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ROUGH SEMI-CONTINUOUS SET-VALUED MAPS

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Abstract. In this paper, we introduce the concepts of rough semi-continuity and rough continuity of set-valued maps. Then we investigate the question whether these concepts may be characterized differently or not.

1. Introduction

The concept of a set-valued map is needed in the solution of problems in sciences such as control theory, economics, business administration / management and biology. In 1982, Neubrunn [7] gave two definitions of the semi-continuity of a set-valued map and proved some results by using these definitions. Moreover, some differences between the classical results for semi-continuous single-valued functions and those for set-valued maps were shown by some examples. In the same year, Hou [4] presented different upper semi-continuity properties of set-valued maps and elaborated their interrelatedness. In 2005, Labuda [6] proved that the active boundary of $F\left(x_0\right)$ is the smallest compact kernel of F at x_0, if x_0 is a $q-$point of a regular space X, Y is a Hausdorff space whose relatively countably compact subsets are relatively compact and $F : X \rightrightarrows Y$ is an upper semi-continuous set-valued map. Then Kanıbir and Reilly [5] introduced two kinds of generalized continuity for set-valued maps.

On the other hand, the idea of rough continuity of the functions on normed spaces was introduced by Phu [8] in 2002 as follows. A mapping $f : X \to Y$ is said to be continuous provided $x' \to x$ always implies $f(x') \to f(x)$, where the two latter arrows denote the classical convergence in the corresponding spaces. Replacing the classical convergence by the rough convergence, we obtain the so-called rough continuity, or $r_X - r_Y-$continuity, where r_X and r_Y denote the convergence degrees in X and Y, respectively. For $r_X = 0$, f is called $0 - r_Y-$continuous if $x' \to x$ always implies $f(x') \overset{r_Y}{\to} f(x)$. By definition, $x' \overset{r_X}{\to} x$ means that $x \to B_{r_X}(x) = \{z \in X : d_X(z, x) \leq r_X\}$, i.e., the classical (ordinary) convergence of x' to the ball $B_{r_X}(x)$. Hence, for $r_X \geq r_Y = 0$, f is called $r_X - 0-$continuous if $x' \overset{r_X}{\to} x$ implies $f(x') \overset{r_Y}{\to} f(B_{r_X}(x))$. For $r_X = r_Y = 0$, the $r_X - r_Y-$continuity is just the classical continuity.

Moreover, Phu [8] proved that if the space Y is finite-dimensional, then all the linear operators $f : X \to Y$ are $r-$continuous at each point of X.

The aim of this paper is to extend the definition of rough continuity for the classical functions to set-valued maps. By this new definition, we will be able to observe the differences between discontinuous set-valued maps, and grade these discontinuities. In order words, although the conditions violating the upper semi-continuity or lower semi-continuity for certain maps may be disregarded with a roughness degree, such a situation is out of the question for certain maps. In this way, we will be able to classify discontinuous maps via certain definitions based on the roughness, instead of defining them as just discontinuous. Thus we will be able to put forward the differences between these discontinuous set-valued maps.

2. Preliminaries

In this section, we briefly recall some of the basic notions in the theory of set-valued analysis and refer to [1–3, 9] for more details.

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Let X and Y be topological spaces. A map that assigns each $x \in X$ to a subset of Y is called a *set-valued map*. The *graph* of a set-valued map F from X to Y is defined by

$$\mathrm{Graph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}.$$

Also, the *domain* of F is defined by

$$\mathrm{Dom}(F) := \{x \in X : F(x) \neq \varnothing\}.$$

The *inverse image* of the set M $(\subset Y)$ is defined by

$$F^{-1}(M) := \{x : F(x) \cap M \neq \varnothing\},$$

while the *core* of the set M $(\subset Y)$ is defined by

$$F^{+1}(M) := \{x : F(x) \subset M\}.$$

The concept of an upper semi-continuous is defined as: "A set-valued map $F : X \rightrightarrows Y$ is called *upper semi-continuous* at $x \in \mathrm{Dom}(F)$ if and only if for any neighbourhood V of $F(x)$, there exists a $\delta > 0$ such that for all $x' \in B^{\circ}(x, \delta) = \{x' \in X : d(x, x') < \delta\}$, we have $F(x') \subset V$. It is said to be *upper semi-continuous* (*usc*) if and only if it is upper semi-continuous at any point of $\mathrm{Dom}(F)$" [1].

