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AN INITIAL-BOUNDARY VALUE PROBLEM FOR SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH A DIFFERENTIAL OPERATOR OF GEGENBAUER TYPE

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Abstract. This article discusses the initial boundary value problem for the systems of linear parabolic equations. The system is written in a matrix form. Its elements are polynomials with the Gegenbauer operator having the same order. The class of functions is located in which a way that the problem under consideration is correct, i.e., there is only a unique solution that depends on the initial function. Explicit formulas for solving the problem are given, while the method of generalized functions is developed by Gelfand and Shilov.

1. Introduction and Statement of the Problem

As is known from the classical analysis, various integral transforms are one of the important methods in solving a number of problems of mathematical analysis and differential equations. The fact is that in studying the initial boundary value problems for systems of parabolic equations, it is required to find a class of functions in which the problem is correct, that is the class in which the system has a solution involved in this class for each $t \geq 0$ and satisfying the initial and boundary conditions. In this class, the solution should be unique and depend continuously on the initial function.

To solve these problems, in [5], the Fourier transform method, and in [14], the Fourier–Bessel transform method were used. The prescribed conditions on the class of functions ensure the existence and uniqueness of solutions. In [11], the conditions for the well-posedness of the Cauchy problem are given under the assumption that the initial functions are bounded. This is an assumption was not connected with the essence of the problem, but only by using the Fourier method. It is known that, for example, for hyperbolic equations, the existence and the uniqueness of the solution to the Cauchy problem takes place without any restrictions on the growth of the initial function at infinity, since the values of the solution in point depend only on those of the initial function inside the corresponding cone of characteristics.

For the heat-conducting equation, the natural area, as is shown in [13], ensuring the existence and uniqueness of the solution of the Cauchy problem is the class of functions that, for each $t \geq 0$ and for $|x| \to \infty$ grow more slowly than e^{x^2}. This result was extended by O. A. Ladyzhenskaya [10] to any parabolic equation and then by S. D. Eidel'man [4] to any parabolic system. Thus, to apply the Fourier method to the questions dealing with the initial boundary value problem for certain equations, it is necessary to construct the theory of the Fourier transform for the corresponding classes of functions. A rigorous substantiation of the Fourier transform for slowing down of growing functions was carried out by Bochner [1] and later by Carleman [2] by using somewhat different method. L. Schwartz in his book [12] treating sequentially the zero of the Fourier transform as a functional, gave a systematic structure of the theory, essential for slowly growing functions.

I. M. Gelfand and G. E. Shilov in [5] constructed the Fourier transform of functions that grow as fast as desired. With this transform, the uniqueness of the solution of the Cauchy problem for equations with partial derivatives with coefficients depends only on time. This was the solution to the problem posed in [11]. The basis of the theorem of uniqueness is essentially the Phragmén–Lindelöf theorem.

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Consideration of generalized functions over arbitrary spaces of basic functions allows us to solve the issues related to the uniqueness, but solved the problem.

The range of these issues and the results in this direction are covered in detail in [5].

In this article, to solve the problem, we use the method of generalized functions developed in [5] in combination with the Gegenbauer transform method introduced in [6], which play the same role as the Fourier transform method in [5].

We consider the following system of linear differential equations:

$$\frac{\partial u(x,t)}{\partial t} = P(G,t)\, u(x,t), \tag{1.1}$$

where $1 \le x < \infty$, $t \ge 0$, $u(x,t) = \{u_1(x,t), \dots, u_m(x,t)\}$ is a vector function and $P(G,t)$ is a matrix of dimension $m \times m$, whose entries are polynomials of the Gegenbauer operator G (see [3])

$$G = (x^2-1)\frac{d^2}{dx^2} + (2\lambda+1)x\frac{d}{dx}, \quad 1 < x < \infty, \quad 0 < \lambda < \frac{1}{2}$$

of the same order $P \ge 0$. The coefficients of the operator G and its powers can be either constant or continuously depend on the time. The initial and boundary conditions for system (1.1)

$$u(x,0) = u_0(x), \quad \frac{\partial u(x,t)}{\partial x}\Big|_{x=1} = 0. \tag{1.2}$$

It is required to find a class of functions in which problem (1.1)–(1.2) is correct, that is, a class in which system (1.1) has a solution $u(x,t)$ included in this class for every $t \ge 0$ and satisfying conditions (1.2). In this class, the solution should be unique and continuously depend on the initial function $u_0(x)$.

