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GLOBAL REGULARITY OF $\bar{\partial}$ ON CERTAIN PSEUDOCONVEXITY

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Abstract. The global boundary regularity for the $\bar{\partial}$-problem on a relatively compact domain with the C^2-smooth boundary in a Kähler manifold that satisfies some "Hartogs-pseudoconvexity" condition is investigated in this paper. The applications to the $\bar{\partial}$-problem and the $\bar{\partial}_b$-problem, are thoroughly discussed.

1. Introduction

The goal of this paper is to prove the existence and regularity of the solution to the Cauchy–Riemann equations, known also as the $\bar{\partial}$-problem, $\bar{\partial}u = f$ on a relatively compact domain with the C^2-smooth boundary in a Kähler manifold that satisfies some "Hartogs-pseudoconvexity" condition.

Previously, many different approaches were employed, including a) the vanishing of the $\bar{\partial}$-cohomology group, which was used by Grauert–Riemenschneider [18], Abdelkader–Saber [1], and Saber [33,34], and b) the abstract L^2-theory of the $\bar{\partial}$-Neumann problem, which was used in two cases: 1) with exact support (see Saber [35, 38, 40, 43]; 2) with regularity up to the boundary ([37, 41, 42, 44] and c) the construction of rather explicit integral solution operators for $\bar{\partial}$, in analogy to the Cauchy transform in C^1 used by Henkin [19] and Grauert–Lieb [17].

In this paper, we investigate global boundary regularity for the $\bar{\partial}$-problem, $\bar{\partial}u = f$ in an n-dimensional Kahler manifold with the C^2-smooth boundary that satisfies some "Hartogs-pseudoconvexity" condition. In [26], Kohn investigated this problem on a pseudoconvexity domain without corners. In [22], Ho solved this problem in $\mathcal{C}^n$ for strongly q-convex domains. In $\mathcal{C}^n$, Zampieri [52] introduced a new kind of q-pseudoconvex notion. He proved local boundary regularity for any degree q under this condition. Heungju [2], Baracco–Zampieri [6, 7], Saber [33, 34, 37, 39], Ahn [3], Henkin–Iordan [21], all obtained the results in this direction. In the case of an annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, this problem has been investigated in the following cases: 1) Ω_1 and Ω_2 are pseudoconvex submanifolds that satisfy property (P) [11], 2) Ω_1 is an internal p-pseudoconcave and Ω_2 is an external q-pseudoconvex in $\mathcal{C}^n$ [4], and 3) Ω_1 and Ω_2 are weakly pseudoconvex with smooth boundaries in C^n [45, 49]. The problem has been studied by the first author in the following domains: a) strongly q-concave and q-convex domains of an n-dimensional Kähler manifold [33], b) annuli domain in a Stein manifold, where the outside boundary is weakly q-convex and the inside boundary is weakly $(n-q-1)$-convex for complex-valued and vector-valued forms [36, 39], c) annuli domain between two pseudoconvex submanifolds of a Stein manifold [35], d) annuli domain between p-pseudoconcave and q-pseudoconvex domains of a complex manifold whose boundaries satisfy property (P) ([42]).

There are many applications to the $\bar{\partial}_b$-problem. Kohn-Rossi [28] proposed this problem in the mid 1960s to investigate the holomorphic extension of CR functions from the boundary of a complex manifold. The following cases have looked into this issue: a) it must be purely pseudoconvex. In 1965, Kohn [24] used subelliptic estimates for $\bar{\partial}_b$ to prove Sobolev estimates for $\bar{\partial}_b$. Folland–Stein [16] proved Holder and L^p-estimates for $\bar{\partial}_b$ in 1974. Skoda [50], Henkin [20], and Romanov [31], all introduced integral kernel methods for $\bar{\partial}_b$) weakly pseudoconvex in 1976 and 1977, respectively. Rosay [32] proved the C^1-solvability for $\bar{\partial}_b$ in 1982. Shaw [45], Boas–Shaw [8], and Kohn [27], Shaw [48] independently proved some results on L^2 and Sobolev estimates for $\bar{\partial}_b$ in 1985 and 1986. Cao–Shaw–Wang [9] studied this problem in 2004 on pseudoconvex domains in $\mathcal{CP}^n$ and c) finite type pseudoconvex domains.

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In 1988, Fefferman–Kohn [15] proved Holder's estimates for $\overline{\partial}_b$ on the boundaries of pseudoconvex domains of finite type in C^2. Hölder and L^p estimates for $\overline{\partial}_b$ on the boundaries of weakly pseudoconvex domains of uniform strict type in C^2 were proved by Shaw [46,47]. Andreea has proved global regularity for $\overline{\partial}_b$ on weakly pseudoconvex CR manifolds [5]. The problem has been studied by the first author on two domains: a) a pseudoconvex domain of order $n-q$ of Kähler manifold with positive holomorphic bisectional curvature [35], and b) weakly q-convex domains in a Kähler manifold for the forms with values in a vector bundle E [38, 40].

2. Preliminaries

Let (X, g) be an n-dimensional Kähler manifold and $\pi : E \longrightarrow X$ be a holomorphic vector bundle of rank p, over X. Let $\Omega \Subset X$ be an open set and let $\delta(z)$ be the distance from $z \in \Omega$ to the boundary $b\Omega$ of Ω with respect to the metric ω. Let ω be the Kähler form associated to the Kähler metric g. Let $\{U_j\}_{j=1}^N$ be a finite covering of X by a local patching. Let $e_1, e_2, \dots, e_p$ be an orthonormal basis on $E_z = \pi^{-1}(z)$, for every $z \in U_j$; $j \in J$. Thus every E-valued differential (r,s)-form u on X can be written locally on U_j, as $u(z) = \sum_{a=1}^{p} u^a(z)\, e_a(z)$, where u^a are the components of the restriction of u on U_j. Let $C^\infty_{r,s}(X, E)$ be the complex vector space of E-valued differential forms of class C^∞ and of type (r,s) on X. For $u, v \in C^\infty_{r,s}(X,E)$, we define a local inner product (u, v) with respect to g and h by

