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AN APPLICATION OF A WIDER CLASS OF INCREASING SEQUENCES

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Abstract. In this paper, we have proved a general theorem dealing with the $\varphi-|C,\alpha|_k$ summability factors of infinite series by using a wider class of power increasing sequences. Some new results are also obtained and some previous results are recovered.

1. Introduction

A positive sequence (X_n) is said to be almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq X_n \leq Nc_n$ (see [2]). A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(X,f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ holds for $n \geq m \geq 1$, where $f = (f_n) = [n^\sigma(\log n)^\gamma$, $\gamma \geq 0$, $0 < \sigma < 1]$ (see [22]). If we take γ=0, then we obtain a quasi-σ-power increasing sequence (see [20]). Let (φ_n) be a sequence of complex numbers and let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n^α the nth Cesàro mean of order α $(\alpha > -1)$ of the sequence (na_n), that is (see [17]),

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1}$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)}, \quad \alpha > -1, \quad A_0^\alpha = 1 \text{ and } A_{-n}^\alpha = 0 \text{ for } n > 0.$$

The series $\sum a_n$ is said to be summable $\varphi - |C,\alpha|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \frac{1}{n^k} |\varphi_n t_n^\alpha|^k < \infty.$$

In the special case, when $\varphi_n = n^{1-1/k}$ (resp., $\varphi_n = n^{\delta+1-1/k}$), the $\varphi - |C,\alpha|_k$ summability is the same as $|C,\alpha|_k$ (see [18]) (resp., $|C,\alpha;\delta|_k$ (see [19])) summability.

2. Known Result

Recently, some new theorems dealing with the absolute Cesàro summability factors of infinite series have been proved(see [3–15]). Among them, in [12], the following theorem has been proved.

Theorem A. *Let $0 < \alpha \leq 1$. Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{2}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{3}$$

$$\sum_{n=1}^{\infty} n |\Delta\beta_n| X_n < \infty, \tag{4}$$

$$|\lambda_n| X_n = O(1) \quad as \quad n \to \infty. \tag{5}$$

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If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}\left|\varphi_n\right|^k)$ is non-increasing and if the sequence (ω_n^α) defined by (see [21])

$$\omega_n^\alpha = \begin{cases} |t_n^\alpha| & (\alpha = 1) \\ \max_{1\leq v\leq n} |t_v^\alpha| & (0 < \alpha < 1) \end{cases}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid w_n^\alpha)^k}{n^k {X_n}^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{6}$$

then the series $\sum a_n\lambda_n$ is $\varphi - \mid C, \alpha \mid_k$ summable, where $k \geq 1$ and $(1 + \alpha k + \epsilon - k) > 1$.

3. Main Result

The aim of this paper is to generalize Theorem A to a wider class of increasing sequences by using a quasi-f-power increasing sequence instead of an almost increasing sequence. Now, we prove the following more general theorem.

Theorem. *Let $0 < \alpha \leq 1$ and let (X_n) be a quasi-f-power increasing sequence. If the sequences (β_n) and (λ_n) satisfy conditions (2), (3), (4), (5) of Theorem A and if there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k}\left|\varphi_n\right|^k)$ is non-increasing and condition (6) holds, then the series $\sum a_n\lambda_n$ is $\varphi - \mid C, \alpha \mid_k$ summable, where $k \geq 1$ and $(1 + \alpha k + \epsilon - k) > 1$.*

4. Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 1 ([16])**.** *If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then*

$$\left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} a_p\right| \leq \max_{1\leq m\leq v} \left|\sum_{p=1}^{m} A_{m-p}^{\alpha-1} a_p\right|.$$

Lemma 2 ([4])**.** *Under the conditions on (X_n), (β_n), and (λ_n) as expressed in the statement of the theorem, we have the following*

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty,$$
$$nX_n\beta_n = O(1).$$

5. Proof of the Theorem

Let (T_n^α) be the nth (C, α) mean, with $0 < \alpha \leq 1$, of the sequence $(na_n\lambda_n)$. Then by (1), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v.$$

First, applying Abel's transformation and then using Lemma 1, we have

$$\begin{aligned} T_n^\alpha =& \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \\ \mid T_n^\alpha \mid \leq& \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^\alpha} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \right| \\ \leq& \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha \omega_v^\alpha \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \omega_n^\alpha = T_{n,1}^\alpha + T_{n,2}^\alpha. \end{aligned}$$

To complete the proof of the theorem by using Minkowski's inequality for $k>1$, it suffices to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^k}\mid\varphi_n T^{\alpha}_{n,r}\mid^k<\infty,\quad\text{for}\quad r=1,2.$$