In obtaining explicit formulas for solving the problem (1.1)–(1.2), the generalized shift operator (GSO, in short) associated with the Gegenbauer operator G considered in [6] is of importance.

The results of this article are presented according to the following scheme. In Section 2, we present some properties of the Gegenbauer transforms and the generalized shift operator.

Section 3 studies Gegenbauer's transformations in the class of generalized functions.

Section 4 is devoted to proving the existence and uniqueness of the solution to the problem posed and finding an explicit formula for the resulting solution.

2. The Gegenbauer Transform and its Properties

A function $\varphi(x)$ is called a main function if it is infinitely differentiable, even and satisfies the estimate $|D^q\varphi(x)| \le \frac{C_{qr}}{(1+x^2)^r}$. The G operator can be applied to the main functions as many times as desired, and the estimate

$$|G^q\varphi(x)| \le \frac{C_{qr}}{(1+x^2)^r},$$

where $q \ge 0$ and $r \le 0$ are arbitrary integers, C_{qr} are the constants, independent of x. By S_G we denote the space of all basic functions. Note that the even functions from the space S of basic functions, introduced in the paper [5], form the space S_G of main functions under consideration. The S - linear space and the convergence of a sequence of functions to zero in S_G is defined in the same way as in [5]. Thus $S_G \equiv S$.

In [6], for the main functions, the direct F_P, F_Q and the inverse F_P^{-1}, F_Q^{-1} Gegenbauer transforms were defined by the following formulas:

$$F_P f(t) \to \hat{f}_P(\alpha) = \int_1^\infty f(t) P_\alpha^\lambda(t)\, dm_\lambda(t),$$

$$F_Q f(t) \to \hat{f}_Q(\alpha) = \int_1^\infty f(t) Q_\alpha^\lambda(t)\, dm_\lambda(t),$$

$$F_P^{-1}\hat{f}_P(\alpha) \to f(x) = C_\lambda \int\limits_1^\infty \hat{f}_P(\alpha))Q_\alpha^\lambda(x)dm_\lambda(\alpha),$$

$$F_Q^{-1}\hat{f}_Q(\alpha) \to f(x) = C_\lambda \int\limits_1^\infty \hat{f}_Q(\alpha))P_\alpha^\lambda(x)dm_\lambda(\alpha),$$

where $dm_\lambda(x) = (x^2-1)^{\lambda-\frac{1}{2}}dx$, and

$$C_\lambda = \frac{2^{\frac{3}{2}-\lambda}\Gamma(\frac{1}{2})F(\lambda+1)\Gamma(\frac{1}{2}-\lambda)\Gamma(\frac{3+2\lambda}{4})[\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{5-2\lambda}{4})\cos\pi\lambda]^{-1}}{{}_2F_1(1,\frac{1}{2}-\lambda;\frac{5-2\lambda}{4};\frac{1}{2}) -{}_2\, F_1(t,\frac{1}{2}-\lambda;\frac{5-2\lambda}{4};\frac{1}{2})}$$

is a normalizing factor, and ${}_2F_1(a;b;c;x)$ is a Gaussian hypergeometric function (see [7]). Here, $P_\alpha^\lambda(x)$ and $Q_\alpha^\lambda(x)$ are linearly independent solutions of the Gegenbauer equation (see [3])

$$(x^2-1)y'' + (2\lambda+1)xy' - \alpha(\alpha+2\lambda)y = 0.$$

In [6], it is proved that for any $f \in S$,

$$F_P F_P^{-1} f = f = F_P^{-1} F_P f,$$

and also

$$F_Q F_Q^{-1} f = f = F_Q^{-1} F_Q f,$$

therefore the direct and inverse Gegenbauer transforms are mutually inverse automorphisms of the space S, i.e., $\check{S} = \hat{S} = S$. Here, $F_P^{-1}S = \check{S}$, $F_P S = \hat{S}$.