$$(u, v)\, dV = \sum_{a=1}^{p} u^a \wedge \star\, \overline{(h\, v)^a},$$

where dV is the volume element with respect to g, $\star : C^\infty_{r,s}(X,E) \longrightarrow C^\infty_{n-s,n-r}(X,E)$ is the Hodge star operator defined by g. Let $C^\infty_{r,s}(\overline{\Omega}, E) = \left\{u\,|_{\overline{\Omega}}; u \in C^\infty_{r,s}(X,E)\right\}$ be the subspace of $C^\infty_{r,s}(\Omega,E)$ whose elements can be extended smoothly up to $b\Omega$. Let ϕ be a locally bounded real-valued function on $\overline{\Omega}$. For $u, v \in C^\infty_{r,s}(\overline{\Omega}, E)$, the associated global inner product $< u, v >_\phi$, with respect to g, h and the weight function ϕ, is defined by

$$< u, v >_\phi = \int_\Omega (u, v)\, e^{-\phi} dV.$$

The norm $\| \,.\, \|_\Omega$ on $C^\infty_{r,s}(\overline{\Omega}, E)$ is defined by

$$\|u\|^2_\phi = < u, u >_\phi = \int_\Omega e^{-\phi}\ |u|^2\ dV,$$

where $|u|^2 = (u, u)$. We shall consider the weighted L^2-spaces

$$L^2_{r,s}(\Omega, e^{-\phi}, E) = \{u : \|u\|_\phi < \infty\}$$

of E-valued differential forms of various degrees. The Laplace–Beltrami operator $\square_{r,s}$ for E-valued forms is defined by

$$\square_{r,s} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \mathrm{dom}(\square_{r,s}, E) \longrightarrow L^2_{r,s}(\Omega, E),$$

where $\mathrm{dom}(\square_{r,s}, E) = \{u \in L^2_{r,s}(\Omega,E) : u \in \mathrm{dom}(\overline{\partial}, E) \cap \mathrm{dom}(\overline{\partial}^*, E); \overline{\partial}u \in \mathrm{dom}(\overline{\partial}^*, E)$ and $\overline{\partial}^* u \in \mathrm{dom}(\overline{\partial}, E)\}$. Thus

$$\mathcal{H}_{r,s}(E) = \{u \in \mathrm{dom}(\square_{r,s}, E);\ \overline{\partial}u = \overline{\partial}^* u = 0\},$$

is a closed subspace of $\mathrm{dom}(\square_{r,s}, E)$, since $\square_{r,s}$ is a closed operator. The $\overline{\partial}$-Neumann operator $N = N_{r,s} : L^2_{r,s}(\Omega,E) \longrightarrow L^2_{r,s}(\Omega,E)$ is defined as the inverse of the restriction of $\square_{r,s}$ to $(\mathcal{H}_{r,s}(E))^\perp$, i.e.,

$$N_{r,s}u = \begin{cases} 0 & \text{if} \quad u \in \mathcal{H}_{r,s}(E), \\ v & \text{if} \quad u = \square_{r,s}v, \text{ and } v \perp \mathcal{H}_{r,s}(E). \end{cases}$$

In other words, $N_{r,s}u$ is the unique solution v to the equations $\square_{r,s}v = u - \Pi_{r,s}\, u$ and $\Pi_{r,s}v = 0$, where $\Pi_{r,s} : L^2_{r,s}(\Omega,E) \longrightarrow \mathcal{H}_{r,s}(E)$ is the orthogonal projection from the space $L^2_{r,s}(\Omega,E)$ onto the space $\mathcal{H}_{r,s}(E)$.

A Hermitian metric h along the fibers of E can be expressed as $h = (h_{a\bar{b}})$; $h_{a\bar{b}} = h(e_a, e_b)$. Let $(h^{a\bar{b}})$ be the inverse matrix of $(h_{a\bar{b}})$. Let $\theta = (\theta_a^c)$, $\theta_a^c = \sum_{\alpha=1}^{n}\sum_{b=1}^{p} h^{c\bar{b}} \frac{\partial h_{a\bar{b}}}{\partial z^\alpha} dz^\alpha = \sum_{\alpha=1}^{n} \mu_{a\alpha}^c dz^\alpha$ and $\Theta = i\overline{\partial}\theta = i\overline{\partial}\partial \log h = \{\Theta_a^c\}$; $\Theta_a^c = i \sum_{\alpha,\beta=1}^{n} \Theta_{a\alpha\overline{\beta}}^c dz^\alpha \wedge d\overline{z}^\beta$ be the connection and the curvature forms associated to the metric h, respectively, where $\Theta_{a\alpha\overline{\beta}}^c = -\frac{\partial}{\partial\overline{z}^\beta}\left(h^{c\bar{b}}\frac{\partial h_{a\bar{b}}}{\partial z^\alpha}\right)$, $1 \le a, c \le p$. Thus the operator

$$A_E^{r,s} = (e(\Theta)\Lambda - \Lambda e(\Theta))$$

acting on $\wedge^{r,s}T^*X \otimes E$. The curvature matrix $\mathcal{H}$, associated to the Hermitian metric h, is given by

$$\mathcal{H} = (\mathcal{H}_{\bar{b}\,\overline{\beta},c\alpha}) = \left(\sum_{a=1}^{p} h_{a\bar{b}}\,\Theta_{c\alpha\overline{\beta}}^a\right).$$

