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WEIGHTED INTEGRAL INEQUALITIES OF EULER-GRÜSS TYPE WITH APPLICATIONS

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Abstract. The main objective of this paper is to present some new bounds for the general weighted n-point integral formulae. This will be done by using some Grüss type inequalities and certain Euler type identities. As special cases, we obtain some error estimates for the Chebyshev–Gauss quadrature rules.

1. Introduction

The integral inequality which establishes a connection between the integral of the product of two functions and the product of the integrals is known in the literature as the Grüss inequality (see [9, p. 296]). That is,

$$\left|\frac{1}{b-a}\int\limits_a^b \phi(t)\,\psi(t)\,\mathrm{d}t-\left(\frac{1}{b-a}\int\limits_a^b \phi(t)\,\mathrm{d}t\right)\left(\frac{1}{b-a}\int\limits_a^b \psi(t)\,\mathrm{d}t\right)\right|$$
$$\leq \frac{1}{4}(\beta-\alpha)(\delta-\gamma),$$

where $\phi,\psi:[a,b]\to\mathbb{R}$ are two bounded integrable functions, α,β,γ and δ are real numbers such that $\alpha\leq\phi(t)\leq\beta$ and $\gamma\leq\psi(t)\leq\delta$, for all $t\in[a,b]$.

Matić et al. [8] proved the following variant of the Grüss inequality:

Lemma 1. *Let $\phi,\psi:[a,b]\to\mathbf{R}$ be two integrable functions on $[a,b]$ such that $\phi\cdot\psi$ is also integrable, α,β are real numbers and*

$$\alpha\leq\phi(t)\leq\beta,\quad \forall t\in[a,b].$$

Then

$$|\mathcal{C}(\phi,\psi)|\leq\frac{\beta-\alpha}{2}\left[\mathcal{C}(\psi,\psi)\right]^{1/2},$$

where

$$\mathcal{C}(\phi,\psi)=\frac{1}{b-a}\int\limits_a^b \phi(t)\,\psi(t)\,\mathrm{d}t-\left(\frac{1}{b-a}\int\limits_a^b \phi(t)\,\mathrm{d}t\right)\left(\frac{1}{b-a}\int\limits_a^b \psi(t)\,\mathrm{d}t\right).$$

The functional $\mathcal{C}(\phi,\psi)$ is well known as the Chebyshev functional (see [9]).

During the last two decades many researchers have focused their attention on the above inequality and functional (see [3–8, 10, 12], and the references cited therein).

In paper [11], the authors have obtained two weighted n-point integral formulae by using certain Euler type identities.

Here, as in the rest of the paper, probability density function $p:[a,b]\to[0,\infty)$ is an integrable function satisfying $\int\limits_a^b p(t)\mathrm{d}t=1$.

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Theorem 1. *Let $f:[a,b]\to \mathbf{R}$ be such that $f^{(m-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $m\geq 1$ and let $p:[a,b]\to[0,\infty)$ be some probability density function. Then the following formulae:*

$$\int_a^b p(t)f(t)dt=\sum_{k=1}^n \Lambda_k f(x_k)+\sum_{i=1}^m \frac{(b-a)^{i-1}}{i!}$$
$$\times\Big(\int_a^b p(t)B_i\Big(\frac{t-a}{b-a}\Big)dt-\sum_{k=1}^n \Lambda_k B_i\Big(\frac{x_k-a}{b-a}\Big)\Big)\times$$
$$\times\Big[f^{(i-1)}(b)-f^{(i-1)}(a)\Big]-\frac{(b-a)^{m-1}}{m!}\int_a^b\Big(\int_a^b p(u)B_m^*\Big(\frac{u-t}{b-a}\Big)\mathrm{d}u$$
$$-\sum_{k=1}^n \Lambda_k B_m^*\Big(\frac{x_k-t}{b-a}\Big)\Big)\mathrm{d}f^{(m-1)}(t) \tag{1.1}$$