The multiplier (see [5]) for the space S is any infinitely differentiable function f that, together with its all derivatives, increases as $x \to \infty$ no faster than some power x.

In [6], the concept of generalized convolution of two functions from S to $[1,\infty)$ is defined as

$$(f * g)(x) = \int\limits_1^\infty g(t)A_t^\lambda f(x)dm_\lambda(t),$$

generated by the generalized shift operator

$$A_t^\lambda f(x) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int\limits_0^\pi f(xt - \sqrt{x^2-1}\sqrt{t^2-1}\cos\varphi)(\sin\varphi)^{2\lambda-1}d\varphi,$$

associated with the Gegenbauer differential operator G.

The propertries of Gegenbauer's GSO are listed in [6]. We will give only those of them that we will need in the future.

1. $A_t^\lambda P_\alpha^\lambda(x) = P_\alpha^\lambda(t)Q_\alpha^\lambda(x)$, $A_t^\lambda Q_\alpha^\lambda(x) = C_\alpha^\lambda(x)Q_\alpha^\lambda(x)$,
where $Q_\alpha^\lambda(x) = \frac{\Gamma(2\lambda)\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda)}C_\alpha^\lambda(x)$,
2. Self-adjointness. For $f, g \in S$ $\int\limits_1^\infty g(t)A_t^\lambda f(x)dm_\lambda(t) = \int\limits_1^\infty f(t)A_t^\lambda g(x)dm_\lambda(t)$,
3. $A_t^\lambda f(x) = A_x^\lambda f(t)$ is a symmetry,
4. $\widehat{(A_t^\lambda)}_P(\alpha) = A_t^\lambda \hat{f}_P(\alpha) = \hat{f}_P(\alpha)Q_\alpha^\lambda(t)$ for $\forall f \in L_{1,\lambda}$,
5. $\widehat{(f*g)}_P(\alpha) = \hat{f}_P(\alpha)\,\hat{g}_Q(\alpha)$ and $\widehat{(f*g)}_Q(\alpha) = \hat{f}_Q(\alpha)\,\hat{g}_Q(\alpha)$
for any $f, g \in S$, where $f * g$ is the convolution of functions which is defined above.

3. Gegenbauer Transform of Generalized Functions

The generalized function, as is customary, is a linear continuous functional defined on the space S of basic functions.

Functionals of the type of the function are linear continuous functionals acting on an arbitrary function $\varphi \in S$ by the formula

$$\langle T, \varphi \rangle = \int_1^\infty T(x)\varphi(x) dm_\lambda(x). \tag{3.1}$$

We say that a locally summable function f defines a principal functional in S if it grows as $x \to \infty$ no faster than some fixed power x, i.e., it has a polynomial growht. Then for $\forall \varphi \in S$,

$$\langle f, \varphi \rangle = \int_1^\infty f(x)\varphi(x) dm_\lambda(x).$$

Theorem 3.1. *If the functions $T_1(x)$ and $T_2(x)$ define principal functionals $\langle T_1, \varphi \rangle$ and $\langle T_2, \varphi \rangle$ differ from each other on the set of positive measures, the functionals $\langle T_1, \varphi \rangle$ and $\langle T_2, \varphi \rangle$ are different, i.e., there is a function $\varphi_0 \in S$ such that $\langle T_1, \varphi_0 \rangle \neq \langle T_2, \varphi_0 \rangle$. Conversely, if the functionals $\langle T_1, \varphi \rangle$ and $\langle T_2, \varphi \rangle$ are different, then the functions T_1 and T_2 differ on a set of positive measure.*

Proof. Let there be given the functions T_1 and T_2 defining the functionals $\langle T_1, \varphi \rangle$ and $\langle T_2, \varphi \rangle$ and coinciding on each function $\varphi \in S$. Then the difference $T = T_1 - T_2$ also determines the main functional, which is identically zero:

$$\langle T, \varphi \rangle = \int_1^\infty T(x)\varphi(x) dm_\lambda(x) = 0 \quad \text{for all} \quad \varphi \in S.$$

