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BITOPOLOGICAL WEAK CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper, we introduce and study the concept of a new type of weak continuous multifunctions between bitopological spaces.

1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. Generalized open sets are basic and important in many topological notions and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Recently, as generalization of closed sets, the notion of ω-closed sets was introduced and studied by Hdeib [9]. Several characterizations and properties of ω-closed sets were provided in [2,3,5,9,10,13]. In this paper, we introduce and study upper (lower) (i,j)-weakly ω-continuous multifunctions on a bitopological space.

2. Preliminaries

Throughout this paper, (X,τ_1,τ_2) and (Y,σ_1,σ_2) always mean bitopological spaces in which no separation axioms are assumed unless explicitly stated. For a subset A of (X,τ), $i\,\mathrm{Cl}(A)$ and $i\,\mathrm{Int}(A)$ denote the closure of A with respect to τ_i and the interior of A with respect to τ_i, respectively. A point $x\in X$ is called a condensation point of A if for each $U\in\tau$ with $x\in U$, the set $U\cap A$ is uncountable. A is said to be ω-closed [9] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X,τ) is ω-open if and only if for each $x\in W$ there exists $U\in\tau$ such that $x\in U$ and $U\backslash W$ is countable. The family of all ω-open subsets of a topological space (X,τ) forms a topology on X finer than τ. The intersection of all ω-closed sets containing A is called the ω-closure [9] of A and is denoted by $\omega\,\mathrm{Cl}(A)$. For each $x\in X$, the family of all ω-open sets containing x is denoted by $\omega O(X,x)$. The family of all ω-open sets of X is denoted by $\omega(\tau)$. By a multifunction $F:(X,\tau)\to(Y,\sigma)$, following [6], we denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B)=\{x\in X: F(x)\subset B\}$ and $F^-(B)=\{x\in X: F(x)\cap B\neq\emptyset\}$. In particular, $F^-(y)=\{x\in X: y\in F(x)\}$ for each point $y\in Y$ and for each $A\subset X$, $F(A)=\cup_{x\in A}F(x)$. Then F is said to be surjective if $F(X)=Y$ and injective if $x\neq y$ implies $F(x)\cap F(y)=\emptyset$.

Definition 2.1 ([11])**.** Let (X,τ_1,τ_2) be a bitopological space and A a subset of X. A point $x\in X$ is said to be in the (i,j)-θ-closure of A, denoted by (i,j)-$\mathrm{Cl}_\theta(A)$ if $A\cap j\,\mathrm{Cl}(U)\neq\emptyset$ for every τ_i-open set U containing x, where $i,j=1,2$ and $i\neq j$.

Definition 2.2 ([11])**.** A subset A of X is said to be (i,j)-θ-closed if $A=(i,j)$-$\mathrm{Cl}_\theta(A)$. A subset A of X is said to be (i,j)-θ-open if $X\backslash A$ is (i,j)-θ-closed. The (i,j)-θ-interior of A, denoted by (i,j)-$\mathrm{Int}_\theta(A)$, is defined as the union of all (i,j)-θ-open sets contained in A. Hence $x\in(i,j)$-$\mathrm{Int}_\theta(A)$, if and only if there exists a τ_i-open set U containing x such that $x\in U\subset j\,\mathrm{Cl}(U)\subset A$.

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Lemma 2.3 ([11]). *For a subset of a bitopological space (X, τ_1, τ_2), we have the following*

(1) $(i,j)\text{-}\mathrm{Cl}_\theta(X\backslash A) = X\backslash(i,j)\text{-}\mathrm{Int}_\theta(A)$;
(2) $(i,j)\text{-}\mathrm{Int}_\theta(X\backslash A) = X\backslash(i,j)\text{-}\mathrm{Cl}_\theta(A)$.