and

$$\int_a^b p(t)f(t)dt=\sum_{k=1}^n \Lambda_k f(x_k)+\sum_{i=1}^{m-1} \frac{(b-a)^{i-1}}{i!}$$
$$\times\Big(\int_a^b p(t)B_i\Big(\frac{t-a}{b-a}\Big)dt-\sum_{k=1}^n \Lambda_k B_i\Big(\frac{x_k-a}{b-a}\Big)\Big)$$
$$\times\Big[f^{(i-1)}(b)-f^{(i-1)}(a)\Big]$$
$$-\frac{(b-a)^{m-1}}{m!}\int_a^b\Big(\int_a^b p(u)\Big(B_m^*\Big(\frac{u-t}{b-a}\Big)-B_m\Big(\frac{u-a}{b-a}\Big)\Big)\mathrm{d}u$$
$$-\sum_{k=1}^n \Lambda_k\Big(B_m^*\Big(\frac{x_k-t}{b-a}\Big)-B_m\Big(\frac{x_k-a}{b-a}\Big)\Big)\Big)\mathrm{d}f^{(m-1)}(t) \tag{1.2}$$

hold, where $\sum_{k=1}^n \Lambda_k=1$, $B_k(t)$, $k\geq 0$, $t\in\mathbf{R}$ are Bernoulli polynomials, $B_k=B_k(0)$, $k\geq 0$ Bernoulli numbers and $B_k^*(t)$, $k\geq 0$ are periodic functions of period 1 such that $B_k^*(t)=B_k(t)$, for $0\leq t<1$.

If we put $m=s+v$, formulae (1.1) and (1.2) are exact for all polynomials of degree $\leq s-1$.

Some other recently obtained bounds for integral formulae of Euler type can be found in monographs [2] and [5].

In this paper, we derive some new estimations of the reminder in weighted integral formulae (1.1) and (1.2) by using Lemma 1 for mappings whose m-derivatives are bounded. This approach generalizes the results developed in papers [3] and [10]. At the end, as applications we obtain error estimates for the Gauss–Chebyshev formulae of the first and second kind.

2. Main Results

Using integrations by parts and properties of Bernoulli polynomials and periodic function B_m^*, the following equalities follow (see [1, p. 804] and [12]).

Lemma 2. *For $m\geq 1$ and $\xi,\eta\in[a,b]$, we have*

$$\int_a^b B_m^*\Big(\frac{\xi-t}{b-a}\Big)dt=0$$

and

$$\int_a^b B_m^*\Big(\frac{\xi-t}{b-a}\Big)B_m^*\Big(\frac{\eta-t}{b-a}\Big)dt=(-1)^{m-1}\frac{(b-a)(m!)^2}{(2m)!}B_{2m}^*\Big(\frac{\xi-\eta}{b-a}\Big).$$

Proof. Since B_m^*, $m\geq 1$ are periodic functions of period 1 such that $B_m^*(t)=B_m(t)$, for $0\leq t<1$, we deduce

$$\int_a^b B_m^*\Big(\frac{\xi-t}{b-a}\Big)dt=\int_a^\xi B_m\Big(\frac{\xi-t}{b-a}\Big)dt+\int_\xi^b B_m\Big(\frac{\xi-t}{b-a}+1\Big)dt.$$

Now, from the properties of Bernoulli polynomials, by elementary calculation, we get

$$\int_a^b B_m^*\Big(\frac{\xi-t}{b-a}\Big)dt=0.$$

To prove the second equality, we use integrations by parts. So, we obtain

$$\begin{aligned}&\int_a^b B_m^*\Big(\frac{\xi-t}{b-a}\Big)B_m^*\Big(\frac{\eta-t}{b-a}\Big)dt\\&=-\frac{m}{m+1}\Big[\int_a^b B_{m-1}^*\Big(\frac{\xi-t}{b-a}\Big)B_{m+1}^*\Big(\frac{\eta-t}{b-a}\Big)dt\Big].\end{aligned}$$