Denote by $e(\omega) : C_{r,s}^\infty(X,E) \longrightarrow C_{r+1,s+1}^\infty(X,E)$ the linear mapping locally defined by $(e(\omega)u)^a = \omega \wedge u^a$ and $\Lambda : C_{r,s}^\infty(X,E) \longrightarrow C_{r-1,s-1}^\infty(X,E)$ the linear mapping locally defined by $\Lambda = (-1)^{r+s} \star e(\omega)\star$. Let $\mathrm{Hom}_{\mathcal{R}}(TX,\mathcal{C})$ be the complex vector space of complex-valued real linear mappings of TX to $\mathcal{C}$. We denote by

$$\wedge\,\mathrm{Hom}_{\mathcal{R}}(TX,\mathcal{C}) = \sum_{t=0}^{2n}\sum_{r+s=t} \wedge^{r,s}T^*X,$$

the $\mathcal{C}$-linear exterior algebra of $\mathrm{Hom}_{\mathcal{R}}(TX,\mathcal{C})$. A linear mapping

$$L : \wedge\mathrm{Hom}_{\mathcal{R}}(TX,\mathcal{C}) \longrightarrow \wedge\mathrm{Hom}_{\mathcal{R}}(TX,\mathcal{C})$$

is defined by $L\phi = e(\omega)\phi = \omega \wedge \phi$, for $\phi \in \wedge^{r,s}T^*X$, i.e., $L : \wedge^{r,s}T^*X \longrightarrow \wedge^{r+1,s+1}T^*X$. The formal adjoint operator $\Lambda : \wedge^{r,s}T^*X \longrightarrow \wedge^{r-1,s-1}T^*X$ of the operator L is defined locally by $\Lambda\phi = (-1)^{r+s} \star L \star \phi$. Let α be a real $(1,1)$-form with values in the vector bundle $\mathrm{Herm}(E;E) = E^* \otimes E$ satisfying $\phi \ge_{n-s+1} 0$. For $\alpha \in \wedge^{n,s}T^*X \otimes E$, we put

$$|f|_\alpha^2 = \sup_{\substack{u\in \wedge^{n,s}T^*X\otimes E,\\ u\neq 0}} \frac{|(f,u)|^2}{(\alpha \wedge \Lambda u, u)}.$$

Lemma 1 ([13, Lemma 3.2]).

(i) *The (n,n)-form $|f|_\alpha^2\, dv$ is a decreasing function of ω.*

(ii) *For any semi-positive real number, c satisfies $\mathcal{H} \ge_{n-s+1} C\,\omega\otimes Id_E$, and for any $f\in\wedge^{n,s}T^*X\otimes E$, one obtains*

$$|f|_\alpha^2 \le \frac{1}{sC}|f|^2.$$

(iii) *Let η be a $(0,1)$-form on X, then*

$$|\,\eta \wedge f|_\alpha \le |\eta|\; |f|_\alpha.$$

Definition 1. Let X be a complex manifold of complex dimension n. For every compact set $K \subset X$, the holomorphically convex hull $\hat{K}_X$ of K is defined by

$$\hat{K}_X := \left\{z \in X : |f(z)| \le \sup_{\xi\in K} |f(\xi)|, \text{ for all } f \in \mathcal{O}(X)\right\}.$$

Definition 2. A complex manifold X is called holomorphically convex if for every compact set $K \subset X$, $\hat{K}_X$ is also compact.

Definition 3. A complex manifold X of complex dimension n which is countable at infinity is said to be a Stein manifold if:

(i) X is holomorphically convex,

(ii) for any point $z \in X$, one can find n functions $f_1, \ldots, f_n \in \mathcal{O}(X)$ which form holomorphic coordinates at z. That is, there exists a neighborhood B of z such that the map $B \ni \xi \longrightarrow (f_1(\xi), \ldots, f_n(\xi)) \in \mathcal{C}^n$ is biholomorphic,

(iii) if z and ξ are two points in X that are different, then there exists an $f \in \mathcal{O}(X)$ such that $f(z) \neq f(\xi)$.

Definition 4. Let X be a complex manifold of complex dimension n. An open subset $\Omega \Subset X$ is said to be a locally Stein domain if for any point $z \in b\Omega$, there exists an open neighborhood V of z such that $V \cap \Omega$ is a Stein manifold.

Definition 5. A complex manifold X is said to be weakly pseudo-convex if there exists an exhaustion plurisubharmonic function $\phi \in C^{\infty}(X, \mathcal{R})$.

Definition 6. Let X be a complex manifold and Ω be a relatively compact domain in X. Ω is said to be a Hartogs pseudo-convex if there exists a Kähler metric on X for which the associated geodesic distance $d(z)$ satisfies the following:

(a) there exists a neighborhood U of $b\Omega$ such that $-\log d(z)$ is strongly plurisubharmonic on U,

(b) the restriction of $-\log d(z)$ to $U \cap \Omega$ admits a C^{∞}-positive strongly plurisubharmonic extension function to Ω.

In what follows, there is a C^{∞}-positive function σ on Ω and a positive constant c such that $\sigma = d$ on U and

$$i\partial\overline{\partial}(-\log \sigma) \geq c\,\omega \ \text{ on } \ \Omega,$$

where ω is the Kähler form associated to the Kähler metric for which (a) of the above definition is satisfied.

In particular, every Hartogs-pseudoconvex domain admits a strictly plurisubharmonic exhaustion function, therefore is a Stein manifold.

Remark 1. Every Hartogs pseudo-convex domain is a weakly pseudoconvex Kähler manifold. In the notation of Definition 6, let c be a positive constant, so that:

$$V = \{z \in \Omega : c < -\log d(z) < +\infty\} \Subset U \cap \Omega.$$

We set

$$\phi(z) = \begin{cases} -\log d(z) & \text{for} \quad z \in V, \\ c & \text{on} \quad z \in \Omega \backslash V. \end{cases}$$

Thus ϕ is a plurisubharmonic exhaustion function on Ω. Then, according to Definition 6, Ω is a weakly pseudoconvex Kähler manifold.

Now we'll look at several Hartogs pseudo-convex domain examples.

Example 1. Let X be a complex manifold such that there exists a continuous strongly plurisubharmonic function on X and let $\Omega \Subset X$ be a locally Stein domain. It was shown in [14] that there exists a Kähler metric on X such that Ω is Hartogs pseudo-convex.

Example 2. Every locally Stein domain in a Stein manifold is Hartogs pseudo-convex.

Example 3. Every pseudo-convex domain in $\mathcal{C}^n$ with a C^2-smooth boundary is a Stein manifold [14]. As a result, (2.2) is satisfied.

Example 4. In a Kähler manifold with positive holomorphic bisectional curvature, any relatively compact locally Stein domain satisfies (2.2) on $U \cap \Omega$ [14] (a Kähler manifold X has positive holomorphic bisectional curvature if the tangent bundle TX of X is positive in sense of Griffiths).

Example 5. Every locally Stein domain in the complex projective space $\mathcal{P}^n$ satisfies (2.2) [51].

