Transactions of A. Razmadze
Mathematical Institute
Vol. **175** (2021), issue 3, 411–415

THE CONSISTENT ESTIMATES OF CHARLIER'S STATISTICAL STRUCTURES

OMAR PURTUKHIA[1,2] AND ZURAB ZERAKIDZE[3]

Abstract. In this paper, we study the Charlier statistical structures $\{E, S, \mu_i, i \in I\}$. The sufficient and necessary conditions are given for the existence of consistent estimates of parameters for various definitions of strong separability of statistical structures and for various σ-algebras S.

1. Introduction

Let (E, S) be a measurable space with a given family of probability measures $\{\mu_i, i \in I\}$.
We recall some definitions from [1–4].

Definition 1.1. An object $\{E, S, \mu_i, i \in I\}$ is called a statistical structure.

Definition 1.2. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal (singular) if the family of probability measures $\{\mu_i, i \in I\}$ consists of pairwise singular measures (i.e. $\mu_i \perp \mu_j$, $\forall i \neq j$).

The example below shows that the relation of orthogonality of pairs of measures is only symmetric and is neither transitive nor reflexive.

Example 1.1. Let $E = [0, 1]$ and let S be a Borel σ-algebra of subsets of $[0, 1]$. Let $\mu_1(B) = 2l(B \cap [0, \frac{1}{2}])$; $\mu_2(B) = 2l(B \cap (\frac{1}{2}, 1])$ and $\mu_3(B) = 3l(B \cap [0, \frac{1}{3}])$ $(B \in S)$, where l is Lebesgue measures on S. Then $\mu_1 \perp \mu_2$ and $\mu_2 \perp \mu_3$, but μ_1 is not orthogonal to μ_3.

Definition 1.3. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly separable if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that

$$\mu_i(X_j) = \begin{cases} 1, & \text{if } \ i = j; \\ 0, & \text{if } \ i \neq j \end{cases} \quad (i, j \in I).$$

Definition 1.4. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called separable if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that

$$1) \ \ \mu_i(X_j) = \begin{cases} 1, & \text{if } \ i = j; \\ 0, & \text{if } \ i \neq j \end{cases} \quad (i, j \in I);$$

$$2) \ \ \forall i, j \in I: \ \ \operatorname{card}(X_i \cap X_j) < c, \ \ if \ i \neq j,$$

where c denotes the power of continuum (in this case, the cardinality of the intersection of the sets is either countable or finite).

Definition 1.5. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called strongly separable if there exists a disjoint family of S-measurable sets $\{X_i, i \in I\}$ such that the relations

$$\mu_i(X_i) = 1, \ \ \forall i \in I$$

are fulfilled.

Remark 1.1. From the strong separability there follows separability, from the separability there follows weak separability and from the weak separability there follows orthogonality, but not vice versa (see [4]).

2020 *Mathematics Subject Classification.* 62H05, 62H12.

Key words and phrases. Statistical structure; Orthogonal structure; Separable structure; Strongly separable structure, Charlier's statistical structure; Consistent estimate.

Let I be the set of hypotheses and let $B(I)$ be σ-algebra of subsets of I which contains all finite subsets of I.

Definition 1.6. We say that the statistical structure $\{E, S, \mu_i, i \in I\}$ admits consistent estimators of parameters $i \in I$ if there exists at least one measurable mapping $\delta : (E, S) \longrightarrow (I, B(I))$ such that

$$\mu_i(\{x : \delta(x) = i\}) = 1, \ \ \forall\, i \in I.$$

Remark 1.2. It is obvious that if the statistical structure $\{E, S, \mu_i, i \in I\}$ admits consistent estimators of parameters $i \in I$, then the statistical structure $\{E, S, \mu_i, i \in I\}$ is strongly separable, but not vice versa.

In the example below, we give the construction of a strongly separable (see Definition 1.5) statistical structure with the cardinality of the continuum that does not admit consistent estimators of parameters.

Example 1.2. As a set of parameters consider the set $I = R = (-\infty, +\infty)$ and let $B(I) = L(I)$ be a Lebesgue σ-algebra on R. Let $\delta : R \to R$ denote some bijective mapping at the axis R which is Lebesgue non-measurable. We divide the segment $[-\frac{1}{2}, \frac{1}{2}]$ into classes as follows: points x and y are included in a certain class if and only if the difference $x - y$ is a rational number. It is evident that the different classes are disjoint. Take one point from each class and denote the set of these points by A. It is obvious that the set A is not $L(R)$-measurable and its cardinality is continuum, $\operatorname{card} A = c$. Therefore there is a one-to-one mapping $f_1 : A \to [0, 1]$ such that $f_1(A) = [0, 1]$. As $A \subset [-\frac{1}{2}, \frac{1}{2}] \subset [-1, 1]$, it is obvious that $\operatorname{card}([-1, 1] \setminus A = c)$ and there exists the bijective reflection $f_2 : [-1, 1] \setminus A \to [-1, 0]$ such that $f_2([-1, 1] \setminus A) = [-1, 0]$. Let

$$\delta(x) = \begin{cases} x, & \text{if } \ x \in R \setminus [-1, 1]; \\ f_1, & \text{if } \ x \in A; \\ f_2, & \text{if } \ x \in [-1, 1] \setminus A. \end{cases}$$