Repeating this procedure $m-1$ times, we derive

$$\begin{aligned}&\int_a^b B_m^*\Big(\frac{\xi-t}{b-a}\Big)B_m^*\Big(\frac{\eta-t}{b-a}\Big)dt\\&=(-1)^{m-1}\frac{m(m-1)\cdots 2}{(m+1)(m+2)\cdots(2m-1)}\Big[\int_a^b B_1^*\Big(\frac{\xi-t}{b-a}\Big)B_{2m-1}^*\Big(\frac{\eta-t}{b-a}\Big)dt\Big].\end{aligned}$$

Finally, we get

$$\int_a^b B_1^*\Big(\frac{\xi-t}{b-a}\Big)B_{2m-1}^*\Big(\frac{\eta-t}{b-a}\Big)dt=\frac{b-a}{2m}B_{2m}^*\Big(\frac{\xi-\eta}{b-a}\Big),$$

which completes the proof. □

Now, we give our main result in the form of a new Grüss type inequality for the weighted integral formulae of Euler type.

Theorem 2. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(m-1)}$ is an absolutely continuous function for some $m\geq 1$ and $\alpha_m\leq f^{(m)}(t)\leq\beta_m$ for all $t\in[a,b]$ and for some real constants α_m and β_m. Let $p:[a,b]\to[0,\infty)$ be some probability density function. Then we have the following inequality:*

$$\begin{aligned}&\Big|\int_a^b p(t)f(t)dt-\sum_{k=1}^n\Lambda_k f(x_k)-\sum_{i=1}^m\frac{(b-a)^{i-1}}{i!}\\&\times\Big(\int_a^b p(t)B_i\Big(\frac{t-a}{b-a}\Big)dt-\sum_{k=1}^n\Lambda_kB_i\Big(\frac{x_k-a}{b-a}\Big)\Big)\Big[f^{(i-1)}(b)-f^{(i-1)}(a)\Big]\Big|\\&\leq\Gamma_m\left(p,f\right)(b-a)^m(\beta_m-\alpha_m),\end{aligned}\tag{2.1}$$

where

$$\Gamma_m(p,f) = \frac{1}{2(m!)}\Bigg\{\frac{1}{b-a}\int_a^b\Big[\int_a^b p(u)B_m^*\Big(\frac{u-t}{b-a}\Big)du\Big]^2 dt$$
$$+(-1)^{m-1}\frac{(m!)^2}{(2m)!}\Big[B_{2m}\sum_{k=1}^{n}\Lambda_k^2 - 2\sum_{k=1}^{n}\Lambda_k\int_a^b p(u)B_{2m}^*\Big(\frac{u-x_k}{b-a}\Big)du$$
$$+\sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n}\Lambda_k\Lambda_l B_{2m}^*\Big(\frac{x_k-x_l}{b-a}\Big)\Big]\Bigg\}^{1/2}.$$

Proof. Identity (1.1) can be written as follows:

$$\int_a^b p(t)f(t)dt - \sum_{k=1}^{n}\Lambda_k f(x_k) - \sum_{i=1}^{m}\frac{(b-a)^{i-1}}{i!}$$
$$\times\left(\int_a^b p(t)B_i\Big(\frac{t-a}{b-a}\Big)dt - \sum_{k=1}^{n}\Lambda_k B_i\Big(\frac{x_k-a}{b-a}\Big)\right)$$
$$\times\Big[f^{(i-1)}(b)-f^{(i-1)}(a)\Big]$$
$$-\frac{(b-a)^{m-2}\Big(f^{(m-1)}(b)-f^{(m-1)}(a)\Big)}{m!}\int_a^b\Phi_m(t)\mathrm{d}t = \mathcal{R}_m(f,p), \tag{2.2}$$

where

$$\Phi_m(t) = \sum_{k=1}^{n}\Lambda_k B_m^*\Big(\frac{x_k-t}{b-a}\Big) - \int_a^b p(u)B_m^*\Big(\frac{u-t}{b-a}\Big)\mathrm{d}u,$$

and

$$\mathcal{R}_m(f,p)$$
$$=\frac{(b-a)^m}{m!}\left[\frac{1}{b-a}\int_a^b\Phi_m(t)f^{(m)}(t)\mathrm{d}t - \frac{1}{(b-a)^2}\int_a^b\Phi_m(t)\mathrm{d}t\int_a^b f^{(m)}(t)\mathrm{d}t\right].$$

