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ON SOME SHARP CONDITIONS FOR GENERALIZED ABSOLUTE CONVERGENCE OF FOURIER SERIES

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Abstract. In the present paper,we give some sufficient conditions for generalized absolute convergence of trigonometric Fourier series in terms of L^p and p-variational best approximations or moduli of smoothness and prove their sharpness. Similar conditions for an arbitrary orthonormal system in $L^2[0,1]$ are considered.

1. Introduction

Let L^p, $1 \le p < \infty$, be the space of 2π-periodic measurable functions with a finite norm $\|f\|_p = \big(\int\limits_0^{2\pi} |f(x)|^p\,dx\big)^{1/p}$ and for $k \in \mathbb{N} = \{1, 2, \dots\}$, $\delta \in [0, 2\pi]$,

$$\omega_k(f, \delta)_p := \sup\{\|\Delta_h^k f(x)\|_p : |h| \le \delta\},$$

where $\Delta_h^k(f)(x) = \sum\limits_{i=0}^{k} (-1)^{k-i}\binom{k}{i} f(x + ih)$, $k \in \mathbb{N}$, is the k-th difference of f with step h. If T_n is the space of trigonometric polynomials of order at most n, then the n-th best approximation in L^p is introduced by

$$E_n(f)_p := \inf_{t_n \in T_n} \|f - t_n\|_p, \quad n \in \mathbb{Z}_+ = \{0, 1, \dots\}.$$

Let f be a 2π-periodic real bounded function, $\xi = \{x_0 < x_1 < \dots < x_n = x_0 + 2\pi\}$ be a partition of a period and $æ_\xi^p(f) := \big(\sum\limits_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p\big)^{1/p}$, $1 \le p < \infty$.

By the definition, for $1 < p < \infty$, we set

$$\omega_{1-1/p}(f, \delta) = \sup\{æ_\xi^p(f) : \lambda(\xi) := \max_i (x_i - x_{i-1}) \le \delta\}$$

and for $k \in \mathbb{N}$, $k \ge 2$,

$$\omega_{k-1/p}(f, \delta) = \sup\{\omega_{1-1/p}(\Delta_h^{k-1} f(x), |h|) : |h| \le \delta\}.$$

For $1 < p < \infty$, let us introduce the space V_p of all 2π-periodic bounded functions with the property

$$\|f\|_{V_p} := \max(\|f\|_\infty, \omega_{1-1/p}(f, 2\pi)) < \infty$$

and $C_p = \{f \in V_p : \lim\limits_{\delta\to 0} \omega_{1-1/p}(f, \delta) = 0\}$. Here, $\|f\|_\infty = \sup_{x\in[0,2\pi]} |f(x)|$. The space V_p of functions of bounded p-variation was introduced for the case $p = 2$ by Wiener [13], while the space C_p of p-absolutely continuous functions in another but equivalent form was considered by Young [14]. Both V_p and C_p are Banach spaces with respect to the norm $\|\cdot\|_{V_p}$. The best approximation $E_n(f)_{V_p}$ in the space C_p, $1 < p < \infty$, is introduced similarly to $E_n(f)_p$. The problems of approximation in C_p and L^p, $1 < p < \infty$, are closely connected (see [6], [7] and lemmas below).

Let $1 \le \alpha < \infty$. We say that a sequence $\{\gamma_k\}_{k=0}^\infty$ belongs to the class $A(\alpha)$ if $\gamma_k > 0$ for all $k \in \mathbb{Z}_+$ and

$$\Big(\sum_{k=2^n}^{2^{n+1}-1} \gamma_k^\alpha\Big)^{1/\alpha} \le C 2^{n(1/\alpha - 1)} \sum_{k=-2^{n-1}}^{2^{n-1}} \gamma_k =: C 2^{n(1/\alpha-1)} \Gamma_n, \tag{1.1}$$

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for all $n \in \mathbb{N}$. In the case $n = 0$ we suppose that (1.1) is valid for $\Gamma_0 = \gamma_0$. This definition due to Gogoladze and Meskhia [2] generalizes a class introduced by Ul'yanov [9]. The class $A(\infty)$ consists of all positive sequences $\{\gamma_k\}_{k=0}^{\infty}$ such that $\max\limits_{2^n \le k < 2^{n+1}} \gamma_k \le C2^{-n}\Gamma_n$, $n \in \mathbb{N}$, $\gamma_1 \le C\gamma_0$. It is known that $A(\alpha_1) \subset A(\alpha_2)$ for $1 \le \alpha_2 < \alpha_1 \le \infty$.