Let us show that the function T vanishes almost everywhere, whence the required statement will follow. We denote by $L_{p,\lambda}[1, \infty) \equiv L_p([1, \infty), dm_\lambda)$, $1 \leq p \leq \infty$ the class of measurable functions with a finite norm

$$\|f\|_{p,\lambda} = \left(\int_1^\infty |f(x)|^p dm_\lambda(x) \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$
$$\|f\|_{\infty,\lambda} = \|f\|_\infty = \operatorname*{ess\,sup}_{x \in [1,\infty)} |f(x)|, \quad p = \infty.$$

Since the product of functions from S belongs also to S, for any sequence $\varphi_n \in S$, we have

$$\int_1^\infty T(x)\varphi(x)\varphi_n(x) dx = 0, \tag{3.2}$$

and since S is everywhere dense in $L_{p,\lambda}$ (see [9, Theorem 4.2]), therefore at least $T\varphi \in L_{2,\lambda}$ and there is a sequence $\varphi_n \in S$ such that $\lim\limits_{n\to\infty} \|T\varphi - \varphi_n\|_{L_{2,\lambda}} = 0$. Then passing to the limit in (3.2) as $n \to \infty$, we get

$$\int_1^\infty |T(x)\varphi(x)|^2 dm_\lambda(x) = 0,$$

whence it follows that the product $T\varphi$ is almost always zero. Since φ is any function from S which, in particular, can be considered as nonzero in a predetermined domain, the function T is almost everywhere zero, which was to be stated.

The converse statement of Theorem 3.1 is obvious, since the functions T_1 and T_2 coincide almost everywhere and lead to equal values of the functionals $\langle T_1, \varphi \rangle$ and $\langle T_2, \varphi \rangle$ for each function $\varphi \in S$. $\square$

Theorem 3.1 allows us to identify principal functionals like the function T with the functions T themselves. In addition, multiplication of generalized functions by a number and by a function (multiplier) is defined as in [5].

In the class S, the Gegenbauer transform and the operator G are symmetric in the certain sense (see [8]),

$$\int_1^\infty f(\alpha)\hat{g}_P(\alpha)dm_\lambda(\alpha) = \int_1^\infty \hat{f}_P(\alpha)g(\alpha)dm_\lambda(\alpha),$$

$$\int_1^\infty (Gf)(x)g(x)dm_\lambda(x) = \int_1^\infty f(x)(Gg)(x)dm_\lambda(x).$$

Therefore we naturally extend $\hat{f}_P(\alpha)$ and G to the space S' of generalized functions as follows: for $\forall T \in S'$ and $\psi \in S$, we set $\langle \hat{T}_P, \psi\rangle = \langle T, \hat{\psi}_2\rangle$ and $\langle GT, \psi\rangle = \langle T, G\psi\rangle$.

In [6], it has been proved that the Gegenbauer transform $\hat{f}_P(\alpha)$ of a generalized function f is a generalized function, that is, if $f \in S'$, then $\hat{f}_P(\alpha) \in S'$ and therefore the following properties hold for the functions $\varphi, \psi \in S'$:

1. $\widehat{(\varphi+\psi)}_P(\alpha) = \widehat{\varphi}_P(\alpha) + \widehat{\psi}_P(\alpha)$,
2. $\widehat{(\alpha\varphi)}_P(\alpha) = \alpha\widehat{\varphi}_P(\alpha)$,
3. If $\varphi_n \to \varphi$, then $\widehat{(\varphi_n)}_P(\alpha) \to \widehat{(\varphi)}_P(\alpha)$,
4. $(\widehat{P(G)\varphi})_P(\alpha) = P\widehat{(G\varphi)}_P(\alpha) = P(\alpha(\alpha+2\lambda))\widehat{\varphi}_P(\alpha)$,
5. $GP_\alpha^\lambda(x) = \alpha(\alpha+2\lambda)P_\alpha^\lambda(x) \Rightarrow P(G)(P_\alpha^\lambda(x)) = P(\alpha(\alpha+2\lambda))P_\alpha^\lambda(x)$.