Further, if we apply Lemma 1 to $\phi\to f^{(m)}$ and $\psi\to\Phi_m$, we obtain

$$|\mathcal{R}_m(f,p)| \le \frac{(b-a)^m}{m!}\cdot\frac{\beta_m-\alpha_m}{2}$$
$$\times\Big[\frac{1}{b-a}\int_a^b\Phi_m^2(t)\mathrm{d}t - \frac{1}{(b-a)^2}\left(\int_a^b\Phi_m(t)\mathrm{d}t\right)^2\Big]^{1/2}. \tag{2.3}$$

Now, from the first part of Lemma 2 there follows

$$\int_a^b\Phi_m(t)\mathrm{d}t = \sum_{k=1}^{n}\Lambda_k\int_a^b B_m^*\Big(\frac{x_k-t}{b-a}\Big)\mathrm{d}t - \int_a^b\left(\int_a^b p(u)B_m^*\Big(\frac{u-t}{b-a}\Big)\mathrm{d}u\right)\mathrm{d}t = 0.$$

Similarly, using the second part of Lemma 2, we derive

$$\int_a^b \Phi_m^2(t)\mathrm{d}t = \sum_{k=1}^n \Lambda_k^2 \int_a^b \Big(B_m^*\Big(\frac{x_k-t}{b-a}\Big)\Big)^2 dt$$
$$-2\sum_{k=1}^n \Lambda_k \int_a^b p(u)\Big(\int_a^b B_m^*\Big(\frac{u-t}{b-a}\Big)B_m^*\Big(\frac{x_k-t}{b-a}\Big)dt\Big)du$$
$$+\sum_{k=1}^n \sum_{l=1,l\neq k}^n \Lambda_k\Lambda_l \int_a^b B_m^*\Big(\frac{x_k-t}{b-a}\Big)B_m^*\Big(\frac{x_l-t}{b-a}\Big)dt$$
$$+\int_a^b \Big[\int_a^b p(u)B_m^*\Big(\frac{u-t}{b-a}\Big)du\Big]^2 dt$$
$$=(-1)^{m-1}\frac{(b-a)(m!)^2}{(2m)!}\times\Big[B_{2m}\sum_{k=1}^n \Lambda_k^2 - 2\sum_{k=1}^n \Lambda_k \int_a^b p(u)B_{2m}^*\Big(\frac{u-x_k}{b-a}\Big)du$$
$$+\sum_{k=1}^n \sum_{l=1,l\neq k}^n \Lambda_k\Lambda_l B_{2m}^*\Big(\frac{x_k-x_l}{b-a}\Big)\Big] + \int_a^b \Big[\int_a^b p(u)B_m^*\Big(\frac{u-t}{b-a}\Big)du\Big]^2 dt.$$

Finally, using identity (2.2) and inequality (2.3), we obtain our main result, inequality (2.1). □

Remark 1. *Applying the same method as in the previous proof, from identity* (1.2) *we get inequality* (2.1)*, too.*

Remark 2. *For a uniform weight function,* $p(t)=1/(b-a)$*,* $t\in[a,b]$*,* $n=2$*,* $\Lambda_k=1/2$*,* $x_1=a$*,* $x_2=b$ *and* $m=1$*, we obtain an inequality related to the following trapezoid formula:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{1}{2}\left[f(a)+f(b)\right]\right|\leq\frac{1}{4\sqrt{3}}\left(b-a\right)\left(\beta_1-\alpha_1\right).$$

This inequality was first published in [8]*.*

Remark 3. *Setting* $p(t)=1/(b-a)$*,* $t\in[a,b]$*,* $n=3$*,* $x_1=a$*,* $x_2=(a+b)/2$*,* $x_3=b$*, and* $m=1,2,3$ *in* (2.1)*, we derive the estimates for the error in Simpson's rule*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{1}{6}\Big[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\Big]\right|$$
$$\leq\Gamma_m\left(b-a\right)^m\left(\beta_m-\alpha_m\right),$$

where

$$\Gamma_1=\frac{1}{12},\quad \Gamma_2=\frac{1}{24\sqrt{30}},\quad \Gamma_3=\frac{1}{96\sqrt{105}}.$$

