*

(1) $Cl_{s\omega}((X-A), Y-B)) = (X,Y) - Cl_{s\omega}(A,B)$.
(2) $\text{Int}_{s\omega}((X-A),(Y-B)) = (X,Y) - \text{Int}_{s\omega}(A,B)$.

Proof. We prove (1) and (2) as follows:

(1) Let $(x,y) \in (X,Y) - Cl_{s\omega}(A,B)$. Then there exits $(U,V) \in BS\omega O(X,Y,M)$ such that $(U,V) \cap (A,B) = \emptyset$, and hence it has $(x,y) \in \text{Int}_{s\omega}(A,B)$. This shows that $(X,Y) - Cl_{s\omega}(A,B) \subset \text{Int}_{s\omega}((X-A),(X-B))$. Now, take $(x,y) \in \text{Int}_{s\omega}((X-A),(X-B))$. Since $\text{Int}_{s\omega}((X-A),(X-B)) \cap (A,B) = \emptyset$, we have $(x,y) \notin Cl_{s\omega}(A,B)$. As a consequence, $Cl_{s\omega}((X-A),(X-B)) = (X,Y) - \text{Int}_{s\omega}(A,B)$.
(2) Let $(x,y) \in (X,Y) - \text{Int}_{s\omega}((X-A),(X-B))$. Since $\text{Int}_{s\omega}((X-A),(X-B)) \cap (A,B) = \emptyset$, we have $(x,y) \notin Cl_{s\omega}(A,B)$ and this implies that $(x,y) \in (X,Y) - Cl_{s\omega}(A,B)$. Now, take $(x,y) \in (X,Y) - Cl_{s\omega}(A,B)$. Then there exist $(U,V) \in BS\omega O(X,Y,M)$ such that $(U,V) \cap (A,B) = \emptyset$. Therefore $\text{Int}_{s\omega}((X-A),(X-B)) = (X,Y) - Cl_{s\omega}(A,B)$. □

Definition 3.6. Let (X,Y,M) be a binary topological space and $(A,B) \subseteq (X,Y)$. Then (A,B) is said to be a binary semi-ω-neighbourhood of a point $(x,y) \in (X,Y)$ if there exists a binary semi-ω-open set (J,K) such that $(x,y) \in (J,K) \subset (A,B)$.

Theorem 3.7. *Let* $(X,Y,M))$ *be a binary topological space and* $(A,B) \subseteq (X,Y)$*. Then* (A,B) *is a binary semi-*ω*-open set if and only if it is a binary semi-*ω*-neighbourhood of each of its points.*

Proof. Necessity. Let (A,B) be a binary semi-ω-open set of (X,Y). Then by Definition 3.6, (A,B) is a binary semi-ω-neighbourhood of each of its points.

Sufficiency. If (A,B) is a binary semi-ω-neighbourhood of each of its points, then for each $(x,y) \in (A,B)$, there exits $(C,D)_x \in BS\omega O(X,Y,M)$ such that $(C,D)_x \subset (A,B)$. As a consequence, $(A,B) = \cup\{(C,D)_x : (x,y) \in (A,B)\}$. Since each $(C,D)_x$ is binary semi-ω-open and an arbitrary union of binary semi-ω-open sets is a binary semi-ω-open set, therefore (A,B) is a binary semi-ω-open set of (X,Y). □

Definition 3.7. Let (X,Y,M) be a binary topological space, then (X,Y,M) is said to be binary semi-ω-compact if every cover of (X,Y) by binary semi-ω-open sets has a finite subcover.

Theorem 3.8. Let (X,Y,M) be a binary topological space, then (X,Y,M) is binary semi-ω-compact if and only if for every collection $\{(A,B)_\alpha : \alpha \in \Delta\}$ of binary semi-ω-closed sets in (X,Y,M) satisfying $\cap\{(A,B)_\alpha : \alpha \in \Delta\} = \emptyset$, there is a finite subcollection $(A,B)_{\alpha_1}, (A,B)_{\alpha_2}, \ldots, (A,B)_{\alpha_n}$ with $\cap\{(A,B)_{\alpha_k} : k = 1, \ldots, n\} = \emptyset$.