Lemma 2.4 ([11]). *Let (X, τ_1, τ_2) be a bitopological space. If U is a τ_j-open subset of X, then $(i,j)\text{-}\mathrm{Cl}_\theta(U) = i\,\mathrm{Cl}(U)$.*

Definition 2.5. A subset A of a bitopological space (X, τ_1, τ_2) is said to be:

(1) (i,j)-preopen [12] if $A \subset i\,\mathrm{Int}(j\,\mathrm{Cl}(A))$;
(2) (i,j)-regular open [7] if $A = i\,\mathrm{Int}(j\,\mathrm{Cl}(A))$.

In each definition above, $i, j = 1, 2$ and $i \neq j$.

The complement of an (i,j)-preopen (resp. (i,j)-regular open) set is called an (i,j)-preclosed (resp., (i,j)-regular closed) set.

Definition 2.6 ([1]). Let (X, τ_1, τ_2) be a bitopological space and let $A \subset X$. Then

(1) A is said to be u-ω-open in (X, τ_1, τ_2) if $A \in \omega(\tau_1) \cup \omega(\tau_2)$;
(2) A is said to be u-ω-closed in (X, τ_1, τ_2) if $X - A$ is u-ω-open in (X, τ_1, τ_2).

The family of all u-ω-open sets in (X, τ_1, τ_2) is denoted by $\omega(\tau_1, \tau_2)$, and the family of all u-ω-closed sets in (X, τ_1, τ_2) is denoted by $\omega C(\tau_1, \tau_2)$.

Definition 2.7.

(1) The u-ω-closure of A in (X, τ_1, τ_2) is denoted by $(\tau_1, \tau_2)\text{-}\omega\,\mathrm{Cl}(A)$ and defined as follows: $(\tau_1, \tau_2)\text{-}\omega\,\mathrm{Cl}(A) = \omega\,\mathrm{Cl}_{\tau_1}(A) \cap \omega\,\mathrm{Cl}_{\tau_2}(A)$.
(2) The u-ω-interior of A in (X, τ_1, τ_2) is denoted by $(\tau_1, \tau_2)\text{-}\omega\,\mathrm{Int}(A)$ and defined as follows: $(\tau_1, \tau_2)\text{-}\omega\,\mathrm{Int}(A) = \omega\,\mathrm{Int}_{\tau_1}(A) \cup \omega\,\mathrm{Int}_{\tau_2}(A)$.

Example 2.8. Let $X = \mathbb{R}$ with the topologies $\tau_1 = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$ and $\tau_2 = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$. Take $A = \mathbb{Q}$, then $\omega\,\mathrm{Cl}_{\tau_1}(\mathbb{Q}) = \mathbb{Q}$ and $\omega\,\mathrm{Cl}_{\tau_2}(\mathbb{Q}) = \mathbb{Q}$; as a consequence, $(\tau_1, \tau_2)\text{-}\omega\,\mathrm{Cl}(\mathbb{Q}) = \mathbb{Q} \cap \mathbb{Q} = \mathbb{Q}$ and $(\tau_1, \tau_2)\text{-}\omega\,\mathrm{Int}(\mathbb{Q}) = \mathbb{Q} \cup \mathbb{Q} = \mathbb{Q}$.

3. On (i,j)-upper (lower) ω-continuous multifunctions

Definition 3.1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) (i,j)-upper weakly ω-continuous if for each $x \in X$ and each σ_i-open set V containing $F(x)$, there exists $U \in \omega(\tau_1, \tau_2)$ containing x such that $F(U) \subset j\,\mathrm{Cl}(V)$,
(2) (i,j)-lower weakly ω-continuous if for each $x \in X$ and each σ_i-open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \omega(\tau_1, \tau_2)$ containing x such that $F(u) \cap j\,\mathrm{Cl}(V) \neq \emptyset$ for each $u \in U$.

Example 3.2. In Example 2.8, take $X = Y$ and $\sigma_1 = \tau_1$ and $\sigma_2 = \tau_2$ and define $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ as

$$F(x) = \begin{cases} \mathbb{Q} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ \mathbb{R} \setminus \mathbb{Q} & \text{if } x \in \mathbb{Q}. \end{cases}$$

It is easy to see that F is (i,j)-upper weakly ω-continuous and (i,j)-lower weakly ω-continuous.