For $m=4$

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]+\frac{\left(f^{(3)}(b)-f^{(3)}(a)\right)(b-a)^3}{2880}\right|$$
$$\leq\frac{1}{5760}\sqrt{\frac{11}{14}}\left(b-a\right)^4\left(\beta_4-\alpha_4\right).$$

These inequalities were established in [10]*.*

Remark 4. *For a uniform weight function and for* $n=1,2,3$ *inequality* (2.1) *reduces to Euler-Grüss inequalities pointed out in paper* [3]*.*

3. Applications for the Chebyshev–Gauss Formulae

In this section, we apply the previous results to obtain some error estimates for Chebyshev–Gauss formulae of the first and the second kind of Euler type.

As a special case of formula (1.1), we have the following Chebyshev–Gauss formula of the first kind of Euler type (see [11]):

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{n} \sum_{k=1}^{n} f(x_k) + \Omega_{s+v}^{CG1}(f,n) + \frac{2^{s+v-1}}{(s+v)!} \int_{-1}^{1} \Phi_{s+v}^{CG1}(t,n) \mathrm{d} f^{(s+v-1)}(t), \tag{3.1}$$

where

$$\Omega_{s+v}^{CG1}(f,n) = \sum_{j=0}^{v} \frac{2^{j+s-1}}{(s+j)!} \times \Big(\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} B_{j+s}\Big(\frac{t+1}{2}\Big) dt$$
$$-\frac{\pi}{n} \sum_{k=1}^{n} B_{j+s}\Big(\frac{x_k+1}{2}\Big)\Big) \Big[f^{(s+j-1)}(1) - f^{(s+j-1)}(-1) \Big]$$

and

$$\Phi_{s+v}^{CG1}(t,n) = \frac{\pi}{n} \sum_{k=1}^{n} B_{s+v}^{*}\Big(\frac{x_k - t}{2}\Big) - \int_{-1}^{1} \frac{1}{\sqrt{1-u^2}} B_{s+v}^{*}\Big(\frac{u-t}{2}\Big) \mathrm{d}u.$$

This formula is exact for all polynomials of degree $\leq s-1$.

Now, applying our main result, we get new estimations of the reminder in formula (3.1).

Theorem 3. *Let $f:[-1,1]\to \mathbf{R}$ be such that $f^{(s+v-1)}$ is an absolutely continuous function for some $s+v\geq 1$ and $\alpha_{s+v} \leq f^{(s+v)}(t) \leq \beta_{s+v}$ for all $t\in[-1,1]$ and for some real constants α_{s+v} and β_{s+v}. Then*

$$\Big| \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt - \frac{\pi}{n} \sum_{k=1}^{n} f(x_k) - \Omega_{s+v}^{CG1}(f,n) \Big|$$
$$\leq \frac{2^{s+v-1}(\beta_{s+v}-\alpha_{s+v})}{\sqrt{2}(s+v)!} \Big\{ \int_{-1}^{1} \Big[\int_{-1}^{1} \frac{1}{\sqrt{1-u^2}} B_{s+v}^{*}\left(\frac{u-t}{2}\right) du \Big]^2 dt$$
$$+(-1)^{s+v-1} \frac{2((s+v)!)^2}{(2(s+v))!} \Big[-\frac{2\pi}{n} \sum_{k=1}^{n} \int_{-1}^{1} \frac{1}{\sqrt{1-u^2}} B_{2(s+v)}^{*}\Big(\frac{u-x_k}{2}\Big) du$$
$$+\frac{\pi^2}{n} B_{2(s+v)} + \frac{\pi^2}{n^2} \sum_{k=1}^{n} \sum_{l=1, l\neq k}^{n} B_{2(s+v)}^{*}\Big(\frac{x_k - x_l}{2}\Big) \Big] \Big\}^{1/2}. \tag{3.2}$$