$\delta(x)$ is a Lebesgue non-measurable because $\delta^{-1}([0, 1]) = f_1^{-1}([0, 1]) = A$. Hence the inverse mapping δ^{-1} will also be Lebesgue non-measurable. Let

$$\mu_i(X) = \begin{cases} 1, & \text{if } \ \delta(i) \in X; \\ 0, & \text{if } \ \delta(i) \notin X \end{cases}$$

for $i \in R$ and $X \in L(R)$. It is easy to see that the statistical structure $\{R, L(R), \mu_i, i \in R\}$ is a strongly separable (see Definition 1.5) statistical structure that does not admit consistent estimators of parameters. This means that there exists the measurable mapping $\overline{\delta} : (R, L(R)) \to (R, L(R))$ such that

$$\mu_i(\{x : \overline{\delta}(x) = i\}) = 1, \ \ \forall i \in R.$$

Therefore $\delta(i) \in \{x : \overline{\delta}(x) = i\}$. Hence, we have $\overline{\delta}(\delta(i)) = i$, $\forall i \in R$. On the other hand, $\delta^{-1}(\delta(i)) = i$, $\forall i \in R$. Consequently, $\delta^{-1} \circ \delta = \overline{\delta} \circ \delta$ and $\delta^{-1} = \overline{\delta} \circ \delta \circ \delta^{-1} = \overline{\delta}$. Thus we find that δ^{-1} is measurable which contradicts the fact that the inverse function of the non-measurable function δ is not measurable.

2. The Charlier Statistical Structure

The normal distribution is symmetrical, that is, the normal distribution density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

is symmetric with respect to the line $x = m$. However, in practice, asymmetric distributions are also often encountered. In the case when the asymmetry in absolute value is not very large, the density can be expressed by using the so-called Charlier's law.

The density of Charlier's law is determined by the equality

$$f_{Ch}(x) = f(x) + \frac{1}{\sigma}\Big[\frac{S_3(X)}{6} \cdot z_u \cdot (u^3 - 3u) + \frac{E_4(X)}{24} \cdot z_u \cdot (u^4 - 6u^2 + 3)\Big], \tag{2.1}$$

where $f(x)$ is the density of the normal distribution, $u = \frac{x-m}{\sigma}$, $z_u = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$, $S_3(X) = \mu_3/\sigma^3$ is asymmetry, and $E_4(X) = \mu_4/\sigma^4 - 3$ is kurtosis.

Thus the second term on the right-hand side of (2.1) is a correction to the normal distribution. Obviously, for $S_3(X) = 0$ and $E_4(X) = 0$, the Charlier distribution coincides with the normal distribution.

Let μ be the probability measure given on $(R, L(R))$ by the formula

$$\mu(A) = \int_A f_{Ch}(x)dx, \quad A \in L(R).$$

The probability measure determined in this way will be called the Charlier measure.

Definition 2.1. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called a statistical Charlier structure if μ_i, $\forall i \in I$ are Charlier measures.

Let $I = \{i_1, i_2, \dots, i_n, \dots\}$ be a countable set of parameters.

Theorem 2.1. *Charlier's statistical structure $\{E, S, \mu_i, i \in I\}$ admits consistent estimators of parameters $i \in I$, if and only if it is either strongly separable, or separable, or weakly separable, or orthogonal.*

Proof. Necessity. Since Charlier's statistical structure $\{E, S, \mu_i, i \in I\}$ admits consistent estimators of parameters $i \in I$, there exists a measurable mapping $\delta : (E, S) \longrightarrow (I, B(I))$ such that

$$\mu_i(\{x : \delta(x) = i\}) = 1, \ \ \forall\, i \in I \ \ (\operatorname{card} I = \chi_0).$$

Let $X_i = \{x : \delta(x) = i\}$, then it is evident that $X_{i'} \cap X_{i''} = \emptyset \ \forall i^{'} \neq i^{''}$ $(i^{'},\ i^{''} \in I)$ and $\mu_i(X_i) = 1$ $\forall i \in I$. Therefore the statistical structure $\{E, S, \mu_i, i \in I\}$ is strongly separable.