For $f \in L^1$, let us consider its Fourier coefficients

$$a_k(f) = \pi^{-1}\int_0^{2\pi} f(x)\cos kx\,dx, \quad k \in \mathbb{Z}_+, \quad b_k(f) = \pi^{-1}\int_0^{2\pi} f(x)\sin kx\,dx, \quad k \in \mathbb{N},$$

partial Fourier sums $S_n(f)(x) = a_0(f)/2 + \sum_{k=1}^{n}(a_k(f)\cos kx + b_k(f)\sin kx)$, $n \in \mathbb{Z}_+$, and $\rho_k(f) = (a_k^2(f) + b_k^2(f))^{1/2}$, $k \in \mathbb{Z}_+$.

Let $\omega(x)$ be a continuous increasing function on $\mathbb{R}_+ = [0, +\infty)$ such that $\omega(0) = 0$ (in this case we write $\omega \in \Omega$). A function $\omega \in \Omega$ belongs to the Bary class B if

$$\sum_{k=n}^{\infty} k^{-1}\omega(k^{-1}) = O(\omega(n^{-1})), \quad n \in \mathbb{N},$$

correspondingly, $\omega \in \Omega$ belongs to the Bary–Stechkin class B_k, $k > 0$ if

$$\sum_{j=1}^{n} j^{k-1}\omega(j^{-1}) = O(n^k\omega(n^{-1})), \quad n \in \mathbb{N}.$$

These definitions may be found in [1].

In [2], the following theorems were proved (the case $r = s$ is treated similarly to the proof in [2]). They generalized the results in the case $\gamma_k = k^{\beta}$ proved by A. A. Konyushkov [3].

Theorem A. *Let $1 < p < \infty$, $1/p + 1/q = 1$, $s = \max(q, 2)$, $0 < r \le s$, $\{\gamma_k\}_{k=0}^{\infty} \in A(s/(s-r))$. If $f \in L^p$ and the series*

$$\sum_{k=1}^{\infty} k^{-r/s}\gamma_k E_k^r(f)_p \tag{1.2}$$

converges, then the series

$$\sum_{k=1}^{\infty} \gamma_k \rho_k^r \tag{1.3}$$

also converges, and for some $C > 0$,

$$\sum_{k=2}^{\infty} \gamma_k \rho_k^r \le C\sum_{k=1}^{\infty} k^{-r/s}\gamma_k E_k^r(f)_p.$$

Theorem B. *If the conditions of Theorem A hold, but instead of the convergence of series* (1.2)

$$\sum_{k=1}^{\infty} k^{-r/s}\gamma_k \omega_l^r(f, 1/k)_p$$

converges for some $l \in \mathbb{N}$, then series (1.3) *converges.*

Note that Theorem B follows from Theorem A and Lemma 2.3.

The aim of the present paper is to establish the sharpness of Theorems A and B and their p-variational analogues (see Theorem 3.1). Also, we investigate similar to (1.3) series in the case of general orthonormal systems and obtain a sharp condition for its convergence.

2. Auxiliary propositions

The first assertion of Lemma 2.1 is proved in [12], while the second one is established in [6].

Lemma 2.1. *Let $f \in V_p$, $1 < p < \infty$, $k \in \mathbb{N}$. Then*
1) $E_n(f)_{V_p} \geqslant Cn^{1/p}E_n(f)_p$, $n \in \mathbb{N}$, *for some* $C > 0$;
2) $\omega_k(f,\delta)_p \le \delta^{1/p}\omega_{k-1/p}(f,\delta)$, $\delta \in [0, 2\pi]$.

Lemma 2.2 is due to W.Rudin and H.S.Shapiro (see [4]). For $t_n(x) = \sum\limits_{k=0}^{n}(\alpha_k \cos kx + \beta_k \sin kx)$, $n \in \mathbb{N}$, we set $\xi(t_n, r) := \big(\sum\limits_{k=0}^{n}(|\alpha_k|^r + |\beta_k|^r)\big)^{1/r}$.

Lemma 2.2. *There exists a sequence $\{\gamma_k\}_{k=0}^{\infty}$ such that $\gamma_n = \pm 1$ for all $n \in \mathbb{Z}_+$, and for all $N \in \mathbb{Z}_+$, one has*

$$\Big|\sum_{n=0}^{N}\gamma_n e^{int}\Big| \le 5\sqrt{N+1}.$$

In particular, $|P_N(t)| := \big|\sum\limits_{n=0}^{N}\gamma_n \cos nt\big| \le 5\sqrt{N+1}$ *and* $\xi(P_N, r) := (N+1)^{1/r}$, $r \ge 1$.

