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ON UNIFORM DISTRIBUTION FOR INVARIANT EXTENSIONS OF THE LEBESGUE MEASURE

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Dedicated to Professor Tengiz Shervashidze on the occasion of his 75th birthday

Abstract. The concept of uniform distribution in [0,1] is extended to a certain mutually singular maximal (in the sense of cardinality) family $\{\lambda_t : t \in [0,1]\}$ of invariant extensions of the linear Lebesgue measure λ in [0,1], and it is shown that the λ_t^∞ measure of the set of all λ_t-uniformly distributed sequences is equal to 1, where λ_t^∞ denotes the infinite power of measure λ_t.

1. Introduction

The theory of uniform distribution is concerned with the distribution of real numbers in the unit interval $[0,1]$ and its development started with Hermann Weyl's celebrated paper [27]. This theory gives a useful tool for exact numerical calculation of the one-dimensional Riemann integral over $[0,1]$.

More precisely, the sequence of real numbers $\{x_n : n \in \mathbf{N}\} \in [0,1]^\infty$ is uniformly distributed in [0,1] if and only if for every real-valued Riemann integrable function f on [0,1] the equality

$$\lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} f(x_k)}{n} = \int\limits_0^1 f(x)dx \tag{1.1}$$

holds (see, e.g., [16, Corollary 1.1, p. 2]). Main corollaries of this assertion were successfully used in Diophantine approximations and widely applied to Monte-Carlo integration (cf. [2, 3, 16, 27]). Note that the set U of all uniformly distributed sequences in [0,1], viewed as a subset of $[0,1]^\infty$, has the λ_t^∞ measure of the set of all λ_t-uniformly distributed sequences equal to 1, where λ_t^∞ denotes the infinite product of the measure λ_t. (cf. [16, Theorem 2.2 (Hlawka), p. 183]). For a fixed Lebesgue integrable function f in [0,1], one can put a question asking what is a maximal subset U_f of U each element of which can be used for calculation of its Lebesgue integral over [0,1] by formula (1.1) and whether this subset a the full λ^∞-measure. This question has been resolved positively by Kolmogorov's Strong Law of Large Numbers. There naturally arises another question asking whether an analogous methodology can be developed for invariant extensions of the Lebesgue measure in [0,1] and whether the main results of the uniform distribution theory hold true in such a situation. In the present paper, we consider this question for a certain mutually singular maximal (in the sense of cardinality) family of invariant extensions of the linear Lebesgue measure in [0,1]. In our investigations, we essentially use the methodology developed in [10, 16, 24].

The rest of the present paper is organized as follows.

In Section 2, we consider some auxiliary facts from the theory of invariant extensions of the Lebesgue measure and from the probability theory. In Section 3, we present our main results. Section 4 presents historical background of the theory of invariant extensions of the n-dimensional Lebesgue measure.

2. Some Auxiliary Notions and Facts from the Theory of Invariant Extensions of the Lebesgue Measure

Throughout this article, we use the following standard notation:

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$\mathbf{R}$ is the set of all real numbers;
$\mathbf{N}$ is the set of all natural numbers;
ω is the first infinite cardinal number (i.e.,$\omega = \mathrm{card}(\mathbf{N})$);
$\mathbf{c}$ is the cardinality of the continuum (i.e., $\mathbf{c} = 2^{\omega}$);
λ is the linear Lebesgue measure on $\mathbf{R}$.
$\mathrm{dom}(\mu)$ is the domain of a given measure μ;
$\mu_1 \supset \mu$ - a measure μ_1 is an extension of the given measure μ.

Lemma 2.1 ([10, Lemma 6, p. 174]). *Let K be a shift-invariant σ-ideal of subsets of the real axis $\mathbf{R}$ such that*

$$(\forall Z)(Z \in K \Rightarrow \lambda_*(Z) = 0),$$

where λ_* denotes inner measure generated by the linear Lebesgue measure λ.

Then the functional μ defined by

$$\mu((X \cup Z^{'}) \setminus Z^{''}) = \lambda(X),$$

where X is a Lebesgue measurable subset of $\mathbf{R}$ and $Z^{'}$ and $Z^{''}$ are elements of the σ-ideal K, is a shift-invariant extension of the Lebesgue measure λ.