3. The L^2-existence Theorem on a Hartogs Pseudo-convex domain

In this section, we prove the L^2-existence theorem on a Hartogs pseudo-convex domain on a Kähler manifold in this section.

Lemma 2 ([12, Lemma 6.3])**.** *Let g, γ be two Hermitian metrics on X such that $\gamma \geq g$. For every $u \in \wedge^{n,s}T^*X \otimes E$, $s \geq 1$, we have*

$$|u|^2_{\gamma}\, dV_{\gamma} \leq |u|^2\, dV, \quad ((A^{r,s}_{E,\gamma})^{-1}u, u)_{\gamma}\, dV_{\gamma} \leq ((A^{r,s}_{E})^{-1}u, u)\, dV, \tag{3.1}$$

where an index γ means that the corresponding term is computed in terms of γ instead of g.

Lemma 3 ([12, Lemma 2.4]). *The properties listed below are equivalent:*

(i) $(X, \mathcal{G})$ *is complete;*

(ii) *There is a function* $\phi \in C^\infty(X, \mathcal{R})$ *that is exhaustive such that* $|d\phi|_{\mathcal{G}} \le 1$;

(iii) *There is an exhaustive sequence* $(K_\nu)_{\nu \in \mathcal{N}}$ *of compact subsets of* X *and the functions* $\phi_\nu \in C^\infty(X, \mathcal{R})$ *such that* $\phi_\nu = 1$ *in a neighborhood of* K_ν, $\operatorname{supp} \phi_\nu \subset K^\circ_{\nu+1}$, $0 \le \phi_\nu \le 1$ *and* $|d\phi_\nu|_g \le 2^{-\nu}$.

Theorem 1 ([12, Theorem 4.5]). *Let* E *be a holomorphic vector bundle over a complete Kähler manifold* $(X, \mathcal{G})$. *Assume that* $A^{r,s}_{E,\mathcal{G}}$ *is a positive definite on* $\wedge^{r,s}T^*X \otimes E$. *Let* $u \in L^2_{r,s}(X, E)$ *with* $s \ge 1$ *satisfy* $\overline{\partial} u = 0$ *and*

$$\int_X ((A^{r,s}_{E,\mathcal{G}})^{-1}u, u)_{\mathcal{G}}\, dV_{\mathcal{G}} < +\infty,$$

then there exists a solution $\psi \in L^2_{r,s-1}(X, E)$ *to the equation* $\overline{\partial}\psi = u$ *such that*

$$\int_X |\psi|^2_{\mathcal{G}} dV_{\mathcal{G}} \le \int_X ((A^{r,s}_{E,\mathcal{G}})^{-1}u, u)_{\mathcal{G}}\, dV_{\mathcal{G}}.$$

Lemma 4 ([12]). *Every weakly pseudoconvex Kähler manifold* (X, g) *carries a complete Kähler metric* $\mathcal{G}$.

Proof. Let $\phi \in C^\infty(X, \mathcal{R})$ be an exhaustion function which is plurisubharmonic on X. After adding a constant to ϕ, we can assume $\phi \ge 0$. Then $\mathcal{G} = g + i\partial\overline{\partial}(\phi^2)$ is a Kähler metric and

$$\mathcal{G} = g + 2i\phi\, \partial\overline{\partial}\phi + 2i\partial\phi \wedge \overline{\partial}\phi \ge g + 2i\partial\phi \wedge \overline{\partial}\phi.$$

Since $d\phi = \partial\phi + \overline{\partial}\phi$, one gets $|d\phi|_{\mathcal{G}} = \sqrt{2}\, |\partial\phi|_{\mathcal{G}} \le 1$. Lemma 3 proves that $\mathcal{G}$ is complete. □

Theorem 2 ([12]). *Let* (X, g) *be a Kähler manifold* (g *is not assumed to be complete*). *Assume that* X *is weakly pseudo-convex. Assume that* E *is a holomorphic vector bundle over* X, *and that a positive continuous function* $\gamma : X \longrightarrow \mathcal{R}$ *exists such that*

$$\Theta(E) \ge \gamma\, \omega \otimes Id_E.$$

Then for $u \in L^2_{\mathrm{loc}}(X, \wedge^{n,s}T^*X \otimes E)$, $s \ge 1$, *satisfying* $\overline{\partial}u = 0$ *and* $\int_X \gamma^{-1}|u|^2_g\, dV_g < +\infty$, *there exists a solution* $\psi \in L^2(X, \wedge^{n,s-1}T^*X \otimes E)$ *to the equation* $\overline{\partial}\psi = u$ *such that*

$$\int_X |\psi|^2_g dV_g \le \int_X \gamma^{-1}|u|^2_g\, dV_g.$$

Proof. Indeed, under the assumption on E, we have

$$((A^{n,s}_{E,g})u, u)_g \ge \gamma |u|^2_g,$$

hence $((A^{n,s}_{E,g})^{-1}u, u)_g \le \gamma^{-1}|u|^2_g$. The assumption that $u \in L^2_{\mathrm{loc}}(X, \wedge^{n,s}T^*X \otimes E)$ instead of $u \in L^2(X, \wedge^{n,s}T^*X \otimes E)$ is not a real problem, since we may restrict ourselves to $X_c = \{x \in X : \varrho(x) < c\} \Subset X$, where ϱ is a plurisubharmonic exhaustion function on X. Then X_c is itself weakly pseudoconvex (with plurisubharmonic exhaustion function $\varrho_c = 1/(c - \varrho)$), hence X_c can be equipped with a complete Kähler metric

$$g_{c,\varepsilon} = g + \varepsilon\, i\partial\overline{\partial}(\varrho_c^2)$$

(cf. the proof of Lemma 4). From (3.1), we obtain

$$\int_{X_c} ((A^{n,s}_{E,g_{c,\varepsilon}})^{-1}u, u)_{g_{c,\varepsilon}}\, dV_{g_{c,\varepsilon}} \le \int_{X_c} ((A^{n,s}_{E,g})^{-1}u, u)_g\, dV_g \le \int_X \gamma^{-1}|u|^2_g\, dV_g < +\infty.$$