The direct Jackson-Stechkin and inverse Bernstein–Salem–Stechkin approximation theorems in L^p, $1 \le p \le \infty$ (see [8, § 5.1, § 6.1]) are combined in the following

Lemma 2.3. *Let $1 \le p < \infty$, $k \in \mathbb{N}$, $f \in L^p$. Then*

$$E_n(f)_p \le C_1\omega_k(f, 1/(n+1))_p, \quad n \in \mathbb{Z}_+,$$

$$\omega_k(f, 1/n)_p \le C_2 n^{-k}\sum_{j=0}^{n}(j+1)^{k-1}E_j(f)_p, \quad n \in \mathbb{N},$$

for some $C_i = C_i(k) > 0$, $i = 1, 2$.

The direct and inverse approximation theorems in C_p were established by A. P. Terekhin. A sketch of proof of the first inequality of Lemma 2.4 may be found in [6], while for the proof of the second one we refer the reader to [10].

Lemma 2.4. *Let $1 < p < \infty$, $k \in \mathbb{N}$, $f \in C_p$. Then*

$$E_n(f)_{V_p} \le C_1\omega_{k-1/p}(f, 1/(n+1)), \quad n \in \mathbb{Z}_+,$$

$$\omega_{k-1/p}(f, 1/n) \le C_2 n^{-k+1/p}\sum_{j=0}^{n}(j+1)^{k-1/p-1}E_j(f)_{V_p}, \quad n \in \mathbb{N},$$

for some $C_1 = C_1(k) > 0$, $C_2 = C_2(k, p) > 0$.

Lemma 2.5 may be derived from the results in [7] (see also [11]).

Lemma 2.5. *Let $1 \le p < \infty$, $t_n \in T_n$, $n \in \mathbb{N}$. Then $\|t_n\|_{V_p} \le C(p)n^{1/p}\|t_n\|_p$.*

3. General absolute convergence of trigonometric Fourier series

From Theorems A and B and Lemma 2.1 we easily deduce

Theorem 3.1. *Let $1 < p < \infty$, $l \in \mathbb{N}$, $1/p + 1/q = 1$, $s = \max(q, 2)$, $0 < r \le s$, $\{\gamma_k\}_{k=0}^{\infty} \in A(s/(s-r))$. If $f \in C_p$ and the series*

$$\sum_{k=1}^{\infty}k^{-r/s-r/p}\gamma_k E_k^r(f)_{V_p}$$

or the series

$$\sum_{k=1}^{\infty}k^{-r/s-r/p}\gamma_k \omega_{l-1/p}^r(f)$$

converges, then the series (1.3) *also converges.*

Theorems 3.2 and 3.3 show the sharpness of Theorem A in the case $1 < p \le 2$ under some additional conditions.

Theorem 3.2. *Suppose that* $1 < p \le 2$, $1/p + 1/q = 1$, $0 < r \le 1$, *and for* $\{\gamma_k\}_{k=0}^{\infty} \in A(q/(q-r))$ *and some* $\alpha \in (0,1)$ *the inequality*

$$(1-\alpha)2^{-kr/q}\Gamma_k \ge 2^{-r(k-1)/q}\Gamma_{k-1}, \quad k \in \mathbb{N}$$

holds. If a sequence $\{\varepsilon_i\}_{i=0}^{\infty}$ *decreases to zero and* $\sum_{i=1}^{\infty} i^{-r/q}\gamma_i\varepsilon_i^r = \infty$, *then there exists* $f \in L^p$ *such that* $E_n(f)_p \le \varepsilon_n$, $n \in \mathbb{N}$, *but the series* (1.3) *diverges.*

Proof. Let $D_n(x) = 1/2 + \sum_{k=1}^{n} \cos kx$, $n \in \mathbb{Z}_+$. It is known that $D_n(x) = \sin(n+1/2)x/(2\sin(x/2))$ for $x \ne 2\pi k$ and

$$\|D_n\|_p^p \le 2\biggl(\int\limits_0^{\pi/n} ((n+1)/2)^p dx + \int\limits_{\pi/n}^{\pi} ((\pi)/2x)^p \, dx\biggr) \le C_1 n^{p-1}, \quad n \in \mathbb{N}. \tag{3.1}$$

We consider the function

$$f_0(x) = 2^{-1}C_1^{-1/p}\sum_{k=1}^{\infty}(\varepsilon_{2^k} - \varepsilon_{2^{k+1}})(D_{2^k}(x) - D_{2^{k-1}}(x))2^{-k/q}.$$