Proof. Let $\{(A,B)_\alpha : \alpha \in \Delta\}$ be a collection of binary semi-ω-closed sets such that $\cap\{(A,B)_\alpha : \alpha \in \Delta\} = \emptyset$, then $\{(X-A, Y-B)_\alpha : \alpha \in \Delta\}$ is a collection of binary semi-ω-open sets such that

$$\begin{aligned}(X,Y) &= (X,Y) - \emptyset = (X,Y) - \cap\{(A,B)_\alpha : \alpha \in \Delta\} \\ &= \cup\{(X-A, Y-B)_\alpha : \alpha \in \Delta\},\end{aligned}$$

that is, $\{(X-A, Y-B)_\alpha : \alpha \in \Delta\}$ is a cover of (X,Y) by binary semi-ω-open sets. Since (X,Y,M) is binary semi-ω-compact, there exists a finite subcollection $(X-A, Y-B)_{\alpha_1}, (X-A, Y-B)_{\alpha_2}, \ldots, (X-A, Y-B)_{\alpha_n}$ such that

$$\begin{aligned}(X,Y) &= \cup\{(X-A, Y-B)_{\alpha_k} : k = 1, \ldots, n\} \\ &= (X,Y) - \cap\{(A,B)_{\alpha_k} : k = 1, \ldots, n\}.\end{aligned}$$

This shows that $\cap\{(A,B)_{\alpha_k} : k = 1,\dots,n\} = \emptyset$. Conversely, suppose that $\{(U,V)_\alpha : \alpha \in \Delta\}$ is a cover of (X,Y) by binary semi-ω-open sets, then $\{(X-U,Y-V)_\alpha : \alpha \in \Delta\}$ is a collection of binary semi-ω-closed sets such that $\cap\{(X-U,Y-V)_\alpha : \alpha \in \Delta\} = (X,Y)-\cup\{(U,V)_\alpha : \alpha \in \Delta\} = (X,Y)-(X,Y) = \emptyset$. By the hypothesis, there exists a finite subcollection $(X-U,Y-V)_{\alpha_1}, (X-U,Y-V)_{\alpha_2}, \dots, (X-U,Y-V)_{\alpha_n}$ such that $\cap\{(X-U,Y-V)_{\alpha_k} : k = 1,\dots,n\} = \emptyset$. Thus $(X,Y) = (X,Y)-\emptyset = (X,Y)-\cap\{(X-U,Y-V)_{\alpha_k} : k = 1,\dots,n\} = (X,Y)-((X,Y)-\cup\{(U,V)_{\alpha_k} : k = 1,\dots,n\}) = \cup\{(U,V)_{\alpha_k} : k = 1,\dots,n\}$, whence it follows that (X,Y,M) is binary semi-ω-compact. □

Definition 3.8. Let (X,Y,M) be a binary topological space, then (X,Y,M) is said to be binary semi-connected if (X,Y) cannot be written as a disjoint union of two non-empty binary semi-open sets.

Definition 3.9. Let (X,Y,M) be a binary topological space, then (X,Y,M) is said to be binary semi-ω-connected if (X,Y) cannot be written as a disjoint union of two non-empty binary semi-ω-open sets.

Theorem 3.9. Let (X,Y,M) be a binary topological space. If (X,Y,M) is binary semi-ω-connected, then (X,Y,M) is binary semi-connected.

Proof. Let (X,Y,M) be binary semi-ω-connected. Now, suppose that (X,Y,M) is not binary semi-connected, then there exist non-empty binary semi-open sets (A,B) and (C,D) such that $(A,B) \cap (C,D) = \emptyset$ and $(A,B) \cup (C,D) = (X,Y)$. Then by Proposition 3.1, we have that (A,B) and (C,D) are binary semi-ω-open sets and so, (X,Y,M) is not binary semi-ω-connected and this is a contradiction, therefore (X,Y,M) is binary semi-connected. □

Theorem 3.10. *For a binary topological space (X,Y,M), the following statements are equivalent:*

1. *(X,Y,M) is binary semi-ω-connected.*
2. *$(\emptyset,\emptyset)$ and (X,Y) are the only subsets of (X,Y) both are binary semi-ω-open and binary semi-ω-closed.*

Proof. (1)⇒(2) Let (V,U) be a subset of (X,Y) which is both binary semi-ω-open and binary semi-ω-closed, then $(X-V,X-U)$ is both binary semi-ω-open and binary semi-ω-closed, so $(X,Y) = (V,U) \cup (X-V,X-U)$. Since (X,Y,M) is binary semi-ω-connected, one of those sets is $(\emptyset,\emptyset)$. Therefore $(V,U) = (\emptyset,\emptyset)$ or $(V,U) = (X,Y)$.
(2)⇒(1) Suppose that (X,Y,M) is not binary semi-ω-connected and let $(X,Y) = (U,N) \cup (V,M)$, where (U,N) and (V,M) are disjoint non-empty binary semi-ω-open sets in (X,Y,M), then $(U,N) = (X,Y)-(V,M)$ both are binary semi-ω-open and binary semi-ω-closed. By the hypothesis, $(U,N) = (\emptyset,\emptyset)$ or $(U,N) = (X,Y)$, which is a contradiction. Therefore $(X,Y,M))$ is binary semi-ω-connected. □

Now, in this part, we define the concept of binary semi-ω-continuous functions. Moreover, we prove some of their properties.