Theorem 3.3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i,j)-upper weakly ω-continuous;*
(2) *$F^+(V) \subset (\tau_1, \tau_2)\text{-}\omega\,\mathrm{Int}(F^+(j\,\mathrm{Cl}(V)))$ for every σ_i-open set V of Y;*
(3) *$(\tau_1, \tau_2)\text{-}\omega\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(K))) \subset F^-(K)$ for every σ_i-closed set K of Y;*
(4) *$(\tau_1, \tau_2)\text{-}\omega\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(i\,\mathrm{Cl}(B)))) \subset F^-(i\,\mathrm{Cl}(B))$ for every subset B of Y;*
(5) *$F^+(i\,\mathrm{Int}(B)) \subset (\tau_1, \tau_2)\text{-}\omega\,\mathrm{Int}(F^+(j\,\mathrm{Cl}(i\,\mathrm{Int}(B))))$ for every subset B of Y;*
(6) *$(\tau_1, \tau_2)\text{-}\omega\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(K))) \subset F^-(K)$ for every (i,j)-regular closed set K of Y;*
(7) *$(\tau_1, \tau_2)\text{-}\omega\,\mathrm{Cl}(F^-(V)) \subset F^-(i\,\mathrm{Cl}(V))$ for every σ_j-open set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any σ_i-open set and $x \in F^+(V)$. Thus we have $F(x) \subset V$. Then there exists $U \in \omega(\tau_1, \tau_2)$ containing x such that $F(U) \subset j\operatorname{Cl}(V))$. Therefore we have $x \in U \subset F^+(j\operatorname{Cl}(V))$. Since $U \in \omega(\tau_1, \tau_2)$, we obtain $x \in (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(j\operatorname{Cl}(V)))$. Hence $F^+(V) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(j\operatorname{Cl}(V)))$.
(2) $\Rightarrow$ (3): Let K be any σ_i-closed set of Y. Then $Y - K$ is σ_i-open in Y and we have $X - F^-(K) = F^+(Y - K) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(j\operatorname{Cl}(Y - K))) = (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(Y - j\operatorname{Int}(K))) = (\tau_1, \tau_2)$-$\omega\operatorname{Int}(X - F^-(j\operatorname{Int}(K))) = X - (\tau_1, \tau_2)$-$\omega\operatorname{Cl}(F^-(j\operatorname{Int}(K)))$. Therefore (τ_1, τ_2)-$\omega\operatorname{Cl}(F^-(j\operatorname{Int}(K))) \subset F^-(K)$.
(3)$\Rightarrow$ (4): Let B be any subset of Y. Then $i\operatorname{Cl}(B)$ is σ_i-closed in Y and by (3), we have (τ_1, τ_2)-$\omega\operatorname{Cl}(F^-(j\operatorname{Int}(i\operatorname{Cl}(B))) \subset F^-(i\operatorname{Cl}(B))$.