Then for $n \in [2^k; 2^{k+1})$, $k \in \mathbb{Z}_+$, by (3.1), we obtain

$$E_n(f_0)_p \le E_{2^k}(f_0)_p \le 2^{-1}\sum_{j=k+1}^{\infty}(\varepsilon_{2^j} - \varepsilon_{2^{j+1}})C_1^{-1/p}2^{-j/q}\|D_{2^j} - D_{2^{j-1}}\|_p$$
$$\le \sum_{j=k+1}^{\infty}(\varepsilon_{2^j} - \varepsilon_{2^{j+1}}) = \varepsilon_{2^{k+1}} \le \varepsilon_n.$$

By the Jensen inequality, we have $(a-b)^r \ge a^r - b^r$ for $a \ge b \ge 0$ and $0 < r \le 1$. Therefore

$$C_2\sum_{i=1}^{\infty}\gamma_i|\widehat{f}_0(i)|^r = \sum_{k=1}^{\infty}\sum_{i=2^{k-1}}^{2^k-1}\gamma_i(\varepsilon_{2^k} - \varepsilon_{2^{k+1}})^r 2^{-kr/q}$$
$$\ge \sum_{k=1}^{\infty}\Gamma_k 2^{-kr/q}(\varepsilon_{2^k}^r - \varepsilon_{2^{k+1}}^r) = \sum_{k=1}^{\infty}\varepsilon_{2^k}^r(\Gamma_k 2^{-kr/q} - \Gamma_{k-1}2^{-(k-1)r/q}) + \Gamma_0$$
$$\ge \alpha\sum_{k=1}^{\infty}\varepsilon_{2^k}^r\Gamma_k 2^{-kr/q}.$$

Since $A(q/(q-r)) \subset A(1)$, the inequality $\Gamma_k \le C_3\Gamma_{k-1}$, $k \in \mathbb{N}$, holds. Using this inequality, we have

$$\sum_{i=2^{k-1}}^{2^k-1}\gamma_i\varepsilon_i^r i^{-r/q} \le C_4\Gamma_{k-1}2^{-(k-1)r/q}\varepsilon_{2^{k-1}}^r,$$

and from the conditions of Theorem 3.2 we deduce the divergence of the series $\sum_{k=1}^{\infty}\varepsilon_{2^k}^r\Gamma_k 2^{-kr/q}$. Thus, the series $\sum_{i=1}^{\infty}\gamma_i|\widehat{f}_0(i)|^r$ diverges. □

Theorem 3.3. *Let* $1 < p \le 2$, $1/p + 1/q = 1$, $0 < r \le q$, $\{\gamma_k\}_{k=0}^{\infty} \in A(q/(q-r))$ *and a sequence* $\{\varepsilon_i\}_{i=1}^{\infty}$ *be decreasing to zero. If* $\{\varepsilon_i\}_{i=1}^{\infty}$ *satisfies the Bary condition*

$$\sum_{i=k}^{\infty}\frac{\varepsilon_i}{i} = O(\varepsilon_k), \quad k \in \mathbb{N}, \tag{3.2}$$

and the series $\sum_{i=1}^{\infty}\gamma_i i^{-r/q}\varepsilon_i^r$ *diverges, then there exists* $f\in L^p$ *such that* $E_n(f)_p=O(\varepsilon_n)$, $n\in\mathbb{N}$, *but series*(1.3) *diverges.*

Proof. From (3.2), by the decreasing of $\{\varepsilon_i\}_{i=1}^{\infty}$, it follows that

$$\sum_{i=l}^{\infty}\varepsilon_{2^i}\le\varepsilon_{2^l}+\sum_{i=l+1}^{\infty}2\sum_{j=2^{i-1}+1}^{2^i}\frac{1}{j}\varepsilon_{2^i}\le\varepsilon_{2^l}+\sum_{j=2^l+1}^{\infty}\frac{\varepsilon_j}{j}\le C_1\varepsilon_{2^l}. \tag{3.3}$$

Let us consider the function

$$f_0(x)=\sum_{k=1}^{\infty}\varepsilon_{2^k}(D_{2^k}(x)-D_{2^{k-1}}(x))2^{-k/q}.$$

Then for $n\in[2^k;2^{k+1})$, $k\in\mathbb{Z}_+$, (3.1) and (3.3) yields

$$E_n(f_0)_p\le E_{2^k}(f_0)_p\le\|f_0-S_{2^k}(f_0)\|_p\le\sum_{i=k+1}^{\infty}\varepsilon_{2^i}2^{-i/q}\|D_{2^i}-D_{2^{i-1}}\|_p$$

$$\le C_2\sum_{i=k+1}^{\infty}\varepsilon_{2^i}\le C_3\varepsilon_{2^{k+1}}\le C_4\varepsilon_n.$$