Sufficiency. The orthogonality of the Charlier statistical structure $\{E, S, \mu_i, i \in I\}$ implies that $\mu_{i'} \perp \mu_{i''}$ for any $i^{'} \neq i^{''}$, $i^{'}$, $i^{''} \in I$. The singularity of probability measures implies the existence of a family of S-measurable sets $X_{i^{'},\, i^{''}}$ such that for any $i^{'} \neq i^{''}$, $i^{'}$, $i^{''} \in I$ we have $\mu_{i''}(X_{i^{'},\, i^{''}}) = 0$ and $\mu_{i'}(E \setminus X_{i^{'},\, i^{''}}) = 0$. If we now consider the sets $X_{i'} = \bigcup_{i' \neq i''} (E \setminus X_{i^{'},\, i^{''}})$, we can see that $\mu_{i'}(X_{i'}) = 0$ and $\mu_{i''}(E \setminus X_{i'}) = 0$, $\forall i^{'} \neq i^{''}$. This implies that the Charlier statistical structure $\{E, S, \mu_i, i \in I\}$ is weakly separable and there exists a family of S-measurable sets $\{\widetilde{X}_i, i \in I\}$ such that

$$\mu_{i''}(\widetilde{X}_{i'}) = \begin{cases} 1, & \text{if } i^{'} = i^{''}; \\ 0, & \text{if } i^{'} \neq i^{''}. \end{cases}$$

Let us consider the sets $\overline{X}_{i'} = \widetilde{X}_{i'} \setminus (\widetilde{X}_{i'} \cap (\bigcup_{i' \neq i''} \widetilde{X}_{i''}))$, $i^{'} \in I$. It is clear that $\overline{X}_{i'} \cap \overline{X}_{i''} = \emptyset \ \forall i^{'} \neq i^{''}$ and $\mu_{i'}(\overline{X}_{i'}) = 1$, $\forall i^{'} \in I$. We define the mapping $\delta : (E, S) \to (I, B(I))$ as follows: $\delta(\overline{X}_i) = i$, $i \in I$. Then we have $\mu_i(\{x : \delta(x)\}) = 1$, $\forall i \in I$, i.e., Charlier's statistical structure $\{E, S, \mu_i, i \in I\}$ admits consistent estimators of parameters. □

As noted above (Remark 1.2), not every strongly separable (Definition 1.6) statistical structure admits consistent estimators of parameters.

Let $\{\mu_i, i \in I\}$ be Charlier probability measures defined on the measurable space (E, S). For each $i \in I$, we denote by $\overline{\mu}_i$ the completion of the measure μ_i, and by $\operatorname{dom}(\overline{\mu}_i)$ – the σ-algebra of all $\overline{\mu}_i$-measurable subsets of E.

We denote

$$S_1 = \bigcap_{i \in I} \operatorname{dom}(\overline{\mu}_i).$$

Definition 2.2. The Charlier statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ is called strongly separable if there exists a family of S_1-measurable sets $\{Z_i, i \in I\}$ such that the relations

1) $\mu_i(Z_i) = 1 \ \forall i \in I$;
2) $Z_{i_1} \cap Z_{i_2} = \emptyset$, $\forall i_1 \neq i_2$, $i_1, i_2 \in I$;
3) $\bigcup_{i \in I} Z_i = E$ are fulfilled.

Definition 2.3. We will say that the Charlier statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ admits consistent estimators of parameters $i \in I$ if there exists at least one measurable mapping $\delta : (E, S_1) \longrightarrow (I, B(I))$ such that

$$\overline{\mu}_i(\{x : \delta(x) = i\}) = 1, \ \ \forall\, i \in I.$$

Theorem 2.2. *In order that the Charlier orthogonal statistical structure* $\{E, S_1, \overline{\mu}_i, i \in I\}$, $\operatorname{card} I = c$, *admitted a consistent estimators of parameters* $i \in I$ (*Definition* 2.3), *it is necessary and sufficient that this statistical structure be strongly separable* (*Definition* 2.2).

Proof. Necessity. The existence of consistent estimators of parameters $i \in I$ means that there exists at least one measurable mapping $\delta : (E, S_1) \longrightarrow (I, B(I))$ such that

$$\overline{\mu}_i(\{x : \delta(x) = i\}) = 1, \ \ \forall\, i \in I.$$

Denoting $Z_i = \{x : \delta(x) = i\}$ for $i \in I$, we get:
1) $\overline{\mu}_i(Z_i) = \overline{\mu}_i(\{x : \delta(x) = i\}) = 1, \ \ \forall i \in I$;
2) $Z_{i_1} \cap Z_{i_2} = \{x : \delta(x) = i_1\} \cap \{x : \delta(x) = i_2\} = \emptyset, \ \ \forall i_1 \neq i_2, \ i_1, i_2 \in I$;
3) $\bigcup\limits_{i \in I} Z_i = E$.

Hence the statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ is strongly separable (Definition 2.2).