Similarly, "A set-valued map $F : X \rightrightarrows Y$ is called *lower semi-continuous* at $x \in \mathrm{Dom}(F)$ if and only if for any $y \in F(x)$ and for any sequence of elements $x_n \in \mathrm{Dom}(F)$ converging to x, there exists a sequence of elements $y_n \in F(x_n)$ converging to y. It is said to be *lower semi-continuous* (*lsc*) if it is lower semi-continuous at every point of $\mathrm{Dom}(F)$" [1].

3. Rough Semi-continuity of Set-valued Maps

First, we give the definitions of *rough usc* and *rough lsc* of a set-valued map.

Definition 3.1 ($r - usc$)**.** A set-valued map $F : X \rightrightarrows Y$ is said to be $r-$upper semi-continuous ($r-usc$) at $x \in \mathrm{Dom}(F)$ if and only if for the closed ball $B(V, r)$ of any neighbourhood V of $F(x)$, $\exists \delta > 0$ such that $F(x') \subset B(V, r)$ for all $x' \in B^{\circ}(x, \delta)$.

Definition 3.2 ($r - lsc$)**.** A set-valued map $F : X \rightrightarrows Y$ is called $r-$lower semi-continuous ($r-lsc$) at $x \in \mathrm{Dom}(F)$ if and only if for any $y \in F(x)$ and any sequence of elements x_n convergent to x, there exists a sequence of elements $y_n \in F(x_n)$ $r-$convergent to y, i.e., $y_n \xrightarrow{r} y$.

Example 3.1. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be defined by

$$F(x) = \begin{cases} [-1, 1], & x \neq 0 \\ \{0\}, & x = 0. \end{cases}$$

This set-valued map $F(x)$ is not *usc* at $x = 0$, but it is $r-usc$ for some r.

First, we show that the map F is $1-usc$ at $x = 0$. Let $\varepsilon > 0$ be given. We have to find a $\delta > 0$ such that $F(x') \subset B((-\varepsilon, \varepsilon), 1)$ for all $x' \in B^{\circ}(x, \delta)$. Hence we get $B((-\varepsilon, \varepsilon), 1) = [-1 - \varepsilon, 1 + \varepsilon]$. Thus we have $F(x') \subset B((-\varepsilon, \varepsilon), 1)$ for all $x' \in (-\delta, \delta)$, where we can choose δ as an arbitrary number.

Now we show that the map F is not $\frac{1}{2}-usc$ at $x = 0$. We have $B\left((-\varepsilon, \varepsilon), \frac{1}{2}\right) = [-\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$. Take $\varepsilon = \frac{1}{3}$. Then we have $F(x') = [-1, 1] \not\subset \left[-\frac{5}{6}, \frac{5}{6}\right]$ for each $x' \in B^{\circ}(0, \delta) - \{0\}$ and each $\delta > 0$.

Remark 3.1. If the closed ball $B(V, r)$ satisfying $F(x) \subset B(V, r)$ was taken instead of the closed ball $B(V, r)$ of any neighbourhood V such that $F(x) \subset V$, the desired definition of $r-usc$ would not be obtained. As is shown in the example above, let $B(V, r) = B((1 - \varepsilon, 1 + \varepsilon), 1) = [-\varepsilon, 2 + \varepsilon]$ such that $F(x) \subset B(V, r)$ at $x = 0$, for $r = 1$. We have $F(x') \not\subset [-\varepsilon, 2 + \varepsilon]$ for $x' \neq 0 \in (-\delta, \delta)$.

Example 3.2. Define a map $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$F(x) = \begin{cases} \{0\}, & x \neq 0 \\ [-1, 1], & x = 0. \end{cases}$$

This map is not *lsc* at $x = 0$, but the map F is $r-lsc$ for some r.

Now we show that this map is $1-lsc$ at $x=0$. That is, we have to find a sequence $\{y_n\}$ such that $y_n \in F(x_n)$, $y_n \xrightarrow{r} y$ for any sequence $x_n \to 0$ and $y \in F(0)$. Take $y \in F(0) = [-1,1]$. Let $\{x_n\}$ be a sequence which converges to 0. If $x_n = 0$ for all $n \in \mathbb{N}$, then we choose $y = y_n \in [-1,1]$, hence we get $y_n \to y$, therefore we have $y_n \xrightarrow{r} y$. Otherwise, if $x_n \neq 0$ for all $n \in \mathbb{N}$, then we choose $y_n = 0 \in \{0\}$ for all $n \in \mathbb{N}$. Hence we get $y_n \xrightarrow{1} y$. Then the map F is $1-lsc$ at $x=0$.