On the other hand,

$$\sum_{i=l}^{\infty}\gamma_i\rho_i^r(f_0)=\sum_{k=1}^{\infty}\sum_{i=2^{k-1}}^{2^k-1}\gamma_i\varepsilon_{2^k}^r2^{-kr/q}=\sum_{k=1}^{\infty}\Gamma_k\varepsilon_{2^k}^r2^{-kr/q}. \tag{3.4}$$

Similarly to the proof of Theorem 3.2, under the condition $r<q$, one can show that from the conditions of Theorem 3.3 follows the divergence of the right-hand side of (3.4). In case $r=q$ and $\gamma\in A(\infty)$, we see that

$$\sum_{i=2^{k-1}}^{2^k-1}\gamma_i\varepsilon_i^q i^{-1}\le 2\max_{i\in[2^{k-1},2^k)}\gamma_i\varepsilon_{2^{k-1}}^q\le C_5\varepsilon_{2^{k-1}}^q2^{-(k-1)}\Gamma_{k-1},$$

whence we obtain the divergence of the right-hand side of (3.4) again. □

Theorem 3.2 is an analogue of Theorem 3 in [2] treating the case $2\le p\le\infty$ (the continuous functions $f\in C_{2\pi}$ were considered for $p=\infty$). Since the condition on Γ_k in the above-mentioned Theorem or Theorem 3.2 is too complicated, we give a corresponding analogue of Theorem 3.3.

Theorem 3.4. *Let* $0<r\le 2$, *a positive sequence* $\{\varepsilon_k\}_{k=0}^{\infty}$ *be decreasing to zero and satisfying the Bary condition* (3.2). *Also, we suppose that* $\varepsilon_n\le C\varepsilon_{2n}$ *for* $n\in\mathbb{N}$ *and* $\{\gamma_k\}_{k=0}^{\infty}\in A(2/(2-r))$. *If the series* $\sum\limits_{k=1}^{\infty}\gamma_k k^{-r/2}\varepsilon_k^r$ *diverges, then there exists* $f_0\in C_{2\pi}$ *such that* $E_n(f_0)_\infty=O(\varepsilon_n)$, *but the series* (1.3) *diverges for* $f=f_0$.

Proof. Let us consider the function

$$f_0(x)=\sum_{k=1}^{\infty}2^{-k/2}\varepsilon_{2^k}(P_{2^k}(x)-P_{2^{k-1}}(x)),$$

where the polynomials $P_{2^k}(x)$ are defined in Lemma 2.2. Then $P_{2^k}(x)-P_{2^{k-1}}(x)=\sum\limits_{i=2^{k-1}+1}^{2^k}\gamma_i\cos ix$, $\gamma_i=\pm1$, and $|P_{2^k}(x)-P_{2^{k-1}}(x)|\le 10(2^k+1)^{1/2}\le C_12^{k/2}$, $x\in[0,2\pi]$. For $n\in[2^k,2^{k+1})$, we have

$$E_n(f_0)_\infty\le E_{2^k}(f_0)_\infty\le\sum_{j=k+1}^{\infty}2^{-j/2}\varepsilon_{2^j}\|P_{2^j}-P_{2^{j-1}}\|_\infty$$

$$\le C_1\sum_{j=k+1}^{\infty}\varepsilon_{2^j}\le C_2\varepsilon_{2^{k+1}}\le C_2\varepsilon_n.$$

On the other hand, for $0 < r < 2$,

$$\sum_{k=1}^{\infty}\sum_{j=2^{k-1}}^{2^k-1}\gamma_j 2^{-kr/2}\varepsilon_{2^k}^r 2^{k-1} = \sum_{k=1}^{\infty}\Gamma_k 2^{-kr/2}\varepsilon_{2^k}^r \ge C_4\sum_{j=1}^{\infty} j^{-r/2}\gamma_j\varepsilon_j^r$$

by the condition $\varepsilon_m \le C_5\varepsilon_n$ for $n \in [m, 2m]$. In the case $r = 2$, we repeat the arguments at the end of the proof of Theorem 3.3. Thus series (1.3) diverges for $f = f_0$. □

Now we can obtain the sharpness of Theorem B.