Proof. This is a special case of Theorem 2 for $[a,b]=[-1,1]$, $m=s+v$, $p(t)=\frac{1}{\pi\sqrt{1-t^2}}$ and $\Lambda_k = \frac{1}{n}$. □

Remark 5. *Taking $n=1$, $x_1=0$, $s=1$ and $v=0$ in inequality (3.2), we obtain*

$$\Big| \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt - \pi f(0) \Big| \leq \sqrt{\pi-2}\,(\beta_1 - \alpha_1).$$

Remark 6. *For* $n=2,\ x_1=-\frac{\sqrt{2}}{2},\ x_2=\frac{\sqrt{2}}{2},\ s_1<4$ *and* $v=0$, *we get the following inequalities:*

$$\Big|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\mathrm{d}t-\frac{\pi}{2}\Big[f\Big(-\frac{\sqrt{2}}{2}\Big)+f\Big(\frac{\sqrt{2}}{2}\Big)\Big]\Big|$$
$$\le K_2^{CG1}(s_1,0)\cdot(\beta_{s_1}-\alpha_{s_1}),$$

where $K_2^{CG1}(1,0)=\sqrt{\frac{\pi\sqrt{2}-4}{2}}\approx 0.470576,\ K_2^{CG1}(2,0)\approx 0.0798678$ *and* $K_2^{CG1}(3,0)\approx 0.0222603.$

Remark 7. *Finally, for* $n=3,\ x_1=-\frac{\sqrt{3}}{2},\ x_2=0,\ x_3=\frac{\sqrt{3}}{2},\ s_1<6$ *and* $v=0$, *we deduce the following inequalities:*

$$\Big|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\mathrm{d}t-\frac{\pi}{3}\Big[f\Big(-\frac{\sqrt{3}}{2}\Big)+f(0)+f\Big(\frac{\sqrt{3}}{2}\Big)\Big]\Big|$$
$$\le K_3^{CG1}(s_1,0)\cdot(\beta_{s_1}-\alpha_{s_1}),$$

where $K_3^{CG1}(1,0)\approx 0.307238,\ K_3^{CG1}(2,0)\approx 0.0344436,\ K_3^{CG1}(3,0)\approx 0.00549894,\ K_3^{CG1}(4,0)\approx 0.00103907$ *and* $K_3^{CG1}(5,0)\approx 0.000246304.$

Further, as a special case of formula (1.1) for $[a,b]=[-1,1],\ m=s+v,\ p(t)=\frac{2\sqrt{1-t^2}}{\pi}$ and $\Lambda_k=\frac{2}{n+1}\sin^2\left(\frac{k\pi}{n+1}\right)$, we have the following Chebyshev–Gauss formula of the second kind of Euler type (see [11]):

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)dt=\frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)f(x_k)+\Omega_{s+v}^{CG2}(f,n)$$
$$+\frac{2^{s+v-1}}{(s+v)!}\int_{-1}^{1}\Phi_{s+v}^{CG2}(t,n)\mathrm{d}f^{(s+v-1)}(t), \tag{3.3}$$

where

$$\Omega_{s+v}^{CG2}(f,n)=\sum_{j=0}^{v}\frac{2^{j+s-1}}{(s+j)!}\times\Bigg(\int_{-1}^{1}\sqrt{1-t^2}B_{j+s}\Big(\frac{t+1}{2}\Big)dt$$
$$-\frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)B_{j+s}\Big(\frac{x_k+1}{2}\Big)\Bigg)\left[f^{(s+j-1)}(1)-f^{(s+j-1)}(-1)\right]$$

and

$$\Phi_{s+v}^{CG2}(t,n)=\frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)B_{s+v}^{*}\Big(\frac{x_k-t}{2}\Big)-\int_{-1}^{1}\sqrt{1-u^2}B_{s+v}^{*}\Big(\frac{u-t}{2}\Big)\mathrm{d}u.$$

This formula is exact for all polynomials of degree $\le s-1$.

Finally, as a special case of Theorem 2, we obtain new estimations of the reminder in formula (3.3).