$$P(G)\hat{f}_P(\alpha) = P(G)\int_1^\infty f(x)P_\alpha^\lambda(x)(x^2-1)^{\lambda-\frac{1}{2}}dx$$

$$= \int_1^\infty f(x)P(\alpha(\alpha+2\lambda))P_\alpha^\lambda(x)(x^2-1)^{\lambda-\frac{1}{2}}dx = (P(\alpha(\alpha+2\lambda))\widehat{f}_P(\alpha).$$

Further (see [6]), $\widehat{(G^r f)}_P(x) = (\alpha(\alpha+2\lambda))^r\widehat{f}_P(x)$. For $r=1$,

$$\widehat{(Gf)}_P(x) = (\alpha(\alpha+2\lambda))\widehat{f}_P(x)$$

$$\Rightarrow (\widehat{P(G)}f)_P(x\alpha) = P(\alpha(\alpha+2\lambda))\widehat{f}_P(\alpha).$$

The convolution of the functions f and g by $R_+ = [0,\infty)$ is given in the equality (see [6])

$$(f*g)(x) = \int_1^\infty g(t)A_t f(x)dm_\lambda(t),$$

moreover, if $f, g \in S$, then $f*g \in S$, as well.

In [6], it is proved that if $f \in S'$ and $g \in S$, then their convolution $f*g \in S'$.

In case $f \in S'$ and $g \in S$, their convolution $f*g$ is defined by the formula

$$(f*g)(x) = \langle f, A_x g\rangle, \quad x \in R_+.$$

The convolution $f*g$ of the generalized function f with the main function g is given by the equality

$$\langle f*g, h\rangle = \langle f, g*h\rangle,\ f \in S', \quad g, h \in S,$$

which at this location has the following properties (see [6]):

1) if $f \in S'$, and $g \in S$, then $f*g = g*f$,
2) $\widehat{(f*g)}_P(\alpha) = \widehat{f}_P(\alpha)\widehat{g}_Q(\alpha)$,
3) if $f \in S'$, and $g, h \in S$, then $(f*g)*h = f*(g*h)$,
4) $G^r(f*g) = (G^r f)*g = f*(G^r g)$,
5) $f*g = F_P^{-1}(\widehat{f}_P(x)\widehat{g}_Q(x))$.

Let T_1 and T_2 be of functional type functions with the usual convolution. Then

$$\langle T_1 * T_2, \varphi\rangle = \int_1^\infty \Big(\int_1^\infty A_t T_1(x) T_2(t) dm_\lambda(t)\Big)\varphi(x) dm_\lambda(x)$$

$$= \int_1^\infty \Big(\int_1^\infty A_t T_1(x)\varphi(x) dm_\lambda(x)\Big) T_2(t) dm_\lambda(t)$$

$$= \int_1^\infty \Big(\int_1^\infty T_1(x) A_t\varphi(x) dm_\lambda(x)\Big) T_2(t) dm_\lambda(t) = \langle T_2, T_1 * A_t\varphi\rangle.$$

A functional T is called a convolution in the space S of basic functions if for $f \in S$,

$$\langle T, A_t f\rangle = g(t) \in S.$$

If T_1 is a convolution in S and $T_2 \in S_1'$, then the convolution $T_1 * T_2$ is defined by the formula

$$\langle T_1 * T_2, \varphi = T_2, T_1' * A_t\varphi\rangle,$$

where T_1' is the matrix, transposed to T_1.

Theorem 3.2. *If a generalized function T is a multiplier in the space S of basic functions, then its Gegenbauer transform $F^{-1}T_P$ is a convolution in the same space.*

Proof. Since T is a multiplier in S, for any function $\varphi \in S$, $\varphi T \in S$. Then the generalized function T defines the main functional by formula (3.1). Therefore, as mentioned above, $\widehat{T}_P \in S'$, and we have

$$\langle F_P^{-1}T, A_t F_P\varphi\rangle = \langle F_P^{-1}T, F_P(A_t\varphi)\rangle$$

$$= \langle F_P^{-1}T, (F_P\varphi)Q_\alpha^\lambda\rangle = \langle T, \varphi \cdot Q_\alpha^\lambda\rangle$$

$$= \int_1^\infty T(t)\varphi(t)Q_\alpha^\lambda(t) dm_\lambda(t) = F_Q(T\varphi) \in \hat{S} = S. \qquad \square$$