Now we show that the map F is not $\frac{1}{2}-lsc$ at $x=0$. Let $V = (1-\delta, 1+\delta)$ such that $F(0) \cap V \neq \varnothing$ for $F(0) = [-1,1]$, $B(V, \frac{1}{2}) = \left[\frac{1}{2} - \delta, \frac{3}{2} + \delta\right]$. For $x' \in (-\delta, \delta)$, define $F(x') = \begin{cases} \{0\}, & x' \neq 0 \\ [-1,1], & x' = 0. \end{cases}$ Therefore $F(x') \cap B(V, \frac{1}{2}) = \{0\} \cap \left[\frac{1}{2} - \delta, \frac{3}{2} + \delta\right] = \varnothing$ for $x' \neq 0$. Thus the map F is not $\frac{1}{2}-lsc$ at $x=0$.

Remark 3.2. If the closed ball $B(V,r)$ satisfying $F(x) \cap B(V,r) \neq \varnothing$ was taken instead of the closed ball $B(V,r)$ of any neighbourhood V such that $F(x) \cap V \neq \varnothing$, the desired definition of $r-lsc$ would not be obtained. As is shown in the example above, let $B(V,r) = B((2,3),1) = [1,4]$ such that $F(x) \cap B(V,r) \neq \varnothing$ at $x=0$, for $r=1$. We have $F(x') \cap [1,4] = \varnothing$ for $x' \neq 0 \in (-\delta, \delta)$.

Now we give an $r-usc$ criterion for a set-valued map. For this criteria we inspire the Geletu's [3] *usc* definition. In order words, the following statement (iii) is the rough generalization of Geletu's definition.

Proposition 3.1. *Let X, Y be metric spaces. For an $F : X \rightrightarrows Y$, the following statements are equivalent:*

(i) *F is $r-usc$ at x, i.e., for the closed ball $B(V,r)$ of any neighbourhood V of $F(x)$, there exists a neighborhood U of x such that $F(x') \subset B(V,r)$ for every $x' \in U$.*

(ii) *Let $\{x_n\}$ be a sequence such that $x_n \to x$, and let $B(V,r)$ be the closed ball of any neighbourhood set $V (\subset Y)$ such that $F(x) \subset V$. Then there exists an $N \geq 1$ such that $F(x_n) \subset B(V,r)$ for any $n \geq N$.*

(iii) *For the closed ball $B(V,r)$ of any neighbourhood V of $F(x)$, there exists a $\delta > 0$ such that $F(x') \subset B(V,r)$ for any $x' \in B_X(x,\delta)$.*

Proof. (i) $\Rightarrow$ (ii) In this case, for the closed ball $B(V,r)$ of any open set $V \subset Y$ such that $F(x) \subset V$, there exists a neighbourhood U of x such that $F(x') \subset B(V,r)$ for any $x' \in U$ (i.e., $U \subset F^{+}(B(V,r))$). Since $x_n \to x$, for this neighbourhood U of x, there exists an $N_U \geq 1$ such that $x_n \in U$ for any $n \geq N_U$. Therefore we have $F(x_n) \subset B(V,r)$ for any $n \geq N_U$.

(ii) $\Rightarrow$ (iii) On the contrary, assume that statement (iii) does not hold. Hence there exists a neighbourhood V of $F(x)$ and an $x' \in B^{\circ}(x,\delta)$ such that $F(x') \not\subset B(V,r)$ for any $\delta > 0$. Let $\delta = \frac{1}{n}$. Let us choose $x_n = x' \in B^{\circ}(x, \frac{1}{n})$ such that $F(x') \not\subset B(V,r)$. Construction of this sequence contradicts statement (ii). Consequently, this contradiction completes the proof.

(iii) $\Rightarrow$ (i) Suppose that the map F is not $r-usc$ at x. Then there exists an open set V such that $F(x) \subset V$. In this case, for every neighbourhood U of x, there exists an $x' \in U$ such that $F(x') \not\subset B(V,r)$. We say that there exists a $\delta > 0$ such that $B^{\circ}(x,\delta) \subset U$. The last inclusion contradicts the fact that $F(x') \subset B(V,r)$. □

Similarly to Proposition 3.1, we give an $r-lsc$ criterion. The following statement (iii) is the rough generalization of Geletu's [3] *lsc* definition.