For each (c,ε), Theorem 1 yields a solution $\psi_{c,\varepsilon} \in L^2_{g_{c,\varepsilon}}(X_c, \wedge^{n,s-1}T^* X\otimes E)$ to the equation $\overline{\partial}\psi_{c,\varepsilon} = u$ on X_c such that

$$\int\limits_{X_c} |\psi_{c,\varepsilon}|^2_{g_{c,\varepsilon}} dV_{g_{c,\varepsilon}} \leq \int\limits_{X_c} ((A^{n,s}_{E,g_{c,\varepsilon}})^{-1}u,u)_{g_{c,\varepsilon}}\, dV_{g_{c,\varepsilon}}.$$

Then, we obtain

$$\int\limits_{X_c} |\psi_{c,\varepsilon}|^2_{g_{c,\varepsilon}} dV_{g_{c,\varepsilon}} \leq \int\limits_{X} \gamma^{-1}|u|^2_g\, dV_g.$$

Thus, $\psi_{c,\varepsilon}$ are the solutions that are uniformly bounded in the L^2-norm on every compact subset of X. Since the closed unit ball of a Hilbert space is weakly compact, we can extract a subsequence

$$\psi_{c_k,\varepsilon_k} \longrightarrow \psi \in L^2_{\mathrm{loc}},$$

converging weakly in L^2 on any compact subset $K \subset X$, for some $c_k \longrightarrow +\infty$ and $\varepsilon_k \longrightarrow 0$. By the weak continuity of differentiations, we get again in the limit $\overline{\partial}\psi = u$. Also, for every compact set $K \subset X$, we get

$$\int\limits_K |\psi|^2_g dV_g \leq \liminf_{k\longrightarrow\infty} \int\limits_K |\psi_{c_k,\varepsilon_k}|^2_{g_{c_k,\varepsilon_k}} dV_{g_{c_k,\varepsilon_k}}$$

by a weak L^2_{loc} convergence. Finally, we let K increase to X and conclude that the desired estimate holds on all of X. □

Remark 2. We have

$$L^2_{r,s}(X,E) \;=\; L^2_{n,s}(X, \wedge^{n-r}TX\otimes E). \tag{3.2}$$

We can see that there is a canonical duality pairing. $\wedge^k TX \otimes \wedge^k T^*X \longrightarrow \mathcal{C}$, hence an (r,s)-form with values in E may be viewed as a section of

$$\wedge^{r,s}T^*X\otimes E = \wedge^{0,s}T^*X \otimes \wedge^r T^*X \otimes E = \wedge^{n,s}T^*X\otimes \tilde{E},$$

where $\tilde{E}$ is the holomorphic vector bundle

$$\tilde{E} = \wedge^n TX \otimes \wedge^r T^*X \otimes E = \wedge^{n-r}TX\otimes E,$$

through the contraction pairing

$$\wedge^n TX\otimes\wedge^r T^*X \simeq \wedge^{n-r}TX.$$

Thus (3.2) follows.

Let $\Theta(\wedge^{n-r}T\Omega\otimes E)$ be the curvature form of the holomorphic vector bundle $\wedge^{n-r}T\Omega\otimes E$. For $0\leq r\leq n$, one defines

$$m_r(\Omega;E) = \sup\{m\in\mathcal{R}|\, \Theta(\wedge^{n-r}T\Omega\otimes E) \geq m\,\omega\otimes \mathrm{Id}_{\wedge^{n-r}T\Omega\otimes E}\},$$

Following Remark 1, every Hartogs pseudo-convex domain is weakly pseudo-convex manifold, then by applying Theorem 2, we obtain the following

Theorem 3. *Let (X,g) be a Kähler manifold of complex dimension n and $\Omega \Subset X$ be a Hartogs pseudo-convex domain. Assume that E is a holomorphic vector bundle over X with the following properties: $m_r(\Omega;E) > 0$. Then, for $\theta \in L^2_{r,s}(\Omega,E)$, $s\geq 1$, satisfying $\overline{\partial}u = 0$, there exists a solution $\psi \in L^2_{r,s-1}(\Omega,E)$ to the equation $\overline{\partial}\psi = \theta$ such that*

$$\int\limits_\Omega |\psi|^2 dV \leq c\int\limits_\Omega |\theta|^2\, dV,$$

with $c = 1/m$.

Proof. Since $m_r(\Omega; E) > 0$, there exists a positive constant m such that

$$\Theta(\wedge^{n-r}T\Omega \otimes E) \geq \ m\,\omega \otimes \mathrm{Id}_{\wedge^{n-r}T\Omega\otimes E}.$$

Thus from (3.2), by using the solution to the $\overline{\partial}$-equation for (n, s)-forms of Theorem 2 with values in the holomorphic vector bundle $\wedge^{n-r}T\Omega \otimes E$, there exists $\psi \in L^2_{r,s-1}(\Omega, E)$ such that $\overline{\partial}\psi = \theta$ and

$$\int_\Omega |\psi|^2\, dV \leq c \int_\Omega |\theta|^2\, dV.$$

Thus the proof follows. □

Corollary 1. *Under the same assumption of Theorem 3, we have*

$$\mathcal{H}_{r,s}(E) = 0. \tag{3.3}$$

Proof. If $\theta \in \mathcal{H}_{r,s}(E)$, thus $\overline{\partial}\theta = 0$. Then from Theorem 3, there exists $\psi \in L^2_{r,s-1}(\Omega, E)$ such that $\overline{\partial}\psi = \theta$. But $\overline{\partial}^*\theta = 0$ implies that

$$0 =< \overline{\partial}^*\theta, \psi >_\Omega =< \overline{\partial}^*\overline{\partial}\psi, \psi >_\Omega = \|\overline{\partial}\psi\|^2_\Omega = \|\theta\|^2_\Omega.$$