Theorem 3.5. *Let $1 < p < \infty$, $l \in \mathbb{N}$, $1/p+1/q = 1$, $s = \max(q, 2)$, $0 < r \le s$, $\{\gamma_k\}_{k=0}^{\infty} \in A(s/(s-r))$. If $\omega \in B\bigcap B_l$ and the series $\sum_{k=1}^{\infty} k^{-r/s}\gamma_k\omega(k^{-1})$ diverges, then there exists $f_0 \in L^p$ such that $\omega_l(f_0, \delta)_p \le C\omega(\delta)$, $\delta \in [0, 2\pi]$, and the series (1.3) for $f = f_0$ diverges.*

Proof. Let $1 < p \le 2$. Let us consider $\varepsilon_n = \omega(1/n)$, $n \in \mathbb{N}$, and the function $f_0(x)$ from the proof of Theorem 3.3. Then $E_n(f_0)_p \le C_1\omega(1/n)$, $n \in \mathbb{N}$, and analogously, $E_0(f_0)_p \le \|f_0\|_p \le C_1\omega(1)$. By the converse approximation theorem in L^p (see Lemma 2.3), we have

$$\omega_l(f, 1/n) \le C_2 n^{-l}\sum_{k=0}^{n}(k+1)^{l-1}\omega((k+1)^{-1}) \le C_3\omega(n^{-1}), \quad n \in \mathbb{N},$$

by the condition $\omega \in B_l$. Note that the condition $\omega \in B_l$ is appropriate to use Theorem 3.3. Since $\omega \in B_l$ satisfies the Δ_2-condition $\omega(2t) \le C_7\omega(t)$, $t \in [0, \pi]$ (see Lemma 3 in [1]), we derive that $\omega_l(f_0, \delta) \le C_5\omega(\delta)$, $\delta \in [0, 2\pi]$. By Theorem 3.3, series (1.3) diverges for $f = f_0$. In the case $p > 2$, we analogously consider $\varepsilon_n = \omega(1/n)$ and the function f_0 from the proof of Theorem 3.4. Further, we proceed as in the case $1 < p \le 2$. □

The following two theorems are devoted to the sharpness of Theorem 3.1.

Theorem 3.6. *Let $1 < p < \infty$, $1/p + 1/q = 1$, $s = \max(q, 2)$, $0 < r \le s$, $\{\gamma_k\}_{k=0}^{\infty} \in A(s/(s-r))$. If $\{\varepsilon_k\}_{k=0}^{\infty}$ decreases to zero, satisfies the Bary condition (3.2), $\varepsilon_n \le C\varepsilon_{2n}$ for $n \in \mathbb{N}$, and the series*

$$\sum_{k=1}^{\infty}\gamma_k k^{-r/s-r/p}\varepsilon_k^r$$

diverges, then there exists $f_1 \in C_p$ such that $E_n(f_1)_{V_p} = O(\varepsilon_n)$, $n \in \mathbb{N}$, and the series (1.3) diverges for $f = f_1$.

Proof. In the case $1 < p \le 2$, similarly to the proof of Theorem 3.3, we consider the function

$$f_1(x) = \sum_{k=1}^{\infty} 2^{-k}\varepsilon_{2^k}(D_{2^k}(x) - D_{2^{k-1}}(x)). \tag{3.5}$$

Then for $n \in [2^k, 2^{k+1})$, $k \in Z_+$, from (3.1), (3.3) and Lemma 2.5 we deduce

$$E_n(f_1)_{V_p} \le \|f_1 - S_{2^k}(f_1)\|_{V_p} \le \sum_{i=k+1}^{\infty} 2^{-i}\varepsilon_{2^i}\|D_{2^k} - D_{2^{k-1}}\|_{V_p}$$

$$\le C_1\sum_{i=k+1}^{\infty}\varepsilon_{2^i}2^{-i/q}\|D_{2^k} - D_{2^{k-1}}\|_p \le C_2\sum_{i=k+1}^{\infty}\varepsilon_{2^i} \le C_3\varepsilon_{2^{k+1}} \le C_3\varepsilon_n.$$

On the other hand, by the condition $\varepsilon_m \le C\varepsilon_n$ for $n \in [m, 2m]$, $m \in \mathbb{N}$, we have

$$\sum_{i=1}^{\infty}\gamma_i\rho_i^r(f_0) = \sum_{k=1}^{\infty}\sum_{i=2^{k-1}}^{2^k-1}\gamma_i 2^{-kr}\varepsilon_{2^k}^r \ge C_4\sum_{i=1}^{\infty}\gamma_i\varepsilon_i^r i^{-r} = \infty.$$