Proposition 3.2. *Let X, Y be metric spaces. For an $F : X \rightrightarrows Y$, the following statements are equivalent:*

(i) *If $\{x_n\}$ is any sequence such that $x_n \to x$ and $B(V,r)$ is the closed ball of any neighbourhood $V \subset Y$ with $F(x) \cap V \neq \varnothing$, then there exists an $N \geq 1$ such that $F(x_n) \cap B(V,r) \neq \varnothing$ for any $n \geq N$.*

(ii) *F is $r-lsc$ at x, i.e., if $\{x_n\}$ is any sequence such that $x_n \to x$ and $y \in F(x)$ is arbitrary, then there exists a sequence $\{y_n\}$ with $y_n \in F(x_n)$ such that $y_n \xrightarrow{r} y$.*

(iii) *For the closed ball $B(V,r)$ of any neighbourhood V with $F(x) \cap V \neq \varnothing$, there exists a neighbourhood U of x such that $F(x') \cap B(V,r) \neq \varnothing$ for any $x' \in U$, i.e., $U \subset F^{-1}(B(V,r))$.*

Proof. (i) $\Rightarrow$ (ii) Let $x_n \to x$ and $y \in F(x)$. Moreover, let $\varepsilon > 0$ and $B^{\circ}(y,\varepsilon) \cap F(x) \neq \varnothing$. By (i), $\exists N : F(x_n) \cap B(B^{\circ}(y,\varepsilon), r) \neq \varnothing$, $\forall n \geq N$. This implies that there exists $y_n \in F(x_n) \cap B(B^{\circ}(y,\varepsilon), r)$

for any $n \geq N$. For $n \in \{1, \ldots, N-1\}$, choosing $y_n \in F(x_n)$, we have a sequence $\{y_n\}$ such that $y_n \in F(x_n)$ and $y_n \xrightarrow{r} y$.

(ii) $\Rightarrow$ (iii) Assume that the map F is not $r-lsc$ at the point $x \in X$. Then there exists a $V \subset Y$ such that $F(x) \cap V \neq \varnothing$, and for any neighbourhood U of x there exists an $x' \in U$ such that $F(x) \cap B(V, r) = \varnothing$. Let $U = B^{\circ}(x, \frac{1}{n})$. Hence we have

$$\forall n \in \mathbb{N},\ \exists x_n \in B^{\circ}\Big(x, \frac{1}{n}\Big) : F(x_n) \cap B(V, r) = \varnothing.$$

This implies that if $y \in F(x_n) \cap V$, then there exist no sequence $\{y_n\}$ such that $y \in F(x_n)$ and $y_n \xrightarrow{r} y$. This contradicts statement (ii).

(iii) $\Rightarrow$ (i) Assume that for the closed ball $B(V, r)$ of any neighbourhood V with $F(x) \cap V \neq \varnothing$, there exists a neighbourhood U of x such that $\forall x' \in U$, $F(x') \cap B(V, r) \neq \varnothing$. For any $\{x_n\}$ such that $x_n \to x$, $\exists N_U > 1 : x_n \in U$ (any neighborhood U of x), $\forall n \geq N_U$. Take $N_U = N$, then we have $F(x_n) \cap B(V, r) \neq \varnothing$, $\forall n \geq N$. □

Proposition 3.3. *Let $F : X \rightrightarrows Y$ and $\mathrm{Dom}(F) = X$. Then the following statements are equivalent:*

(i) *F is $r-usc$.*

(ii) *For the closed ball $B(V, r)$ of each neighbourhood $V \subset Y$ such that $F(x) \subset V$, $F^{+1}(B(V, r))$ is an open set in X.*

Proof. (i) $\Rightarrow$ (ii) Let $x \in F^{+1}(B(V, r))$ and F be $r-usc$ at x. By definition, for the closed ball $B(V, r)$ of any neighbourhood V of $F(x)$, there exists a neighbourhood U of x such that $F(x') \subset B(V, r)$, $\forall x' \in U$, i.e. $U \subset F^{+1}(B(V, r))$. Since F is $r-usc$ at $x \in F^{+}(B(V, r))$, there exists a neighbourhood U of x such that $U \subset F^{+1}(B(V, r))$. Consequently, $F^{+1}(B(V, r))$ is an open set.