Definition 3.10. Let (X,Y,M) be a binary topological space and (Z,τ) be a topological space. Now, let $f : (Z,\tau) \to X \times Y$ be a function, then f is said to be binary semi-ω-continuous if $f^{-1}(A,B)$ is open in (Z,τ) for every binary semi-ω-open set (A,B) in $X \times Y$.

Theorem 3.11. *Every binary semi-continuous function is binary semi-ω-continuous.*

Proof. It follows from the fact that every binary semi-open is binary semi-ω-open. □

The converse of the above Theorem need not be true as can be seen in the following

Example 3.2. Let $X = \{a,b,c\}$, $Y = \{0,1\}$, $M = \{(\emptyset,\emptyset), (\{a\},\{1\}), (\{b\},\{0,1\}), \text{k}(X,Y)\}$, $Z = \{q,w,e\}$ and $\tau = \{\emptyset, Z, \{q\}, \{w\}, \{q,w\}\}$. Define the function $f : (Z,\tau) \to X \times Y$ by $f(q) = (a,1)$, $f(w) = (b,0)$ and $f(e) = (c,1)$. Then f is a binary semi-ω-continuous function, but it is not binary semi-continuous, because $f^{-1}(\{a,c\},\{1\}) = \{q,e\}$ is not a semi-open set.

Theorem 3.12. *For a function $f : (Z,\tau) \to X \times Y$, the following statements are equivalent:*

(1) *f is binary semi-ω-continuous.*
(2) *$f^{-1}(A,B)$ is a closed set in (Z,τ) for each binary semi-ω-closed set (A,B) in $X\times Y$.*
(3) *For each $(x,y)\in(X,Y)$ and each binary semi-ω-open set (V,U) in $X\times Y$ containing $(f(x),f(y))$ there exists an open set (N,M) in (Z,τ) containing (x,y) such that $f(N,M)\subset(V,U)$.*

Proof. (1)⇒(2) Let (A,B) be any binary semi-ω-closed set in $X\times Y$, then $(V,U)=(X,Y)\setminus(A,B)$ is a binary semi-ω-open set in $X\times Y$ and since f is binary semi-ω-continuous, $f^{-1}((V,U))$ is an open subset in (Z,τ), but $f^{-1}((V,U))=f^{-1}((X,Y)\setminus(A,B))=f^{-1}((X,Y))\setminus f^{-1}((A,B))=(X,Y)\setminus f^{-1}((A,B))$, and hence $f^{-1}((A,B))$ is a closed set in (Z,τ).
(2) ⇒ (1) Let (V,U) be any binary semi-ω-open set in $X\times Y$, then $(A,B)=(X,Y)\setminus(V,U)$ is a binary semi-ω-closed set in $X\times Y$. By the hypothesis, we have $f^{-1}((A,B))$ is a closed set in (Z,τ), but $f^{-1}((A,B))=f^{-1}((X,Y)\setminus(V,U))=f^{-1}((X,Y))\setminus f^{-1}((V,U))=(X,Y)\setminus f^{-1}((V,U))$ and so, $f^{-1}((V,U))$ is an open set in (Z,τ). This shows that f is binary semi-ω-continuous.
(1) ⇒ (3) Let $(x,y)\in(X,Y)$ and (V,U) be any binary semi-ω-open set in $X\times Y$ such that $(f(x),f(y))\in(V,U)$, then $(x,y)\in f^{-1}((V,U))$ and since f is a binary semi-ω-continuous function, $f^{-1}((V,U))$ is an open set in (Z,τ). If $(N,M)=f^{-1}((V,U))$, then (N,M) is an open set in (Z,τ) containing (x,y) such that $f((N,M))=f(f^{-1}((V,U)))\subset(V,U)$.
(3) ⇒ (1) Let (V,U) be any binary semi-ω-open set in $X\times Y$ and $(x,y)\in f^{-1}((V,U))$, then $(f(x),f(y))\in(V,U)$ and by (3) there exists an open set $(N,M)_x$ in (Z,τ) such that $(x,y)\in(N,M)(x,y)$ and $f((N,M)_{(x,y)})\subset(V,U)$. Thus $(x,y)\in(N,M)_{(x,y)}\subset f^{-1}(f(N,M)_{(x,y)}))\subset f^{-1}((V,U))$ and hence $f^{-1}((V,U))=\cup\{(N,M)_{(x,y)}:(x,y)\in f^{-1}((V,U))\}$. Then, we find that $f^{-1}((V,U))$ is an open set in (Z,τ) and so, f is a binary semi-ω-continuous function. □

