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ON SOME LOCAL PROPERTIES OF THE CONJUGATE FUNCTION AND THE MODULUS OF SMOOTHNESS OF FRACTIONAL ORDER

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Abstract. In the present paper, we study the behavior of the smoothness of fractional order of the conjugate functions of many variables at a fixed point in the space C if the global smoothness, as well as the behavior of the original functions at this point are known. The direct estimates are obtained and exactness of these estimates are established by proper examples.

1. Introduction

Let R^n $(n = 1, 2, \dots;\ R^1 \equiv R)$ be the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$ with real coordinates. Let B be an arbitrary non-empty subset of the set $M = \{1, \dots, n\}$. Denote by $|B|$ the cardinality of B. Let x_B be such a point in R^n whose coordinates with indices in $M \backslash B$ are zero.

As usual, $C(T^n)$ $(C(T^1) \equiv C(T))$, where $T = [-\pi, \pi]$, denotes the space of all continuous functions $f : R^n \to R$ that are 2π-periodic in each variable endowed with the norm

$$\|f\| = \max_{x \in T^n} |f(x)|.$$

If $f \in L(T^n)$, then following Zhizhiashvili [14], we call the expression

$$\widetilde{f}_B(x) = \Big(-\frac{1}{2\pi}\Big)^{|B|} \int\limits_{T^{|B|}} f(x + s_B) \prod_{i \in B} \cot\frac{s_i}{2}\, ds_B$$

the conjugate function of n variables with respect to those variables whose indices form the set B (with $\widetilde{f}_B \equiv \widetilde{f}$ for $n = 1$).

Suppose that $f \in C(T^n)$, $1 \le i \le n$, and $h \in R$. For each $x \in R^n$, let us consider the difference of fractional order α $(\alpha > 0)$

$$\Delta_i^\alpha(h)\, f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x_1, \dots, x_{i-1}, x_i + j\,h, x_{i+1}, \dots, x_n),$$

where

$$\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} \ \text{ for } j > 1, \quad \binom{\alpha}{j} = \alpha \ \text{ for } \ j = 1, \quad \binom{\alpha}{j} = 1 \ \text{ for } j = 0.$$

Then define the partial modulus of smoothness of order α $(\alpha > 0)$ of the function f with respect to the variable x_i by the equality

$$\omega_{\alpha,i}\,(f; \delta) = \sup_{|h| \le \delta} \|\Delta_i^\alpha(h)\, f\|\,.$$

$(\Delta_i^\alpha(h)\, f(x) \equiv \Delta^\alpha(h)\, f(x)$ and $\omega_{\alpha,i}\,(f; \delta) \equiv \omega_\alpha\,(f; \delta)$ for $n = 1)$.

Let Φ_α $(\alpha > 0)$ be the set of nonnegative, continuous functions $\varphi(\delta)$ on $[0, 1)$ such that

1. $\varphi(\delta) = 0$,
2. $\varphi(\delta)$ is nondecreasing,
3. $\int\limits_0^\delta \frac{\varphi(t)}{t} dt = O(\varphi(\delta))$,

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4. $\delta^{\alpha}\int\limits_{\delta}^{1}\frac{\varphi(t)}{t^{\alpha+1}}dt = O(\varphi(\delta))$.

Note that when $\alpha = k$ is an integer, then the class Φ_{α} coincides with the well-known Bari–Stechkin class of order k.

Let φ be a nonnegative, nondecreasing continuous function defined on $[0,1)$ with $\varphi(\delta) = 0$. Then we denote by $H_i^{\alpha}(\varphi; C(T^n))$ $(i = 1,\dots,n)$ the set of all functions $f \in C(T^n)$ such that

$$\omega_{\alpha,i}(f;\delta) = O\big(\varphi(\delta)\big), \quad \delta \to 0+, \quad i = 1,\dots,n.$$

We set

$$H^{\alpha}\big(\varphi; C(T^n)\big) = \bigcap_{i=1}^{n} H_i^{\alpha}\big(\varphi; C(T^n)\big).$$

By I we denote the following subset of the set $\mathbb{R}^n$: $\big\{x : x = \big(\underbrace{\overline{x},\dots,\overline{x}}_{n}\big); \overline{x} \in T\big\}$.

Definition 1. We say that a function φ is almost decreasing in $[a,b]$ if there exists a positive constant A such that $\varphi(t_1) \geq A\,\varphi(t_2)$ for $a \leq t_1 \leq t_2 \leq b$.