(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4), we have $X - (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(j\operatorname{Cl}(i\operatorname{Int}(B)))) = (\tau_1, \tau_2)$-$\omega\operatorname{Cl}(X - F^+(j\operatorname{Cl}(i\operatorname{Int}(B)))) = (\tau_1, \tau_2)$-$\omega\operatorname{Cl}(F^-(Y - j\operatorname{Cl}(i\operatorname{Int}(B))) = (\tau_1, \tau_2)$-$\omega\operatorname{Cl}(F^-(j\operatorname{Int}(i\operatorname{Cl}(Y - B))) \subset F^-(i\operatorname{Cl}(Y - B)) = X - F^+(i\operatorname{Int}(B))$. Therefore we obtain $F^+(i\operatorname{Int}(B)) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(j\operatorname{Cl}(i\operatorname{Int}(B))))$.
(5) $\Rightarrow$ (6): Let K be any (i, j)-regular closed set of Y. Then by (5), we have $X - F^-(K) = X - F^-(i\operatorname{Cl}(j\operatorname{Int}(K))) = F^+(Y - i\operatorname{Cl}(j\operatorname{Int}(K))) = F^+(i\operatorname{Int}(Y - j\operatorname{Int}(K))) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(j\operatorname{Cl}(i\operatorname{Int}(Y - j\operatorname{Int}(K)))) = (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(Y - j\operatorname{Int}(i\operatorname{Cl}(j\operatorname{Int}(K))))) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(Y - j\operatorname{Int}(K))) = (\tau_1, \tau_2)$-$\omega\operatorname{Int}(X - F^-(j\operatorname{Int}(K))) = X - (\tau_1, \tau_2)$-$\omega\operatorname{Cl}(F^-(j\operatorname{Int}(K))$. Therefore we obtain (τ_1, τ_2)-$\omega\operatorname{Cl}(F^-(j\operatorname{Int}(K))) \subset F^-(K)$.
(6) $\Rightarrow$ (7): Let V be any σ_j-open set of Y. Then $i\operatorname{Cl}(V)$ is (i, j)-regular closed. By (6), we have (τ_1, τ_2)-$\omega\operatorname{Cl}(F^-(V)) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Cl}(F^-(j\operatorname{Int}(i\operatorname{Cl}(V)))) \subset F^-(i\operatorname{Cl}(V))$.
(7) $\Rightarrow$ (1): Let $x \in X$ and V be a σ_i-open set containing $F(x)$. Then $Y - j\operatorname{Cl}(V)$ is σ_j-open in Y and we have $X - (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(j\operatorname{Cl}(V))) = (\tau_1, \tau_2)$-$\omega\operatorname{Cl}(F^-(Y - j\operatorname{Cl}(V))) \subset F^-(i\operatorname{Cl}(Y - j\operatorname{Cl}(V))) = F^-(Y - i\operatorname{Int}(j\operatorname{Cl}(V))) = X - F^+(i\operatorname{Int}(j(Cl(V))) \subset X - F^+(V)$. Therefore, $x \in F^+(V) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+ j\operatorname{Cl}(V)))$. There exists $U \in \omega(\tau_1, \tau_2)$ such that $x \in U \subset F^+(j\operatorname{Cl}(V))$, hence $F(U) \subset j\operatorname{Cl}(V)$. This shows that F is (i, j)-upper weakly ω-continuous. $\square$