Thus $\theta = 0$, i.e., $\mathcal{H}_{r,s}(E) = 0$, and so, (3.3) follows. □

Theorem 4. *Let (X, g) be a Kähler manifold of complex dimension n and $\Omega \Subset X$ be a Hartogs pseudo-convex domain. Assume that E is a holomorphic vector bundle over X with the following properties: $m_r(\Omega; E) > 0$. Then there exists a bounded linear operator*

$$N_{r,s} : L^2_{r,s}(\Omega, E) \longrightarrow L^2_{r,s}(\Omega, E)$$

such that

(i) rang $(N_{r,s}, E) \subset \mathrm{dom}(\Box_{r,s}, E)$, $N_{r,s}\Box_{r,s} = I - \Pi_{r,s}$ *on* $\mathrm{dom}(\Box_{r,s}, E)$,

(ii) *for* $\theta \in L^2_{r,s}(\Omega, E)$, *we have* $\theta = \overline{\partial}\,\overline{\partial}^* N_{r,s}\theta \oplus \overline{\partial}^*\overline{\partial}N_{r,s}\theta \,\oplus\, \Pi_{r,s}\theta$,

(iii) $N_{r,s}$ *commutes with* $\overline{\partial}$ *and* $\overline{\partial}^*$, $\Pi_{r,s}N_{r,s} = N_{r,s}\Pi_{r,s} = 0$,

(iv) $N_{r,s}(C^\infty_{r,s}(\overline{\Omega}, E) \subset C^\infty_{r,s}(\overline{\Omega}, E)$ *and* $\Pi_{r,s}(C^\infty_{r,s}(\overline{\Omega}, E)) \subset C^\infty_{r,s}(\overline{\Omega}, E)$.

Proof. Under the assumption on E, we have

$$((A^{r,s}_E)\theta, \theta) \geq \ m\,|\theta|^2. \tag{3.4}$$

Since Ω carries a complete Kähler metric $\mathcal{G}$ (cf. Lemma 4), by using Nakano inequality [29], for $\theta \in \mathfrak{D}_{r,s}(\Omega, E)$,

$$\|\overline{\partial}\theta\|^2_{\mathcal{G},\Omega} + \|\overline{\partial}^*\theta\|^2_{\mathcal{G},\Omega} \geq < A^{r,s}_{E,\mathcal{G}}\theta, \theta >_{\mathcal{G},\Omega} .$$

Since $\mathcal{G}$ is complete, $\mathfrak{D}_{r,s}(\Omega, E)$ is dense in $\mathrm{dom}(\overline{\partial}, E) \cap \mathrm{dom}(\overline{\partial}^*, E)$. Since $\mathcal{G} \geq g$, then, by using (3.1), we obtain

$$\|\overline{\partial}\theta\|^2_\Omega + \|\overline{\partial}^*\theta\|^2_\Omega \geq \|\overline{\partial}\theta\|^2_{\mathcal{G},\Omega} + \|\overline{\partial}^*\theta\|^2_{\mathcal{G},\Omega} \geq < A^{r,s}_{E,\mathcal{G}}\theta, \theta >_{\mathcal{G},\Omega} \geq < A^{r,s}_E\theta, \theta >_\Omega,$$

for $\theta \in \mathrm{dom}(\overline{\partial}, E) \cap \mathrm{dom}(\overline{\partial}^*, E)$. Thus from (3.4), one obtains

$$\|\theta\|^2_\Omega \leq c\left(\|\overline{\partial}\theta\|^2_\Omega + \|\overline{\partial}^*\theta\|^2_\Omega\right) = c\ < \Box_{r,s}\theta, \theta >_\Omega \leq c\|\Box_{r,s}\theta\|_\Omega\|\theta\|_\Omega,$$

for $\theta \in \mathrm{dom}(\Box_{r,s}, E)$, with $c = 1/m$. Hence for $\theta \in \mathrm{dom}(\Box_{r,s}, E)$, one obtains

$$\|\theta\|_\Omega \leq c\,\|\Box_{r,s}\theta\|_\Omega. \tag{3.5}$$

Since $\Box_{r,s}$ is a linear closed densely defined operator, from Theorem 1.1.1 in [23], $\mathrm{Ran}(\Box_{r,s}, E)$ is closed. Then from (3.3), one obtains

$$L^2_{r,s}(\Omega, E) = \mathrm{Ran}(\Box_{r,s}, E) = \overline{\partial}\,\overline{\partial}^*\mathrm{dom}(\Box_{r,s}, E) \oplus \overline{\partial}^*\overline{\partial}\,\mathrm{dom}(\Box_{r,s}, E).$$

It follows that the range of $\Box_{r,s}$ is the whole space $L^2_{r,s}(\Omega, E)$. Since from (3.5), $\Box_{r,s} : \mathrm{dom}(\Box_{r,s}, E) \longrightarrow L^2_{r,s}(\Omega, E)$ is one-to-one on $\mathrm{dom}(\Box_{r,s}, E)$, there exists a unique bounded inverse operator $N_{r,s} : L^2_{r,s}(\Omega, E) \longrightarrow \mathrm{dom}(\Box_{r,s}, E)$ such that $N_{r,s}\Box_{r,s} = I$ on $\mathrm{dom}(\Box_{r,s}, E)$. Also, from the definition of N, we obtain $\square_{r,s}N_{r,s} = I$ on $L^2_{r,s}(\Omega, E)$. Therefore (i) and (ii) follows. The proof is complete as in [10]. $\square$

Theorem 5. *Let (X, g) be a Kähler manifold of complex dimension n and $\Omega \Subset X$ be a Hartogs pseudo-convex domain. Assume that E is a holomorphic vector bundle over X with the following properties: $m_r(\Omega; E) > 0$. Therefore there is a bounded linear operator $N_{r,0} : L^2_{r,0}(\Omega, E) \longrightarrow L^2_{r,0}(\Omega, E)$ satisfying*

(i) $\mathrm{Ran}(N_{r,0}, E) \subset \mathrm{dom}(\square_{r,0}, E)$, *$\square_{r,0}N_{r,0} = N_{r,0}\square_{r,0} = I - \mathcal{B}_{r,0}$ on $\mathrm{dom}(\square_{r,0}, E)$,*

(ii) *for $\theta \in L^2_{r,0}(\Omega, E)$, the decomposition $\theta = \overline{\partial}^*\overline{\partial}N_{r,0}\theta \oplus \mathcal{B}_{r,0}\theta$.*