Definition 2. If for $f \in \mathbb{C}(\mathbb{T})$ there exists $g \in \mathbb{C}(\mathbb{T})$ such that $\lim\limits_{h\to 0+}\|h^{-\alpha}\Delta^{\alpha}(h)f - g\| = 0$, then g is called the Liouville–Grunwald derivative of order $\alpha > 0$ of f in the $\mathbb{C}(\mathbb{T})$-norm, denoted by $D^{\alpha}f$.

Moduli of smoothness play a basic role in approximation theory, Fourier analysis and their applications. For a given function f, they essentially measure the structure or smoothness of the function via the k-th difference $\Delta_i^k(h)\,f(x)$. In fact, for the functions f belonging to the Lebesgue space L^p or the space of continuous functions C, the classical k-th modulus of continuity has turned out to be a rather good measure for determining the rate of convergence of best approximation. In this direction one can see books [7] and [13].

In 1977, P. L. Butzer, H. Dyckhoff, E. Goerlich, R. L. Stens and R. Tabersky ([1,12]) introduced the modulus of smoothness of fractional order. This notion can be considered as a direct generalization of the classical modulus of smoothness and is more natural to use for a number of problems in harmonic analysis.

In the theory of real functions there is the well-known theorem of Privalov on the invariance of the Lipschitz classes under the conjugate function $\widetilde{f}$. Namely, in 1916, Privalov proved that if the function f belongs to the class $\mathrm{Lip}(\alpha, C(T))(0 < \alpha < 1)$, then $\widetilde{f} \in \mathrm{Lip}(\alpha, C(T))$. For $\alpha = 1$, Privalov's theorem is not valid. Analogous problem in terms of modulus of smoothness of fractional order was considered by Samko and Yakubov [11]. They proved that the generalized Hölder class $H^{\alpha}\big(\varphi; C(T)\big)$ $(\varphi \in \Phi_{\alpha}, \alpha > 0)$ is invariant under the operator $\widetilde{f}$.

In 1924, Zygmund obtained more strong result than Privalov's theorem. Namely, if the function $f \in C(T)$ and

$$\int\limits_{0}^{\pi}\frac{\omega(f;t)}{t}dt < +\infty,$$

then $\widetilde{f}$ exists everywhere, $\widetilde{f} \in C(T)$ and

$$\omega(\widetilde{f};\delta) \leq A\bigg[\int\limits_{0}^{\delta}\frac{\omega(f;t)}{t}dt + \delta\int\limits_{\delta}^{\pi}\frac{\omega(f;t)}{t^2}dt\bigg], \ \ 0 < \delta \leq \frac{\pi}{2}.$$

From this result we get that if the modulus of continuity ω satisfies the so-called Zygmund's condition, then the class $H(\omega; C(T))$ is invariant under the operator $\widetilde{f}$.

In 1945, Zygmund proved that the analogous theorem of the Privalov theorem is valid for the modulus of continuity of second order in case $\alpha = 1$.

Afterwards, Bari and Stechkin studied the necessary and sufficient conditions for the invariance of classes $H(\varphi; C(T))$ under the conjugate operator.They showed that Zygmund's condition is exact for the invariance of $H(\omega; C(T))$ classes under the operator $\widetilde{f}$.

As to the functions of many variables, the first result in this direction belongs to Cesari and Zhak. They showed that the class $\mathrm{Lip}(\alpha, C(T^2))(0 < \alpha < 1)$ is not invariant under the conjugate operators of two variables.

Later, there were obtained the sharp estimates for partial moduli of continuity of different orders in the space of continuous functions [3, 4, 10].

The case $k = 1$ is considered by Okulov in [10, Theorem 2, Theorem 3] and the cases $k \geq 2$ were considered by us in [3, Theorem] and [4, Theorem].

The cases when moduli of continuity of different orders satisfy Zygmund's condition were considered in works [2, 8, 9].

In [6], the exact estimates of the partial moduli of smoothness of fractional order of the conjugate functions of several variables are obtained in the space $H\big(\varphi; C(T^n)\big)$ with the condition $\varphi \in \Phi_\alpha$, $\alpha > 0$.