Theorem 3.4. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-lower weakly ω-continuous;*
(2) *$F^-(V) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^-(j\operatorname{Cl}(V)))$ for every σ_i-open set V of Y;*
(3) *(τ_1, τ_2)-$\omega\operatorname{Cl}(F^+(j\operatorname{Int}(K))) \subset F^+(K)$ for every σ_i-closed set K of Y;*
(4) *(τ_1, τ_2)-$\omega\operatorname{Cl}(F^+(j\operatorname{Int}(i\operatorname{Cl}(B)))) \subset F^+(i\operatorname{Cl}(B))$ for every subset B of Y;*
(5) *$F^-(i\operatorname{Int}(B)) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^-(j\operatorname{Cl}(i\operatorname{Int}(B))))$ for every subset B of Y;*
(6) *(τ_1, τ_2)-$\omega\operatorname{Cl}(F^+(j\operatorname{Int}(K))) \subset F^+(K)$ for every (i, j)-regular closed set K of Y;*
(7) *(τ_1, τ_2)-$\omega\operatorname{Cl}(F^+(V)) \subset F^+(i\operatorname{Cl}(V))$ for every σ_j-open set V of Y.*

Proof. The proof is similar to that of Theorem 3.3. $\square$

Theorem 3.5. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-upper weakly ω-continuous;*
(2) *(τ_1, τ_2)-$\omega\operatorname{Cl}(F^-(j\operatorname{Int}(i\operatorname{Cl}(V)))) \subset F^-(i\operatorname{Cl}(V))$ for every (j, i)-preopen set V of Y;*
(3) *(τ_1, τ_2)-$\omega\operatorname{Cl}(F^-(V)) \subset F^-(i\operatorname{Cl}(V))$ for every (j, i)-preopen set V of Y;*
(4) *$F^+(V) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(j\operatorname{Cl}(V)))$ for every (i, j)-preopen set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any (j, i)-preopen set of Y. Since $j\operatorname{Int}(i\operatorname{Cl}(V))$ is σ_j-open, by Theorem 3.5, we obtain (τ_1, τ_2)-$\omega\operatorname{Cl}(F^-(j\operatorname{Int}(i\operatorname{Cl}(V)))) \subset F^-(i\operatorname{Cl}(j\operatorname{Int}(i\operatorname{Cl}(V)))) \subset F^-(i\operatorname{Cl}(V))$.
(2) $\Rightarrow$ (3): Let V be any (j, i)-preopen set of Y. Then we have (τ_1, τ_2)-$\omega\operatorname{Cl}(F^-(V)) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Cl}(F^-(j\operatorname{Int}(i\operatorname{Cl}(V)))) \subset F^-(i\operatorname{Cl}(V))$.
(3) $\Rightarrow$ (4): Let V be any (i, j)-preopen set of Y. By (3), we have $X - (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(j\operatorname{Cl}(V))) = (\tau_1, \tau_2)$-$\omega\operatorname{Cl}(X - F^+(j\operatorname{Cl}(V))) = (\tau_1, \tau_2)$-$\omega\operatorname{Cl}(F^-(Y - j\operatorname{Cl}(V))) \subset F^-(i\operatorname{Cl}(Y - j\operatorname{Cl}(V))) = X - F^+(i\operatorname{Int}(j\operatorname{Cl}(V))) \subset X - F^+(V)$. Therefore we obtain $F^+(V) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(j\operatorname{Cl}(V)))$.
(4) $\Rightarrow$ (1): Let V be any σ_i-open set of Y. Then V is (i, j)-preopen in Y and $F^+(V) \subset (\tau_1, \tau_2)$-$\omega\operatorname{Int}(F^+(j\operatorname{Cl}(V)))$. By Theorem 3.5, F is (i, j)-upper weakly ω-continuous. $\square$

Theorem 3.6. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-lower weakly ω-continuous;*
(2) *(τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^+(j\,\mathrm{Int}(i\,\mathrm{Cl}(V)))) \subset F^+(i\,\mathrm{Cl}(V))$ for every (j, i)-preopen set V of Y;*
(3) *(τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^+(V)) \subset F^+(i\,\mathrm{Cl}(V))$ for every (j, i)-preopen set V of Y;*
(4) *$F^-(V) \subset (\tau_1, \tau_2)$-$\omega\,\mathrm{Int}(F^-(j\,\mathrm{Cl}(V)))$ for every (i, j)-preopen set V of Y.*

Proof. The proof is similar to that of Theorem 3.5. □

Lemma 3.7. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-lower weakly ω-continuous, then for each $x \in X$ and each subset B of Y with $F(x) \cap (i, j)$-$\mathrm{Int}_\theta(B) \neq \emptyset$, there exists $U \in \omega(\tau_1, \tau_2)$ containing x such that $U \subset F^-(B)$.*

Proof. Since $F(x) \cap (i, j)$-$\mathrm{Int}_\theta(B) \neq \emptyset$, there exists a σ_i-open set V of Y such that $V \subset j\,\mathrm{Cl}(V) \subset B$ and $F(x) \cap V \neq \emptyset$. Since F is (i, j)-lower weakly ω-continuous, there exists $U \in \omega(\tau_1, \tau_2)$ containing x such that $F(u) \cap j\,\mathrm{Cl}(V) \neq \emptyset$ for every $u \in U$ and hence $U \subset F^-(B)$. □