(ii) $\Rightarrow$ (i) Assume that for the closed ball $B(V, r)$ of each $V \subset Y$ such that $F(x) \subset V$, $F^{+1}(B(V, r))$ is an open set in X. For $x \in X$ with $F(x) \subset V$, $x \in F^{+1}(B(V, r))$. Since $F^{+1}(B(V, r))$ is an open set, there exists an U of x such that $U \subset F^{+1}(B(V, r))$, i.e. for $x' \in U$, $F(x') \subset B(V, r)$. Thus F is $r-usc$. □

Remark 3.3. In Proposition 3.3 (ii), if the condition $F(x) \subset V$ is removed, then the equivalence may not hold. In Example 3.1, if we choose $V = (1 + \varepsilon, 1 - \varepsilon)$, then $B(V, 1) = [2 + \varepsilon, -\varepsilon]$ and $F^{+1}(B(V, 1)) = \{0\}$ is a closed set.

Remark 3.4. It is not true that F is $r-usc$ if and only if $F^{-1}(B(V, r))$ is a closed set in X for the closed ball $B(V, r)$ of an open set $V \subset Y$. In Example 3.1, take $W = [2, 1 + \varepsilon]$. Hence $B(W, 1) = [3, \varepsilon]$ and $F^{-1}(B(W, 1)) = \mathbb{R} - \{0\}$ is an open set. Conversely, although the map F is $1-usc$, $F^{+1}(B(W, 1))$ is not a closed set.

Proposition 3.4. *Let $F : X \rightrightarrows Y$ and $\mathrm{Dom}(F) = X$. Then the following statements are equivalent:*

(i) *F is $r-lsc$.*

(ii) *For closed ball $B(V, r)$ of each neigborhood $V \subset Y$ with $F(x) \cap V \neq \varnothing$, $F^{-1}(B(V, r))$ is an open set in X.*

Proof. (i) $\Rightarrow$ (ii) Let $x \in F^{-1}(B(V, r))$. For the closed ball $B(V, r)$ of any neighbourhood V such that $F(x) \cap V \neq \varnothing$, there exists a neighbourhood U of x such that

$$F(x') \cap B(V, r) \neq \varnothing,\ \forall x' \in U,\ \text{i.e.,}\ U \subset F^{-1}\left(B(V, r)\right).$$

Since F is $r-lsc$ at any point x, then there exists a neighbourhood U of x such that $U \subset F^{-1}(B(V, r))$. At least a neighbourhood of any point of $F^{-1}(B(V, r))$ is included in $F^{-1}(B(V, r))$. Consequently, $F^{-1}(B(V, r))$ is an open set.

(ii) $\Rightarrow$ (i) Assume that for the closed ball $B(V, r)$ of each $V \subset Y$ with $F(x) \cap V \neq \varnothing$, $F^{-1}(B(V, r))$ is an open set in X. For $x \in F^{-1}(B(V, r))$, there exists a neighbourhood U of x such that $x \in U \subset F^{-1}(B(V, r))$. For $B(V, r)$ with $F(x) \cap V \neq \varnothing$, there exists a neighbourhood U of x such that $F(x') \cap B(V, r) \neq \varnothing$, $x' \in U$. □

Remark 3.5. (ii) In Proposition 3.4 (ii), if the condition $F(x) \cap V \neq \varnothing$ is removed, then the equivalence may not hold. In Example 3.2, if we choose $V = (\frac{3}{2}, 2)$, then $B(V, 1) = \left[\frac{1}{2}, 3\right]$ and $F^{-1}(B(V, 1)) = \{0\}$ is a closed set.

Remark 3.6. It is not true that F is $r-lsc$ if and only if for the closed ball $B(V, r)$ of an open set $V \subset Y$, $F^{-1}(B(V, r))$ is a closed set in X. In Example 3.2, if we take $W = [\frac{1}{2}, 1]$, then $B(W, 1) = [-\frac{1}{2}, 2]$ and $F^{+1}(B(W, 1)) = \mathbb{R} - \{0\}$ is an open set. Conversely, although F is $1-lsc$, $F^{+1}(B(W, 1))$ is not a closed set.

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