Proof. If $\psi \in \mathrm{dom}(\square_{r,0}, E) \cap (\ker(\square_{r,0}, E))^\perp$, thus $\psi \perp \ker(\overline{\partial}, E)$ and $\psi \in \mathrm{Ran}(\overline{\partial}, E)$. Let $\theta = \overline{\partial}\psi$, then $\theta \in L^2_{r,1}(\Omega, E)$, because $\psi \in \mathrm{dom}(\square_{r,0}, E)$. From Theorem 4, $\psi = \overline{\partial}^*N_{r,1}\theta$ is the solution of $\overline{\partial}\psi = \theta$ and $\psi \perp \ker(\overline{\partial}, E)$. Using Theorem 4,

$$\|\psi\|^2_\Omega \le c\,\|\theta\|^2_\Omega = c\,\|\overline{\partial}\psi\|^2_\Omega = c < \square_{r,0}\psi, \psi >_\Omega \le c\,\|\square_{r,0}\psi\|_\Omega\,\|\psi\|_\Omega,$$

i.e.,

$$\|\psi\|_\Omega \le c\,\|\square_{r,0}\psi\|_\Omega. \tag{3.6}$$

Since $\square_{r,0}$ is a linear closed densely defined operator, from Theorem 1.1.1 in [23], $\mathrm{Ran}(\square_{r,0}, E)$ is closed. Thus

$$L^2_{r,0}(\Omega, E) = \mathrm{Ran}(\square_{r,0}, E) \oplus \ker(\square_{r,0}, E).$$

Following (3.6), $\square_{r,0}$ is one-to-one. Thus there exists a unique bounded inverse operator

$$N_{r,0} : \mathrm{Ran}(\square_{r,0}, E) \longrightarrow \mathrm{dom}(\square_{r,0}, E) \cap (\ker(\square_{r,0}, E))^\perp$$

such that $N_{r,0}\square_{r,0}\psi = \psi$ on $\mathrm{dom}(\square_{r,0}, E)$. Write $N_{r,0}\square_{r,0} = I - \mathcal{B}_{r,0}$ on $\mathrm{dom}(\square_{r,0}, E) \cap (\ker(\square_{r,0}, E))^\perp$. We extending $N_{r,0}$ to $L^2_{r,0}(\Omega, E)$ by setting $N_{r,0}\mathcal{B}_{r,0} = 0$. Following the definition of N, we have $\square_{r,0}N_{r,0} = I - \mathcal{H}_{r,0}$ on $L^2_{r,0}(\Omega, E)$. Thus $N_{r,0}$ satisfies (i) and (ii). $\square$

4. Global Regularity of $\overline{\partial}$ and its Applications

By Theorem 4 (ii) and the density of $C^\infty_{r,s}(\overline{\Omega}, E)$ in $W^k_{r,s}(\Omega, E)$, the following theorem is immediate.

Theorem 6. *Let (X, g) be a Kähler manifold of complex dimension n and $\Omega \Subset X$ be a Hartogs pseudo-convex domain. Assume that E is a holomorphic vector bundle over X with the following properties: $m_r(\Omega; E) > 0$. If $\theta \in C^\infty_{r,s}(\overline{\Omega}, E)$ with $1 \le s \le n-2$, $n \ge 3$ and $N\theta \in C^\infty_{r,s}(\overline{\Omega}, E)$, then for any nonnegative integer k, there exist the constants C_k such that*

$$\|N\theta\|_{W^k(E)} \le C_k\|\theta\|_{W^k(E)}. \tag{4.1}$$

Proof. The proof is identical to that given in [25]. $\square$

We can pass from a priori estimates (4.1) to actual estimates using the elliptic regularisation method presented in [25] and prove the following

Theorem 7. *Under the same assumption of Theorem 6, for every integer $k \ge 0$, the weighted $\overline{\partial}$-Neumann operator N is bounded from $W^k_{r,s}(\Omega, E)$ into itself for $1 \le s \le n-1$.*

By Theorem 7 and the density of $C^\infty_{r,s}(\overline{\Omega}, E)$ in $W^k_{r,s}(\Omega, E)$, the following is immediate.

Corollary 2. *If $\theta \in W^k_{r,s}(\Omega, E)$, $k = 0, 1, 2, 3, \dots$ satisfies $\overline{\partial}\theta = 0$, where $1 \le s \le n-2$, $n \ge 3$, then there exits $u \in W^k_{r,s-1}(\Omega, E)$ so that $\overline{\partial}u = \theta$ on Ω with the estimate*

$$\|u\|_{W^k(E)} \le C_k\|\theta\|_{W^k(E)}.$$

Theorem 8. *Let (X, g) be a Kähler manifold of complex dimension n and $\Omega \Subset X$ be a Hartogs pseudo-convex domain. Assume that E is a holomorphic vector bundle over X with the following properties: $m_r(\Omega; E) > 0$. Therefore for $\theta \in C^\infty_{r,s}(\overline{\Omega}, E)$, $s \ge 1$, with $\overline{\partial}\theta = 0$, we can find $u \in C^\infty_{r,s-1}(\overline{\Omega}, E)$ satisfying $\overline{\partial}u = \theta$.*

Proof. From Corollary 2, there is $u_k \in W^k_{r,s-1}(\Omega, E)$ satisfying $\overline{\partial}\, u_k = \theta$ for each positive integer k. We shall modify u_k to generate a new sequence that converges to a smooth solution. Since $u_k - u_{k+1}$ is in $W^k_{r,s-1}(\Omega, E) \cap \ker \overline{\partial}$, there exists $v_{k+1} \in W^{k+1}_{r,s-1}(\Omega, E) \cap \ker \overline{\partial}$ such that

$$\|u_k - u_{k+1} - v_{k+1}\|_{W^k} \leq 2^{-k}, \quad k = 1, 2, 3, \dots .$$

Setting $\widetilde{u}_{k+1} = u_{k+1} + v_{k+1}$, then $\widetilde{u}_{k+1} \in W^{k+1}_{r,s-1}(\Omega, E)$ and $\overline{\partial}\widetilde{u}_k = \theta$. Inductively, we can choose a new sequence $\widetilde{u}_k \in W^k_{r,s-1}(\Omega, E)$ such that $\overline{\partial}\widetilde{u}_k = \theta$ and

$$\|\widetilde{u}_{k+1} - \widetilde{u}_k\|_{W^k} \leq 2^{-k}, \quad k = 1, 2, 3, \dots .$$

Set $u_\infty = \widetilde{u}_t + \sum_{k=t}^{\infty} (\widetilde{u}_{k+1} - \widetilde{u}_k)$, $t \in N$. Then u_∞ is well defined and is in $W^k_{r,s-1}(\Omega, E)$. Thus, by the Sobolev embedding theorem, $u_\infty \in C^\infty_{r,s}(\overline{\Omega}, E)$ and $\overline{\partial} u_\infty = \theta$. □

The global solutions of the $\overline{\partial}$-problem, as well as their regularity properties, are presented.