Sufficiency. Since the Charlier statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$, $\operatorname{card} I = c$, is strongly separable (Definition 2.2), there exists a family $\{Z_i, \ i \in I\}$ of elements of the σ-algebra $S_1 = \bigcap\limits_{i \in I} \operatorname{dom}(\overline{\mu}_i)$ such that:
1) $\overline{\mu}_i(Z_i) = 1, \ \ \forall i \in I$;
2) $Z_{i_1} \cap Z_{i_2} = \emptyset, \ \ \forall i_1 \neq i_2, \ i_1, i_2 \in I$;
3) $\bigcup\limits_{i \in I} Z_i = E$.

For $x \in E$, we put $\delta(x) = i$, where i is a unique hypothesis from the set I for which $x \in Z_i$. The existence and uniqueness of such hypothesis i can be proved by using conditions 2) and 3).

Take now $Y \in B(I)$. Then $\{x : \delta(x) \in Y\} = \bigcup\limits_{i \in Y} Z_i$. We have to show that $\{x : \delta(x) \in Y\} \in \operatorname{dom}(\overline{\mu}_{i_0})$ for each $i_0 \in I$.

If $i_0 \in Y$, then

$$\{x : \delta(x) \in Y\} = \bigcup_{i \in I} Z_i = Z_{i_0} \cup (\bigcup_{i \in Y \setminus \{i_0\}} Z_i).$$

On the one hand, from conditions 1), 2) and 3) it follows that

$$Z_{i_0} \in S_1 = \bigcap_{i \in I} \operatorname{dom}(\overline{\mu}_i) \subseteq \operatorname{dom}(\overline{\mu}_{i_0}).$$

On the other hand, the inclusion

$$\bigcup_{i \in Y \setminus \{i_0\}} Z_h \subseteq (E \setminus Z_{i_0})$$

implies that $\overline{\mu}_{i_0}\big(\bigcup\limits_{i \in Y \setminus \{i_0\}} Z_i\big) = 0$, and hence

$$\bigcup_{i \in Y \setminus \{i_0\}} Z_i \in \operatorname{dom}(\overline{\mu}_{i_0}).$$

Since $\operatorname{dom}(\overline{\mu}_{i_0})$ is a σ-algebra, we conclude that

$$\{x : \delta(x) \in Y\} = Z_{i_0} \cup \big(\bigcup_{i \in Y \setminus \{i_0\}} Z_i\big) \in \operatorname{dom}(\overline{\mu}_{i_0}).$$

If $i_0 \notin Y$, then $\{x : \delta(x) \in Y\} = \bigcup\limits_{i \in I} Z_i \subseteq (E \setminus Z_{i_0})$ and we conclude that $\overline{\mu}_{i_0}(\{x : \delta(x) \in Y\}) = 0$. The last relation implies that

$$\{x : \delta(x) \in Y\} \in \operatorname{dom}(\overline{\mu}_{i_0}), \ \ \forall Y \in B(I).$$

Thus, we have proved the validity of the relation $\{x : \delta(x) \in Y\} \in \operatorname{dom}(\overline{\mu}_{i_0})$ for any $i_0 \in I$. Hence

$$\{x : \delta(x) \in Y\} \in \bigcap_{i \in I} \operatorname{dom}(\overline{\mu}_i) = S_1.$$

Therefore the mapping $\delta : (E, S_1) \longrightarrow (I, B(I))$ is measurable.

Since $B(I)$ contains all finite subsets of I, we ascertain that

$$\overline{\mu}_i(\{x : \delta(x) = i\}) = \overline{\mu}_i(Z_i) = 1, \quad \forall\, i \in I. \qquad \square$$

Remark 2.1. If we consider Definition 1.5, then Theorem 2.1 is valid for strongly separable Charlier statistical structures of countable cardinality, but if we consider Definition 2.2, then Theorem 2.2 is true for strongly separable statistical structures with the cardinality of the continuum.

References

1. P. E. Danko, A. G. Popov, T. Ya. Kozhevnikova, S. P. Danko, *Higher Mathematics in Exercises and Problems, II.* (Russian) Mir i Obrazovanie, Moscow, 2009.
2. I. Š. Ibramhalilov, A. V. Skorohod, *Consistent Estimators of Parameters of Random Processes.* (Russian) Naukova Dumka, Kiev, 1980.
3. A. Kharazishvili, *Topological Aspects of Measure Theory.* (Russian) Naukova Dumka, Kiev, 1984.
4. Z. Zerakidze, O. Purtukhia, The weakly consistent, strongly consistent and consistent estimates of the parameters. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.* **31** (2017), 151–154.

(Received 25.06.2021)

[1]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0186, Georgia

[2]Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University, 13 University Str., Tbilisi, 0143, Georgia

[3]Gori State University, Gori, Georgia
E-mail address: o.purtukhia@gmail.com
E-mail address: zura.zerakidze@mail.ru