Theorem 3.8. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-lower weakly ω-continuous;*
(2) *(τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^+(B)) \subset F^+((i, j)$-$\mathrm{Cl}_\theta(B))$ for every subset B of Y;*
(3) *$F((\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(A)) \subset (i, j)$-$\mathrm{Cl}_\theta(F(A))$ for every subset A of X.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \notin F^+((i, j)$-$\mathrm{Cl}_\theta(B))$. Then $x \in F^-(Y \setminus (i, j)$-$\mathrm{Cl}_\theta(B)) = F^-((i, j)$-$\mathrm{Int}_\theta(Y - B))$. Then, by Lemma 3.7, there exists $U \in \omega(\tau_1, \tau_2)$ containing x such that $U \subset F^-(Y - B) = X - F^+(B)$. Thus we have $U \cap F^+(B) = \emptyset$. Then $x \in X - (\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(F^+(B))$ and hence (τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^+(B)) \subset F^+((i, j)$-$\mathrm{Cl}_\theta(B))$.
(2) $\Rightarrow$ (3): Let A be a subset of X. By (2), we have (τ_1, τ_2)-$\omega\,\mathrm{Cl}(A) \subset (\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(F^+(F(A))) \subset F^+((i, j)$-$\mathrm{Cl}_\theta(F(A)))$. Hence $F((\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(A)) \subset (i, j)$-$\mathrm{Cl}_\theta(F(A))$.
(3) $\Rightarrow$ (1): Let V be any σ_j-open set of Y. Then (i, j)-$\mathrm{Cl}_\theta(V) = i\,\mathrm{Cl}(V)$ and by (3), we have $F((\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(F^+(V))) \subset (i, j)$-$\mathrm{Cl}_\theta(F(F^+(V))) \subset (i, j)$-$\mathrm{Cl}_\theta(V) = i\,\mathrm{Cl}(V)$. Therefore we obtain (τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^+(V)) \subset F^+(i\,\mathrm{Cl}(V))$. By Theorem 3.4, F is (i, j)-lower weakly ω-continuous. □

Theorem 3.9. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-upper weakly ω-continuous;*
(2) *(τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^-(j\,\mathrm{Int}((i, j)$-$\mathrm{Cl}_\theta(B)))) \subset F^-((i, j)$-$\mathrm{Cl}_\theta(B))$ for every subset B of Y;*
(3) *(τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(i\,\mathrm{Cl}(B)))) \subset F^-((i, j)$-$\mathrm{Cl}_\theta(B))$ for every subset B of Y.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Then (i, j)-$\mathrm{Cl}_\theta(B)$ is σ_i-closed in Y. Therefore by Theorem 3.3, (τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^-(j\,\mathrm{Int}((i, j)$-$\mathrm{Cl}_\theta(B)))) \subset F^-((i, j)$-$\mathrm{Cl}_\theta(B))$.
(2) $\Rightarrow$ (3): This is obvious since $i\,\mathrm{Cl}(B) \subset (i, j)$-$\mathrm{Cl}_\theta(B)$.
(3) $\Rightarrow$ (1): Let K be any (i, j)-regular closed set of Y. Then by Lemma 2.4, (i, j)-$\mathrm{Cl}_\theta(j\,\mathrm{Int}(K)) = i\,\mathrm{Cl}(j\,\mathrm{Int}(K))$ and we have (τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(K))) = (\tau_1, \tau_2)$-$\omega\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(i\,\mathrm{Cl}(j\,\mathrm{Int}(K))))) \subset F^-((i, j)$-$\mathrm{Cl}_\theta(j\,\mathrm{Int}(K))) = F^-(K)$. Therefore (τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^-(j\,\mathrm{Int}(K))) \subset F^-(K)$ and by Theorem 3.3, F is (i, j)-upper weakly ω-continuous. □

Theorem 3.10. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-lower weakly ω-continuous;*
(2) *(τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^+(j\,\mathrm{Int}((i, j)$-$\mathrm{Cl}_\theta(B)))) \subset F^+((i, j)$-$\mathrm{Cl}_\theta(B))$ for every subset B of Y,*
(3) *(τ_1, τ_2)-$\omega\,\mathrm{Cl}(F^+(j\,\mathrm{Int}(i\,\mathrm{Cl}(B)))) \subset F^+((i, j)$-$\mathrm{Cl}_\theta(B))$ for every subset B of Y.*

Proof. The proof is similar to that of Theorem 3.9. □

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