Lemma 5. *Let (X, g) be a Kähler manifold of complex dimension n and $\Omega \Subset X$ be a Hartogs pseudo-convex domain. Assume that E is a holomorphic vector bundle over X with the following properties: $m_r(\Omega; E) > 0$. Then for $\theta \in C^\infty_{r,s}(b\Omega, E)$, $s \geq 1$, satisfying $\overline{\partial}_b\theta = 0$, there exists $\Gamma \in C^\infty_{r,s}(\overline{\Omega}, E)$ such that $\Gamma|_{b\Omega} = \theta$ and $\overline{\partial}\Gamma = 0$.*

Proof. Let $\theta \in C^\infty_{r,s}(b\Omega, E)$, $s \geq 1$, with $\overline{\partial}_b f = 0$, then there exists $\theta' \in C^\infty_{r,s}(\overline{\Omega}, E)$ such that $\theta'|_{b\Omega} = \theta$ and $\overline{\partial}\theta'$ vanishes to infinite order on $b\Omega$. Following Theorem 3.4 in [1], there exists $u \in C^\infty_{r,s}(\overline{\Omega}, E)$ with $\operatorname{supp} u \subset \overline{\Omega}$ such that $\overline{\partial} u = \overline{\partial}\theta'$. Then the form $\Gamma = \theta' - u$ such that $\Gamma \in C^\infty_{r,s}(\overline{\Omega}, E)$, $\Gamma|_{b\Omega} = \theta$ and $\overline{\partial}\Gamma = 0$. □

Theorem 9. *Under the same assumption of Lemma 5, if $\theta \in C^\infty_{r,s}(b\Omega, E)$, $s \geq 1$, with $\overline{\partial}_b\theta = 0$, there exists $u \in C^\infty_{r,s-1}(b\Omega, E)$, such that $\overline{\partial}_b u = \theta$.*

Proof. Let $\theta \in C^\infty_{r,s}(b\Omega, E)$, $s \geq 1$, with $\overline{\partial}_b\theta = 0$. Then from Lemma 5, there exists $\Gamma \in C^\infty_{r,s}(\overline{\Omega}, E)$ such that $\Gamma|_{b\Omega} = f$ and $\overline{\partial}\Gamma = 0$. There exists, according to Lemma 5, $U \in C^\infty_{r,s-1}(\overline{\Omega}, E)$ satisfying $\overline{\partial} U = \theta$ in Ω. Then $u = U|_{b\Omega}$ satisfies $\overline{\partial}_b u = \theta$. □

Theorem 10 ([12, (2.3) Corollary, page 417])**.** *Let X be an n-dimensional compact Kähler manifold and E be a holomorphic vector bundle over X. If the Hermitian operator $A^{r,s}_E$ is positive definite on $\wedge^{r,s} T^\star X \otimes E$, with $s \geq 1$, then the E-valued $\overline{\partial}$-cohomology groups*

$$H^{r,s}(X, E) = \frac{C^\infty_{r,s}(X, E) \cap \ker(\overline{\partial}, E)}{\overline{\partial}(C^\infty_{r,s-1}(X, E))} \equiv 0.$$

We show the following results using Theorem 10 (as in [21]).

Theorem 11. *Let (X, g) be a Kähler manifold of complex dimension n and $\Omega \Subset X$ be a Hartogs pseudo-convex domain. Assume that E is a holomorphic vector bundle over X with the following properties: $m_r(\Omega; E) > 0$ and let $D = X \backslash \overline{\Omega}$. Then for $\theta \in C^\infty_{r,s}(\overline{D}, E)$, $\overline{\partial}\theta = 0$, $s \geq 1$, there exists $u \in C^\infty_{r,s-1}(\overline{D}, E)$ such that $\overline{\partial} u = \theta$.*

Proof. Let $\theta \in C^\infty_{r,s}(\overline{D}, E)$, with $\overline{\partial}\theta = 0$, then there exists $\theta' \in C^\infty_{r,s}(X, E)$ such that $\theta'|_{\overline{D}} = \theta$ and $\overline{\partial}\theta'$ vanishes to infinite order on $b\Omega$ (see, for example, [30]; Lemma 2.4 for the existence of such an extension). Using [1]; Theorem 3.4, there exists $u \in C^\infty_{r,s}(X, E)$ with $\operatorname{supp} u \subset \overline{\Omega}$ and such that $\overline{\partial} u = \overline{\partial}\theta'$. Define a (r, s)-form Γ on X by

$$\Gamma = \begin{cases} \theta' & \text{on } \overline{D}, \\ \theta' - u & \text{on } \Omega. \end{cases}$$

Then Γ is a $\overline{\partial}$-closed C^∞ extension of θ to X. Following Theorem 10,

$$H^{r,s}(X, E) = 0.$$

When the Hodge theory is applied to compact complex manifolds, it follows that $\overline{\partial}u = \Gamma$ for some $u \in C^{\infty}_{r,s-1}(X, E)$. Thus $u|_{\overline{D}}$ has the desired properties. □

Corollary 3. *If $H^{r,s}(X, E) = 0$, then for $\theta \in C^{\infty}_{r,s}(\overline{D}, E)$, $\overline{\partial}\theta = 0$, $s \geq 1$, there exists $u \in C^{\infty}_{r,s-1}(\overline{D}, E)$ such that $\overline{\partial}u = \theta$.*

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