Theorem 4. *Let* $f:[-1,1]\to\mathbf{R}$ *be such that* $f^{(s+v-1)}$ *is an absolutely continuous function for some* $s+v\ge 1$ *and* $\alpha_{s+v}\le f^{(s+v)}(t)\le\beta_{s+v}$ *for all* $t\in[-1,1]$ *and for some real constants* α_{s+v} *and* β_{s+v}. *Then*

$$\Big|\int_{-1}^{1}\sqrt{1-t^2}f(t)dt-\frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)f(x_k)-\Omega_{s+v}^{CG2}(f,n)\Big|$$

$$
\begin{aligned}
&\le \frac{2^{s+v}\left(\beta_{s+v}-\alpha_{s+v}\right)}{2(s+v)!}\Bigg\{\int_{-1}^{1}\left[\int_{-1}^{1}\sqrt{1-u^2}B^*_{s+v}\Big(\frac{u-t}{2}\Big)du\right]^2 dt\\
&+(-1)^{s+v-1}\frac{2((s+v)!)^2}{(2(s+v))!}\Bigg[-\frac{2\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)\int_{-1}^{1}\sqrt{1-u^2}B^*_{2(s+v)}\Big(\frac{u-x_k}{2}\Big)du\\
&+\frac{\pi^2}{(n+1)^2}B_{2(s+v)}\sum_{k=1}^{n}\sin^4\left(\frac{k\pi}{n+1}\right)\\
&+\frac{\pi^2}{(n+1)^2}\sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)\sin^2\left(\frac{l\pi}{n+1}\right)B^*_{2(s+v)}\Big(\frac{x_k-x_l}{2}\Big)\Bigg]\Bigg\}^{1/2}. \qquad (3.4)
\end{aligned}
$$

Proof. This is a special case of Theorem 2 for $[a,b]=[-1,1]$, $m=s+v$, $p(t)=\frac{2\sqrt{1-t^2}}{\pi}$ and $\Lambda_k=\frac{2}{n+1}\sin^2\left(\frac{k\pi}{n+1}\right)$. □

Remark 8. *For $n=1$, $x_1=0$, $s=1$ and $v=0$, we derive the following inequality:*

$$
\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{2}f(0)\right|\le K_1^{CG2}(1,0)\cdot(\beta_1-\alpha_1),
$$

where $K_1^{CG2}(1,0)\approx 0.409931$.

Remark 9. *Furthermore, for $n=2$, $x_1=-\frac{1}{2}$, $x_2=\frac{1}{2}$, $s_1<4$ and $v=0$, we consider the following inequalities:*

$$
\begin{aligned}
&\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{4}\Big[f\Big(-\frac{1}{2}\Big)+f\Big(\frac{1}{2}\Big)\Big]\right|\\
&\le K_2^{CG2}(s_1,0)\cdot(\beta_{s_1}-\alpha_{s_1}),
\end{aligned}
$$

where $K_2^{CG2}(1,0)\approx 0.227524$, $K_2^{CG2}(2,0)\approx 0.0342287$ and $K_2^{CG2}(3,0)\approx 0.0077412$.

Remark 10. *Finally, for $n=3$, $x_1=-\frac{\sqrt{2}}{2}, x_2=0$ and $x_3=\frac{\sqrt{2}}{2}$, $s_1<6$ and $v=0$, we get the following inequalities:*

$$
\begin{aligned}
&\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{8}\Big[f\Big(-\frac{\sqrt{2}}{2}\Big)+2f(0)+f\Big(\frac{\sqrt{2}}{2}\Big)\Big]\right|\\
&\qquad\le K_3^{CG2}(s_1,0)\cdot(\beta_{s_1}-\alpha_{s_1}),
\end{aligned}
$$

where $K_3^{CG2}(1,0)\approx 0.169113$, $K_3^{CG2}(2,0)\approx 0.0174633$, $K_3^{CG2}(3,0)\approx 0.00238238$, $K_3^{CG2}(4,0)\approx 0.000382337$ and $K_3^{CG2}(5,0)\approx 0.0000773027$.

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