For $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\frac{1}{n^k}\mid\varphi_n T^{\alpha}_{n,1}\mid^k &\le\sum_{n=2}^{m+1}n^{-k}(A^{\alpha}_n)^{-k}|\varphi_n|^k\left\{\sum_{v=1}^{n-1}A^{\alpha}_v\omega^{\alpha}_v\beta_v\right\}^{k-1}\\
&\le\sum_{n=2}^{m+1}\frac{1}{n}(A^{\alpha}_n)^{-k}|\varphi_n|^k\sum_{v=1}^{n-1}(A^{\alpha}_v)^k(\omega^{\alpha}_v)^k\beta^k_v\times\left\{\frac{1}{n}\sum_{v=1}^{n-1}1\right\}^{k-1}\\
&=O(1)\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^{1+\alpha k}}\sum_{v=1}^{n-1}v^{\alpha k}(\omega^{\alpha}_v)^k\beta^k_v=O(1)\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^{1+\alpha k}}\sum_{v=1}^{n-1}v^{\alpha k}(\omega^{\alpha}_v)^k\beta^k_v\\
&=O(1)\sum_{v=1}^{m}v^{\alpha k}(\omega^{\alpha}_v)^k\beta_v\beta^{k-1}_v\sum_{n=v+1}^{m+1}\frac{n^{\epsilon-k}|\varphi_n|^k}{n^{1+\alpha k+\epsilon-k}}\\
&=O(1)\sum_{v=1}^{m}v^{\alpha k}(\omega^{\alpha}_v)^k\beta_v\frac{v^{\epsilon-k}|\varphi_v|^k}{v^{k-1}X^{k-1}_v}\int_v^{\infty}\frac{dx}{x^{1+\alpha k+\epsilon-k}}\\
&=O(1)\sum_{v=1}^{m}v\beta_v\frac{(\omega^{\alpha}_v|\varphi_v|)^k}{v^kX^{k-1}_v}=O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{(\omega^{\alpha}_r|\varphi_r|)^k}{r^kX^{k-1}_r}\\
&+O(1)m\beta_m\sum_{v=1}^{m}\frac{(\omega^{\alpha}_v|\varphi_v|)^k}{v^kX^{k-1}_v}=O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v+O(1)m\beta_mX_m\\
&=O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)m\beta_mX_m\\
&=O(1)\quad\text{as}\quad m\to\infty,\end{aligned}$$

by the hypotheses of the theorem and Lemma 2. Again, by using (5), we have

$$\begin{aligned}\sum_{n=1}^{m}\frac{1}{n^k}\mid\varphi_n T^{\alpha}_{n,2}\mid^k&=\sum_{n=1}^{m}n^{-k}\,|\varphi_n|^k\,|\lambda_n|\,|\lambda_n|^{k-1}\,(\omega^{\alpha}_n)^k=O(1)\sum_{n=1}^{m}|\lambda_n|\,\frac{(\mid\varphi_n\mid w^{\alpha}_n)^k}{n^kX_n{}^{k-1}}\\
&=O(1)\sum_{n=1}^{m-1}\Delta\,|\lambda_n|\sum_{v=1}^{n}\frac{(\mid\varphi_v\mid w^{\alpha}_v)^k}{v^kX_v{}^{k-1}}+O(1)\,|\lambda_m|\sum_{n=1}^{m}\frac{(\mid\varphi_n\mid w^{\alpha}_n)^k}{n^kX_n{}^{k-1}}\\
&=O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|\,X_n+O(1)\,|\lambda_m|\,X_m\\
&=O(1)\sum_{n=1}^{m-1}\beta_nX_n+O(1)\,|\lambda_m|\,X_m=O(1)\quad as\quad m\to\infty,\end{aligned}$$

by the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

6. Conclusions

If we take $\epsilon=1$ and $\varphi_n=n^{1-1/k}$ (resp., $\epsilon=1$ and $\varphi_n=n^{\delta+1-1/k}$), then we obtain two new results dealing with the $\mid C,\alpha\mid_k$ (resp., $\mid C,\alpha;\delta\mid_k$) summability factors. Also, if we take (X_n) as a positive non-decreasing sequence, then we obtain a theorem of Bor (see [3]) under weaker conditions. Furthermore, if we take (X_n) as an almost increasing sequence, then we obtain Theorem A. Finally, if we take $\gamma=0$, we obtain a new result dealing with an application of quasi-σ-power increasing sequences.

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