In the case $p > 2$, similarly to the proof of Theorem 3.4, let us consider the function

$$f_1(x) = \sum_{k=1}^{\infty} 2^{-k/2-k/p}\varepsilon_{2^k}(P_{2^k}(x) - P_{2^{k-1}}(x)),$$

where $P_{2^k}(x)$ is defined in Lemma 2.2. Then for $n \in [2^k, 2^{k+1})$, $k \in \mathbb{Z}_+$, we have

$$E_n(f_1)_{V_p} \le E_{2^k}(f_1)_{V_p} \le \sum_{j=k+1}^{\infty} 2^{-j/2-j/p} \|P_{2^j} - P_{2^{j-1}}\|_{V_p}$$

$$\le C_1 \sum_{j=k+1}^{\infty} \varepsilon_{2^j} 2^{-j/2} ||P_{2^j} - P_{2^{j-1}}||_p \le C_4 \sum_{j=k+1}^{\infty} \varepsilon_{2^j} 2^{-j/2} ||P_{2^j} - P_{2^{j-1}}||_\infty$$

$$\le C_5 \sum_{j=k+1}^{\infty} \varepsilon_{2^j} \le C_6 \varepsilon_{2^{k+1}} \le C_7 \varepsilon_n.$$

Since $\lim_{n\to\infty} \varepsilon_n = 0$, the last relation and the completeness of C_p imply that $f_1 \in C_p$.

On the other hand,

$$\sum_{j=1}^{\infty} \gamma_j \rho_j^r(f_1) = \sum_{k=1}^{\infty} \sum_{j=2^{k-1}}^{2^k-1} \gamma_j 2^{-kr/2-kr/p} \varepsilon_{2^k}^r \ge C_4 \sum_{j=1}^{\infty} j^{-r/2-r/p} \gamma_j \varepsilon_j^r = \infty$$

and the series (1.3) diverges for $f = f_1$. □

Theorem 3.7. *Let $1 < p < \infty$, $1/p + 1/q = 1$, $s = \max(q, 2)$, $0 < r \le s$, $\{\gamma_k\}_{k=0}^{\infty} \in A(s/(s-r))$, $l \in \mathbb{N}$, $\omega \in B \cap B_{l-1/p}$. If the series $\sum_{k=1}^{\infty} \gamma_k k^{-r/s-r/p} \omega^r(k^{-1})$ diverges, then there exists a function $f_1 \in C_p$ such that $\omega_{l-1/p}(f_1, \delta) \le \omega(\delta)$, $\delta \in [0, 2\pi]$, and the series* (1.3) *diverges for $f = f_1$.*

Proof. Let us consider $\varepsilon_k = \omega(1/k)$, $k \in \mathbb{N}$ and the function $f_1(x)$ from (3.5). Then $E_n(f_1))_{V_p} \le C_1\omega((n+1)^{-1})$, $n \in \mathbb{Z}_+$. By the converse approximation theorem in C_p (see Lemma 2.4) we have

$$\omega_{l-1/p}(f_1, 1/n) \le C_2 n^{-l+1/p} \sum_{k=0}^{n} (k+1)^{l-1/p-1} \omega((k+1)^{-1}) \le C_3 \omega(1/n). \tag{3.6}$$

Since $\omega \in B_{l-1/p}$ satisfies the Δ_2-condition (see Lemma 3 in [1]), from (3.6) and the monotonicity of ω we easily deduce the inequality $\omega_{l-1/p}(f, \delta) \le C_4\omega(\delta)$, $\delta \in [0, 2\pi]$. On the other hand, by Theorem 3.6, we have $\sum_{k=1}^{\infty} \gamma_k \rho_k^r(f_1) = \infty$. □

4. The results for general orthonormal systems

Let $\{\varphi_k(x)\}_{k=1}^{\infty}$ be a complete in $L^2[0,1]$ orthonormal system. For $f \in L^2[0,1]$ we set

$$c_n(f) = \int_0^1 f(x)\overline{\varphi_n(x)}\, dx, \quad S_n^{\varphi}(f)(x) = \sum_{k=1}^{n} c_k(f)\varphi_k(x),$$

$$E_n^{\varphi}(f)_2 = \inf_{\alpha_i \in \mathbb{C}} \left\| f - \sum_{k=1}^{n} \alpha_k \varphi_k \right\|_{L^2[0,1]}, \quad n \in \mathbb{N}.$$

It is well known that

$$E_n^{\varphi}(f)_2 = \|f - S_n^{\varphi}(f)\|_{L^2[0,1]} = \left(\sum_{k=n+1}^{\infty} |c_k(f)|^2 \right)^{1/2}. \tag{4.1}$$