In the present work, we study the behavior of the smoothness of conjugate functions $\widetilde{f}_B$ on the set I. If we restrict the function $\widetilde{f}_B$ on the set I, we can consider it as a function of one variable. There arises the following question: what can we say about the smoothness of this "new function" if the function f belongs to $H(\varphi; C(T^n))$ and $\varphi \in \Phi_\alpha$, $\alpha > 0$.

Note that the analogous problem is considered in [5] when the function f belongs to $H(\omega_k; C(T^n))$ and the modulus of continuity ω_k satisfies Zygmund's condition.

We now state the facts based on the proof of the main result.

Lemma 1 (see [6]). *If $\varphi \in \Phi_\alpha (\alpha > 0)$, then the function $\frac{\varphi(t)}{t^\alpha}$ is almost decreasing in $[0, 1]$.*

Lemma 2 (see [6]). *If $\varphi \in \Phi_\alpha (\alpha > 0)$, then there exists a real number β $(0 < \beta < \alpha)$ such that the function $\frac{\varphi(t)}{t^\beta}$ is almost decreasing in $[0, 1]$.*

2. Main result

The following theorem is valid.

Theorem.

a) *Let $f \in H^\alpha(\varphi; C(T^n))$ and $\varphi \in \Phi_\alpha$, $\alpha > 0$. Then*

$$\sup_{h \in I, |\overline{h}| \leq \delta} \sup_{x \in I} | \Delta_j^\alpha(\overline{h}) \, \widetilde{f}_B(x) | = O(\varphi(\delta) \mid \ln \delta \mid^{|B|-1}), \;\; j \in B, \;\; \delta \to 0+,$$

$$\sup_{h \in I, |\overline{h}| \leq \delta} \sup_{x \in I} | \Delta_j^\alpha(\overline{h}) \, \widetilde{f}_B(x) | = O(\varphi(\delta) \mid \ln \delta \mid^{||}), \;\; j \in M \setminus B, \;\; \delta \to 0+.$$

b) *For each $B \subseteq M$, there exist the functions F and G such that $F, G \in H^\alpha(\varphi; C(T^n))$ and*

$$\sup_{h \in I, |\overline{h}| \leq \delta} \sup_{x \in I} | \Delta_j^\alpha(\overline{h}) \, \widetilde{F}_B(x) | \geq C\varphi(\delta) \mid \ln \delta \mid^{|B|-1}, \;\; j \in B, \;\; 0 \leq \delta \leq \delta_0, \tag{1}$$

$$\sup_{h \in I, |\overline{h}| \leq \delta} \sup_{x \in I} | \Delta_j^\alpha(\overline{h}) \, \widetilde{G}_B(x) | \geq C\varphi(\delta) \mid \ln \delta \mid^{|B|}, \;\; j \in M \setminus B, \;\; 0 \leq \delta \leq \delta_0, \tag{2}$$

where C and δ_0 are positive constants.

Proof.

a) Part a) is the particular case of the first part of the theorem given in [6].

b) Without loss of generality, we shall carry out the proof of part (b) for the case $B = \{1, \dots, n-1\}$.

We consider a strictly decreasing sequence of positive numbers $(b_l)_{l \geq 1}$ such that

1. $\sum\limits_{l=0}^{\infty} b_l \leq 1 \quad (b_0 = 0)$;
2. $\sum\limits_{i=l+1}^{\infty} b_i < b_l$;
3. $\varphi^{-1}(b_{l+1}) < (\varphi^{-1}(b_l))^{\frac{\alpha}{\alpha - \beta}}$, where $\varphi^{-1}(b_l) (l = 1, 2, \dots)$ is a certain element of the set $\{t : \varphi(t = b_l)\}$ and β $(0 < \beta < \alpha)$ satisfies the condition of Lemma 2.

We set

$$\tau_l = 2\sum_{j=0}^{l-1} \varphi^{-1}(b_j),$$
$$\tau_l^* = \tau_l + \varphi^{-1}(b_l).$$