By Property 5, we have $\widehat{f}_P = T_1 \Rightarrow f = F_P^{-1}T_1$; $\widehat{g}_Q = T_2 \Rightarrow g = F_Q^{-1}T_2$. Then we obtain

$$F_P^{-1}(T_1 \cdot T_2) = F_P^{-1}T_1 * F_Q^{-1}T_2.$$

4. Initial Boundary Value Problem

To solve the initial boundary value problem (1.1)–(1.2), consider system (1.1) as a system of relatively generalized vector functions $u(x,t)$ acting over the space of basic functions S. Using the Gegenbauer transform, we can go from (1.1)–(1.2) to the system

$$\frac{dv(\alpha(\alpha+2\lambda),t)}{dt} = P(\alpha(\alpha+2\lambda),t)\, v(\alpha(\alpha+2\lambda),t)$$

and the initial condition

$$v(\alpha(\alpha+2\lambda),0) = v_0(\alpha(\alpha+2\lambda)) = (u_0)_P^\wedge(\alpha),$$

the boundary condition $\frac{dv(\alpha(\alpha+2\lambda),t)}{d\alpha}\Big|_{\alpha(\alpha+2\lambda)=1} = 0$ is fulfilled due to the fact that

$$P_\alpha^\lambda(1) = \frac{\Gamma(\alpha+2\lambda)\cos\pi\lambda}{2^{\alpha+2\lambda}\Gamma(\lambda)\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha+1}{2})}.$$

Let $Q(\alpha(\alpha+2\lambda),0,t)$ be the matrix of the normal fundamental system of solutions of system (1.1):

$$Q(\alpha(\alpha+2\lambda),0,t)=\left\|\begin{matrix} v_1^{(1)}(\alpha(\alpha+2\lambda),0,t) & \dots & v_1^{(m)}(\alpha(\alpha+2\lambda),0,t) \\ v_2^{(1)}(\alpha(\alpha+2\lambda),0,t) & \dots & v_2^{(m)}(\alpha(\alpha+2\lambda),0,t) \\ \dots\dots & \dots & \dots\dots \\ v_m^{(1)}(\alpha(\alpha+2\lambda),0,t) & \dots & v_m^{(m)}(\alpha(\alpha+2\lambda),0,t) \end{matrix}\right\|.$$

Here, $v^{(j)}(\alpha(\alpha+2\lambda),0,t)=\left\{v_1^{(j)}(\alpha(\alpha+2\lambda),0,t),\dots,v_m^{(1)}(\alpha(\alpha+2\lambda),0,t)\right\}$ is the solution satisfying the initial condition $v_k^{(j)}(\alpha(\alpha+2\lambda),0,t)\Big|_{t=0}=\delta_k^j$, and δ_k^j is the Kronecker symbol. Together with each of its columns and the entire matrix $Q(\alpha(\alpha+2\lambda),0,t)$ it satisfies the equation

$$\frac{dQ(\alpha(\alpha+2\lambda),0,t)}{dt}=P(\alpha(\alpha+2\lambda),t)\,Q(\alpha(\alpha+2\lambda),0,t)$$

and, by the construction, the initial conditions are satisfied

$$Q(\alpha(\alpha+2\lambda),0,t)=E,$$

where E is the identity matrix. This is also due to the fact that there are linearly independent solutions $P_\alpha^\lambda(x)$ and $Q_\alpha^\lambda(x)$. Then the solution of the system (1.1)–(1.2) can be written in the form

$$v(\alpha(\alpha+2\lambda),0,t)=Q(\alpha(\alpha+2\lambda),0,t)\cdot\widehat{(u_0)}_P(\alpha)$$

and for the case of one equation ($m=1$), $Q(\alpha(\alpha+2\lambda),0,t)$ transforms into the function

$$Q(\alpha(\alpha+2\lambda),0,t)=e^{\int_0^t P(\alpha(\alpha+2\lambda),t)dt}.$$

Let $T^{[m]}[S]$ be the collection of all vector functions $\varphi(x)=\{\varphi_1(x),\dots,\varphi_m(x)\}$ with components taken from S (see [5]). As in [5], the following theorems hold.