S. Stechkin [5] established a sharp condition of convergence of the series $\sum_{k=1}^{\infty} |c_k(f)|$. Using the method of proof of Theorem A in [2] and the first equality in (4.1), one can easily obtain

Theorem 4.1. *Let $0 < r < 2$, $\{\gamma_k\}_{k=0}^{\infty} \in A(2/(2-r))$, $f \in L^2[0,1]$ and the series*

$$\sum_{k=1}^{\infty} k^{-r/2} \gamma_k (E_k^{\varphi}(f)_2)^r$$

converge. Then we have

$$\sum_{k=2}^{\infty}\gamma_k|c_k(f)|^r \le C\sum_{k=1}^{\infty}k^{-r/2}\gamma_k(E_k^{\varphi}(f)_2)^r<\infty.$$

The following counterpart of Theorem 3.3 shows the sharpness of Theorem 4.1.

Theorem 4.2. *Suppose that $\{\varepsilon_i\}_{i=1}^{\infty}$ decreases to zero and satisfies the Bary condition (3.2), $0<r< 2$, $\{\gamma_k\}_{k=0}^{\infty}\in A(2/(2-r))$. If the series*

$$\sum_{i=1}^{\infty}\gamma_i i^{-r/2}\varepsilon_i^r$$

diverges, then there exists $f_0\in L^2[0,1]$ such that $E_n^{\varphi}(f_0)_2\le C\varepsilon_n$, $n\in\mathbb{N}$, but the series

$$\sum_{k=1}^{\infty}\gamma_k|c_k(f)|^r$$

diverges.

Proof. Let us consider Dirichlet kernels $D_n^{\varphi}(x)=\sum\limits_{k=1}^{n}\varphi_k(x)$. Since $\{\varphi_k(x)\}_{k=1}^{\infty}$ is orthonormal on $[0,1]$, we have

$$\|D_n^{\varphi}\|_{L^2[0,1]}=n^{1/2},\quad \|D_n^{\varphi}-D_m^{\varphi}\|_{L^2[0,1]}=(n-m)^{1/2},\quad n,m\in\mathbb{N},\ n\ge m. \tag{4.2}$$

Just as in the proof of Theorem 3.3, we consider

$$f_0(x)=\sum_{k=1}^{\infty}\varepsilon_{2^k}(D_{2^k}^{\varphi}(x)-D_{2^{k-1}}^{\varphi}(x))2^{-k/2}.$$

Using (4.2) and (3.3), we find for $n\in[2^k,2^{k+1})$, $k\in\mathbb{Z}_+$, that

$$E_n^{\varphi}(f_0)_2\le E_{2^k}^{\varphi}(f_0)_2=\|f_0-S_{2^k}(f_0)\|_{L^2[0,1]}$$
$$\le\sum_{i=k+1}^{\infty}\varepsilon_{2^i}2^{-i/2}\|D_{2^i}^{\varphi}-D_{2^{i-1}}^{\varphi}\|_{L^2[0,1]}\le\sum_{i=k+1}^{\infty}\varepsilon_{2^i}\le C_1\varepsilon_{2^{k+1}}\le C_1\varepsilon_n.$$

On the other hand,

$$\sum_{i=1}^{\infty}\gamma_i|c_i(f_0)|^r\ge\sum_{k=1}^{\infty}\sum_{i=2^{k-1}+1}^{2^k}\gamma_i\varepsilon_{2^k}^r2^{-kr/2}.$$

As in the proof of Theorem 3.2, we have

$$\sum_{i=2^{k-1}+1}^{2^k}\gamma_i\varepsilon_i^r i^{-r/2}\le C_2\Gamma_{k-1}2^{-(k-1)r/2}\varepsilon_{2^{k-1}}^r$$

and from the embedding $A(2/(2-r))\subset A(1)$, we can see that $\Gamma_k\le C_3\Gamma_{k-1}$, $k\in\mathbb{N}$. Thus we obtain

$$\sum_{i=1}^{\infty}\gamma_i|c_i(f_0)|^r\ge\sum_{k=1}^{\infty}\Gamma_{k-1}2^{-kr/2}\varepsilon_{2^k}^r\ge C_3^{-1}\sum_{k=1}^{\infty}\Gamma_k2^{-kr/2}\varepsilon_{2^k}^r$$
$$\ge(C_2C_3)^{-1}\sum_{k=1}^{\infty}\sum_{i=2^{k-1}+1}^{2^k}\gamma_i\varepsilon_i^r i^{-r/2}=\infty.$$

□

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