Proposition 3.1. *Let $f:(Z,\tau)\to X\times Y$ be binary semi-ω-continuous if and only if for each $A\subseteq X$ and $B\subseteq Y$, $f^{-1}(\mathrm{Int}_{s\omega}(A,B))\subseteq\mathrm{Int}_{s\omega}(f^{-1}(A,B))$.*

Proof. Necessity. Let $f:(Z,\tau)\to X\times Y$ be binary semi-ω-continuous and let $A\subseteq X$ and $B\subseteq Y$. Then $\mathrm{Int}_{s\omega}(A,B)$ is a binary semi-ω-open set of (X,Y,M) contained in (A,B). Hence $f^{-1}(\mathrm{Int}_{s\omega}(A,B))$ is a pen set of (Z,τ). Now,

$$\begin{aligned}
&\mathrm{Int}_{s\omega}(A,B)\subseteq(A,B)\\
&\Rightarrow f^{-1}(\mathrm{Int}_{s\omega}(A,B))\subseteq f^{-1}(A,B)\\
&\Rightarrow \mathrm{Int}_{s\omega}(f^{-1}(\mathrm{Int}_{s\omega}(A,B)))\subseteq\mathrm{Int}_{s\omega}(f^{-1}(A,B))\\
&\Rightarrow f^{-1}(\mathrm{Int}_{s\omega}(A,B))\subseteq\mathrm{Int}_{s\omega}(f^{-1}(A,B)).
\end{aligned}$$

Sufficiency. Suppose that $f^{-1}(\mathrm{Int}_{s\omega}(A,B))\subseteq\mathrm{Int}_{s\omega}(f^{-1}(A,B))$ for each $A\subseteq X$ and $B\subseteq Y$. Now, let $(A,B)\in(X,Y,M)$, this implies that $\mathrm{Int}_{s\omega}(A,B)=(A,B)$. Hence $f^{-1}(A,B)=f^{-1}(\mathrm{Int}_{s\omega}(A,B))\subseteq\mathrm{Int}_{s\omega}(f^{-1}(A,B))$. Therefore $\mathrm{Int}_{s\omega}(f^{-1}(A,B))$ is an open set of (Z,τ). □

Definition 3.11. Let $f:(Z,\tau)\to X\times Y$ be a function, then f is said to be:
(1) strongly binary semi-ω-continuous if the inverse image of every binary semi-ω-closed set in $X\times Y$ is closed set in (Z,τ);
(2) perfectly binary semi-ω-continuous if the inverse image of every binary semi-ω-closed set in $X\times Y$ is both open and closed in (Z,τ).

Theorem 3.13. *Let $f:(Z,\tau)\to X\times Y$ be strongly binary semi-ω-continuous, then f is binary continuous.*

Proof. Let (V,U) be any binary closed set in $X\times Y$, since every binary closed set is binary semi-open and it is well known that every binary semi-open is binary semi-ω-closed set, (V,U) is binary semi-ω-closed set in $X\times Y$. Since f is strongly binary semi-ω-continuous, $f^{-1}(V,U)$ is closed set in (Z,τ). Therefore f is binary continuous. □

Theorem 3.14. *Let $f:(Z,\tau)\to X\times Y$ be perfectly binary semi-ω-continuous, then f is strongly binary semi-ω-continuous.*

Proof. Let (V, U) be any binary semi-ω-closed set in $X \times Y$. Since f is perfectly binary semi-ω-continuous, $f^{-1}(V, U)$ is closed in (Z, τ). Therefore f is strongly binary semi-ω-continuous. □

4. Conclusion

The main idea of this article was to define a new notion of binary sets which were called binary semi-open and binary semi-ω-open. Furthermore, some notions associated to functions among these sets were defined. For future works, it is recommended to study these sets on sets b-open, α-open, β-open and so on. In addition and to establish new properties and relationships among them.

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