For any $l = 1, 2, \ldots$ let us consider the functions g_l and h_l in $\mathbb{T}$ as follows:

$$g_l(u) = \begin{cases} 0, & -\pi \le u \le \frac{\tau_l^*+\tau_l}{2}, \\ \dfrac{(u - \frac{\tau_l^*+\tau_l}{2})^\alpha}{(\tau_l^* - \tau_l)^\alpha}, & \frac{\tau_l^*+\tau_l}{2} < u \le \frac{3\tau_l^*-\tau_l}{2}, \\ 1, & \frac{3\tau_l^*-\tau_l}{2} < u \le \pi - \tau_l^* + \tau_l, \\ \dfrac{(\pi - u)^\alpha}{(\tau_l^* - \tau_l)^\alpha}, & \pi - \tau_l^* + \tau_l < u \le \pi \ . \end{cases}$$

$$h_l(u) = \begin{cases} \dfrac{(u - \tau_l)^\alpha (\tau_l^* - u)^\alpha}{(\tau_l^* - \tau_l)^{2\alpha}}, & \tau_l \le u \le \tau_l^*, \\ 0, & \text{otherwise.} \end{cases}$$

We define the function G_l $(l = 1, 2, \ldots)$ in $\mathbb{T}^n$ as follows:

$$G_l(x_1, \ldots, x_n) = b_l \prod_{i=1}^{n-1} g_l(x_i) h_l(x_n).$$

Consider the function G defined by the series

$$G(x_1, \ldots, x_n) = \sum_{l=1}^{\infty} G_l(x_1, \ldots, x_n).$$

We extend this function G 2π-periodically in each variable to the whole space $\mathbb{R}^n$.

We claim that

$$G \in H^\alpha\big(\varphi; \mathbb{C}(\mathbb{T}^n)\big).$$

Let $0 < h < \varphi^{-1}(b_1)$. Then

$$\|\Delta_n^\alpha(h) G\| \le \sum_{l=1}^{\infty} \|\Delta_n^\alpha(h) G_l\| = \sum_{l=1}^{\infty} I_l(h).$$

Let us estimate each $I_l(h)$ $(l = 1, 2, \ldots)$ from above.

For the given h there exists the number N such that $\tau_{N+1}^* - \tau_{N+1} \le h < \tau_N^* - \tau_N$.

Let $l = 1, \ldots, N$.

It is known [2] that if the function of one variable $f \in \mathbb{C}(\mathbb{T})$ has fractional derivative of order $\alpha (\alpha > 0)$, then

$$\omega_\alpha(f; \delta) \le C\delta^\alpha \|D^\alpha f\| (\delta > 0), \quad C = \text{const} > 0.$$

In our situation, using the definition of the function G_l and this fact for the variable x_n, we can conclude that

$$I_l(h) \le A_1 h^\alpha \frac{b_l}{(\tau_l^* - \tau_l)^\alpha}, A_1 = \text{const}.$$

If $l = N + 1, \ldots,$

$$I_l(h) \le A_2 b_l, A_2 = \text{const}.$$

Hence

$$\|\Delta_n^\alpha(h) G\| \le A_1 \sum_{l=1}^{N} h^\alpha \frac{b_l}{(\tau_l^* - \tau_l)^\alpha} + A_2 \sum_{l=N+1}^{\infty} b_l.$$

If $\tau_{N+1}^* - \tau_{N+1} \le h \le (\tau_N^* - \tau_N)^{\frac{\alpha}{\alpha-\beta}}$, then by Lemma 2 and by the construction of the sequence $(b_l)_{l \ge 1}$, we have

$$\|\Delta_n^\alpha(h)G\| \le A_1 \sum_{l=1}^{N} \frac{b_l}{(\tau_l^* - \tau_l)^\alpha} h^\beta h^{\alpha-\beta} + A_2 \sum_{l=N+1}^{\infty} b_l \le A_3 \varphi(h),\ A_3 = \text{const}\,.$$

If $(\tau_N^* - \tau_N)^{\frac{\alpha}{\alpha-\beta}} \le h \le \tau_N^* - \tau_N$, then by Lemma 1 and Lemma 2 and by the construction of the sequence $(b_l)_{l\ge1}$, we get

$$\|\Delta_n^\alpha(h)G\| \le A_1 \sum_{l=1}^{N-1} \frac{b_l}{(\tau_l^* - \tau_l)^\alpha} h^\beta h^{\alpha-\beta} + A_1 \frac{b_N}{(\tau_N^* - \tau_N)^\alpha} h^\alpha + A_2 \sum_{l=N+1}^{\infty} b_l$$
$$\le A_3 \varphi(h),\ A_3 = \text{const}\,.$$

Hence

$$\omega_{\alpha,n}(G;\delta)(h)G = O(\varphi(\delta)), \delta \to 0+\,.$$

Analogously, we can conclude that

$$\omega_{\alpha,i}(G;\delta)(h)G = O(\varphi(\delta)), \delta \to 0+, \quad i = 1, \dots, n-1.$$

Hence

$$G \in H^\alpha\big(\varphi; \mathbb{C}(\mathbb{T}^n)\big).$$

Let us now prove inequalities (1) and (2).