Theorem 4.1. *If in the space S of basic functions $\varphi(S)$ the elements of the matrix $Q(\alpha(\alpha+2\lambda),0,t)$ are the multipliers for any $t\geq 0$, then the system* (1.1) *has a solution for any initial generalized vector of the function $(u_0)_P^\wedge(\alpha)\in T^{[m]}[S]$,*

$$v(\alpha(\alpha+2\lambda),t)=Q(\alpha(\alpha+2\lambda),0,t)\cdot\widehat{(u_0)}_P(\alpha)$$

and this solution continuously depends on the initial vector of the function $\widehat{(u_0)}_P(\alpha)$ in the sense of continuity, established for the space $T^{[m]}[S]$.

Theorem 4.2. *If in the space S of basic functions $\varphi(S)$ the elements of the matrix $Q(\alpha(\alpha+2\lambda),0,t)$ are the multipliers for any t, $0\leq t\leq t_0$, then system* (1.1)–(1.2) *may have only a unique solution in the class $T^{[m]}[S]$.*

Note 4.1. In these theorems, to solve the problem, we interpret equation (1.1) as an equation for a generaized vector function depending on t as a parameter and belonging to α for some space $T^{[m]}[S]$.

The solution to the problem (1.1)–(1.2) under the conditions of Theorem 4.1 can be obtained on the basis of Theorem 3.2. However, Theorem 4.1 puts restrictions on the growth of the matrix $Q(\alpha(\alpha+2\lambda),0,t)$ as $\alpha\to\infty$. Indeed, for the matrix $Q(\alpha(\alpha+2\lambda),0,t)$ to be a multiplier in the space S of basic functions, it is necessary that it and all its derivatives grow on the real axis no faster than some polynomial

$$|D^qQ(\alpha(\alpha+2\lambda),0,t)|\leq C(1+\alpha^2)^k, \tag{4.1}$$

where $k>0$ is fixed and $q>0$ is arbitrary.

If condition (4.1) is satisfied for a system, then this system is called regular.

For regular systems $F_P^{-1}Q(\alpha(\alpha+2\lambda),0,t)$ there is, according to Theorem 3.2, a convolution in the space $\widehat{T}_P[S]=T[\widehat{S}_P]=T[S]$ and the convolution formula

$$u(x,t)=F_P^{-1}v(\alpha(\alpha+2\lambda),t)=F_P^{-1}[Q(\alpha(\alpha+2\lambda),0,t)\widehat{(u_0)}_P(\alpha)]$$

$$= F_P^{-1} Q(\alpha(\alpha+2\lambda),0,t) * u_0(x)$$

holds. Thus

$$u(x,t) = F_P^{-1} Q(\alpha(\alpha+2\lambda),0,t) * u_0(x). \tag{4.2}$$

Formula (4.2) gives a solution to problem (1.1)–(1.2) for arbitrary systems of the form (1.1).

Based on Theorems 4.1 and 4.2, we can formulate the following theorems defining the class of initial data in which the solution to problem (1.1)–(1.2) exists and is unique.

Theorem 4.3. *If the vector function $u_0(x)$ satisfies the inequality*

$$|u_0(x)| \le C_1(1+x^2)^k, \tag{4.3}$$

where $k > 0$ is an arbitrary integer, then the solution to problem (1.1)–(1.2) *exists in the class of generalized vector functions $u(x,t)$, which for each $t \ge 0$ belongs to the space $T[S]$.*

Theorem 4.4. *If the vector function $u_0(x)$ satisfies inequality* (4.3), *then the solution to problem* (1.1)–(1.2) *is always unique in class of generalized vector functions $T[S]$.*

Remark 4.2. The results of Theorems 4.1–4.4 are also true for systems of a more general form

$$\frac{\partial u(x,t)}{\partial t} = P(G_1, G_2, \ldots, G_n, t)u(x,t)$$

with the initial and boundary conditions

$$u(x,0) = u_0(x), \quad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=1} = 0,$$

where $G_i = (x_i^2-1)\frac{\partial^2}{\partial x_i^2} + (2\lambda+1)x_i\frac{\partial}{\partial x_i}$, $(i = 1,2,\ldots,n)$.