Let $h = \tau_l^* - \tau_l$.

According to the definition of the conjugate function and the function G, we obtain

$$\Delta_n^\alpha(h)\, \widetilde{G}_{\{1,\dots,n-1\}}\Big(\frac{\tau_l^* + \tau_l}{2}, \dots, \frac{\tau_l^* + \tau_l}{2}, \frac{\tau_l^* + \tau_l}{2}\Big)$$
$$= \Big(-\frac{1}{2\pi}\Big)^{n-1} \sum_{j=1}^{\infty} \int_{\mathbb{T}^{n-1}} \Delta_n^\alpha(h) G_j\Big(s_1 + \frac{\tau_l^* + \tau_l}{2}, \dots, s_{n-1} + \frac{\tau_l^* + \tau_l}{2}, \frac{\tau_l^* + \tau_l}{2}\Big)$$
$$\times \prod_{i=1}^{n-1} \cot \frac{s_i}{2}\, ds_i = \Big(-\frac{1}{2\pi}\Big)^{n-1}$$
$$\times \int_{\mathbb{T}^{n-1}} \Big[\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} G_j\Big(s_1 + \frac{\tau_l^* + \tau_l}{2}, \dots, s_{n-1} + \frac{\tau_l^* + \tau_l}{2}, \frac{\tau_l^* + \tau_l}{2} + kh\Big)\Big]$$
$$\times \prod_{i=1}^{n-1} \cot \frac{s_i}{2}\, ds_i.$$

Now, using the fact that $\left|\binom{\alpha}{k}\right| \le C_1 k^{-\alpha-1} (k = 1, 2, \dots)$ [9], the construction of the sequence $(b_l)_{l\ge1}$ and definition of the function G_j, we have

$$\Big|\sum_{j=l+1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} G_j(s_1 + \frac{\tau_l^* + \tau_l}{2}, \dots, s_{n-1} + \frac{\tau_l^* + \tau_l}{2}, \frac{\tau_l^* + \tau_l}{2} + kh)\Big|$$
$$\le C_2 \sum_{j=l+1}^{\infty} b_j \prod_{i=1}^{n-1} g_j(s_i),$$
$$\Big|\sum_{j=1}^{l-1} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} G_j(s_1 + \frac{\tau_l^* + \tau_l}{2}, \dots, s_{n-1} + \frac{\tau_l^* + \tau_l}{2}, \frac{\tau_l^* + \tau_l}{2} + kh)\Big|$$
$$\le \sum_{j=1}^{l-1} b_j \prod_{i=1}^{n-1} g_j(s_i) \sum_{k=[\frac{2\pi + \tau_1 - \frac{\tau_l^* + \tau_l}{2}}{h}]+1}^{\infty} \left|\binom{\alpha}{k}\right| \le C_3 h^\alpha \sum_{j=1}^{l-1} b_j \prod_{i=1}^{n-1} g_j(s_i),$$

$$\Big|\sum_{k=1}^{\infty}(-1)^k\binom{\alpha}{k}G_j(s_1+\frac{\tau_l^*+\tau_l}{2},\dots,s_{n-1}+\frac{\tau_l^*+\tau_l}{2},\frac{\tau_l^*+\tau_l}{2}+kh)\Big|$$
$$\le b_l\prod_{i=1}^{n-1}g_l(s_i)\sum_{k=[\frac{2\pi}{h}-\frac{1}{2}]+1}^{\infty}\Big|\binom{\alpha}{k}\Big|\le C_4h^{\alpha}b_l\prod_{i=1}^{n-1}g_l(s_i),$$

where $C_i (i=1,\dots,4)$ are the positive constants, and the symbol $[a]$ denotes the integer part of the real number a.

Hence we can conclude that

$$\left|\Delta_n^{\alpha}(h)\,\widetilde{G}_{\{1,\dots,n-1\}}\Big(\frac{\tau_l^*+\tau_l}{2},\dots,\frac{\tau_l^*+\tau_l}{2},\frac{\tau_l^*+\tau_l}{2}\Big)\right|$$
$$\ge C_5\int\limits_{[0,\pi]^{n-1}}G_l\Big(s_1+\frac{\tau_l^*+\tau_l}{2},\dots,s_{n-1}+\frac{\tau_l^*+\tau_l}{2},\frac{\tau_l^*+\tau_l}{2}\Big)\prod_{i=1}^{n-1}s_i^{-1}\,ds_i$$
$$\ge C_6b_l|\ln(\tau_l^*-\tau_l)|^{n-1};\quad C_5,\ C_6=\text{const}.$$

Therefore inequality (4) is proved.

Let us now prove inequality (3). Without loss of generality, we take $i=n-1$.

Let $h=\tau_l^*-\tau_l$.

By the definition of a conjugate function, we have

$$\Delta_{n-1}^{\alpha}(-h)\,\widetilde{G}_{\{1,\dots,n-1\}}\Big(\frac{\tau_l^*+\tau_l}{2},\dots,\frac{\tau_l^*+\tau_l}{2},\frac{\tau_l^*+\tau_l}{2}\Big)$$
$$=\Big(-\frac{1}{2\pi}\Big)^{n-1}\int\limits_{\mathbb{T}^{n-1}}\Delta_{n-1}^{\alpha}(-h)\,G\Big(s_1+\frac{\tau_l^*+\tau_l}{2},\dots,s_{n-1}+\frac{\tau_l^*+\tau_l}{2},\frac{\tau_l^*+\tau_l}{2}\Big)$$
$$\times\prod_{i=1}^{n-1}\cot\frac{s_i}{2}\,ds_i=\Big(-\frac{1}{2\pi}\Big)^{n-1}$$
$$\times\int\limits_{[0,\pi]^{n-2}}\int\limits_{\mathbb{T}}\Delta_{n-1}^{\alpha}(-h)\,G\Big(s_1+\frac{\tau_l^*+\tau_l}{2},\dots,s_{n-1}+\frac{\tau_l^*+\tau_l}{2},\frac{\tau_l^*+\tau_l}{2}\Big)\prod_{i=1}^{n-1}\cot\frac{s_i}{2}\,ds_i$$
$$=\Big(-\frac{1}{2\pi}\Big)^{n-1}\times$$
$$\times\int\limits_{[0,\pi]^{n-2}}\int\limits_{0}^{\pi}G\Big(s_1+\frac{\tau_l^*+\tau_l}{2},\dots,s_{n-1}+\frac{\tau_l^*+\tau_l}{2},\frac{\tau_l^*+\tau_l}{2}\Big)\Big(\Delta^{\alpha}(h)\cot\frac{s_{n-1}}{2}\Big)ds_{n-1}$$
$$\times\prod_{i=1}^{n-2}\cot\frac{s_i}{2}\,ds_i.$$

Further, by the definition of the function G, we have

$$\left|\Delta_{n-1}^{\alpha}(-h)\,\widetilde{G}_{\{1,\dots,n-1\}}\Big(\frac{\tau_l^*+\tau_l}{2},\dots,\frac{\tau_l^*+\tau_l}{2},\frac{\tau_l^*+\tau_l}{2}\Big)\right|$$
$$=\Big(\frac{1}{2\pi}\Big)^{n-1}$$
$$\times\Big|\int\limits_{[0,\pi]^{n-2}}\Big[\int\limits_{0}^{\pi}G_l(s_1+\frac{\tau_l^*+\tau_l}{2},\dots,s_{n-1}+\frac{\tau_l^*+\tau_l}{2},\frac{\tau_l^*+\tau_l}{2})(\Delta^{\alpha}(h)\cot\frac{s_{n-1}}{2})ds_{n-1}\Big]$$
$$\times\prod_{i=1}^{n-2}\cot\frac{s_i}{2}\,ds_i\Big|\ge C_7b_l\int\limits_{\tau_l*-\tau_l}^{1}\cdots\int\limits_{\tau_l*-\tau_l}^{1}\prod_{i=1}^{n-2}s_i^{-1}\frac{h^{\alpha}}{s_{n-1}^{\alpha+1}}\,ds_i\ge C_8b_l|\ln(\tau_l^*-\tau_l)|^{n-2},$$

where C_7 and C_8 are positive constants.

Therefore inequality (1) is proved. □

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