However, this remark does not imply that the solution of problem (1.1)–(1.2) for arbitrary regular systems of the form (1.1) will be an ordinary function. The latter is true, for example, for parabolic systems of the following form.

System (1.1) will be called parabolic if the matrix $Q(\alpha(\alpha+2\lambda),0,t)$, defined above, satisfies the inequality

$$|Q(\alpha(\alpha+2\lambda),0,t)| \le C_1 e^{-|\alpha|^{2q}}, \tag{4.4}$$

for arbitrary $t, t_0 \le t \le T, t_0 \ge 0$, where q is the highest degree of the Gegenbauer operators in system (1.1). We represent the matrix $P(G,t)$ on the right-hand side of system (1.1) in the form

$$P(G,t) = P_1(G,t) + P_2(G,t),$$

where $P_1(G,t)$ is the matrix whose elements contain the highest powers of the Gegenbauer operator, $P_2(G,t)$ is the matrix of the remaining terms. Using the theorem of I. G. Petrovsky [11], we can assert that for condition (4.4) to be satisfied, it is sufficient that the real parts of all roots of the matrix $\lambda E - P_2(\alpha(\alpha+2\lambda),t)$ are negative numbers for $\alpha(\alpha+2\lambda) = 2$.

Since the elements of matrix $Q(\alpha(\alpha+2\lambda),0,t)$ are integer analytic functions of $\alpha(\alpha+2\lambda)$ and condition (4.4) holds, $Q(\alpha(\alpha+2\lambda),0,t)$ belongs to S and, therefore it is a multiplier in the space S. Thus $F_P^{-1}Q(\alpha(\alpha+2\lambda),0,t)$ is a convolution in S. Since $F_P^{-1}Q(\alpha(\alpha+2\lambda),0,t) \in S$, we can get the convolution $F_P^{-1}Q(\alpha(\alpha+2\lambda),0,t) * u_0(x)$ which gives a solution to problem (1.1)–(1.2) written as an ordinary integral

$$\begin{aligned}
&\langle F_P^{-1}Q(\alpha(\alpha+2\lambda),0,t) * u_0(x), \varphi\rangle \\
&= \int_1^\infty u_0(y)\Big(\int_1^\infty F_P^{-1}Q(\alpha(\alpha+2\lambda),0,t)(x)A_y^\lambda\varphi(x)d\mu_\lambda(x)\Big)d\mu_\lambda(y) \\
&= \int_1^\infty \varphi(x)\Big(\int_1^\infty A_y^\lambda F_P^{-1}Q(\alpha(\alpha+2\lambda),0,t)(x)u_0(y)d\mu_\lambda(y)\Big)d\mu_\lambda(x)
\end{aligned}$$

$$= \int_1^\infty \left(F_P^{-1} Q(\alpha(\alpha+2\lambda), 0, t) * u_0(x)\right) \varphi(x) d\mu_\lambda(x).$$

So, the convolution $F_P^{-1}Q(\alpha(\alpha+2\lambda), 0, t)*u_0(x)$ acts on an arbitrary function $\varphi \in S$ as a functional like an ordinary function $F_P^{-1}Q(\alpha(\alpha+2\lambda), 0, t)*u_0(x)$. Therefore, by Theorem 3.2, $F_P^{-1}Q(\alpha(\alpha+2\lambda), 0, t)*u_0(x)$ is an ordinary function, and the following formula

$$u(x,t) = F_P^{-1}Q(\alpha(\alpha+2\lambda), 0, t) * u_0(x)$$
$$= \int_1^\infty A_y^\lambda F_P^{-1} Q(\alpha(\alpha+2\lambda), 0, t)(x) u_0(y) d\mu_\lambda(y)$$

holds.

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