Lemma 2.2 ([10, Lemma 4, p. 164]). *There exists a family $\{Y_i : i \in [0,1]\}$ of subsets of the real line $\mathbf{R}$ such that:*

(1) $(\forall i)(\forall i')(i \in [0,1], i' \in [0,1], i \neq i' \Rightarrow Y_i \cap Y_{i'} = \emptyset)$;
(2) $(\forall i)(\forall F)(i \in [0,1]$, *$F$ is a closed subset of the real line $\mathbf{R}$ with* $\lambda(F) > 0 \Rightarrow \mathrm{card}(X_i \cap F) = \mathbf{c})$;
(3) $(\forall I^{'})(\forall g)(I^{'} \subseteq [0,1], g \in \mathbf{R} \Rightarrow \mathrm{card}((g + (\cup_{i \in I'} Y_i)) \triangle (\cup_{i \in I'} Y_i)) < \mathbf{c})$.

Lemma 2.3 ([10, Lemma 5, p. 166]). *There exists a family $\{X_i : i \in [0,1]\}$ of subsets of the real line $\mathbf{R}$ such that:*

(a) *for any sequence $\{i_k : k \in \mathbf{N}\} \subset [0,1]$, the intersection*

$$\cap_{k \in \mathbf{N}} \overline{X_{i_k}},$$

where

$$\overline{X_{i_k}} = X_{i_k} \vee \overline{X_{i_k}} = \mathbf{R} \setminus X_{i_k},$$

is almost invariant set.

(b) *for any sequence $\{i_k : k \in \mathbf{N}\} \subset [0,1]$ and for any closed subset F of the real line $\mathbf{R}$ with $\lambda(F) > 0$, we have*

$$\mathrm{card}((\cap_{k \in \mathbf{N}} \overline{X_{i_k}}) \cap F) = \mathbf{c}.$$

Lemma 2.4 ([10, Corollary 5, p. 174]). *There exists a family $\{\mu_t : t \in [0,1]\}$ of measures defined on some shift-invariant σ-algebra $S(\mathbf{R})$ of subsets of the real axis $\mathbf{R}$ such that:*

(1) $(\forall t)(t \in [0,1] \Rightarrow$ *the measure μ_t is a shift-invariant extension of the linear Lebesgue measure λ);*
(2) $(\forall t)(\forall t^{'})(t \in [0,1], t^{'} \in [0,1], t \neq t^{'} \Rightarrow \mu_t$ *and $\mu_{t'}$ are mutually singular measures.*

Moreover, $\mu_t(\mathbf{R} \setminus X_t) = 0$ for each $t \in [0,1]$, where $\{X_t : t \in [0,1]\}$ comes from Lemma 2.2.

Remark 2.5. Let us consider the family $\{\mu_t : t \in [0,1]\}$ of shift-invariant extensions of the measure λ which comes from Lemma 2.4. We denote by λ_t the restriction of the measure μ_t to the class

$$S[0,1] = \{Y \cap [0,1] : Y \in S(\mathbf{R})\},$$

where $S(\mathbf{R})$ comes from Lemma 2.4. It is obvious that for each $t \in [0,1]$, the measure λ_t is concentrated on the set $C_t = X_t \cap [0,1]$, provided that

$$\lambda_t([0,1] \setminus C_t) = 0.$$

The next proposition is useful for our further consideration.

Lemma 2.6 (Kolmogorov Strong Law of Large Numbers, [24, Theorem 3, p. 379]). *Let (Ω, S, P) be a probability space and $\{\xi_k : k \in \mathbf{N}\}$ be a sequence of independent equally distributed random variables for which mathematical expectation m of $\|\xi_1\|$ is finite. Then the following condition*

$$P\left(\left\{\omega : \omega \in \Omega \wedge \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \xi_k(\omega)}{n} = m\right\}\right) = 1$$

holds.

3. Uniform Distribution for Invariant Extensions of the Lebesgue Measure Defined by Remark 2.5

Let us consider the family of probability measures $\{\lambda_t : t \in [0,1]\}$ and the family $\{C_t : t \in [0,1]\}$ of subsets of [0,1] which come from Remark 2.5.

Lemma 3.1. *For $t \in [0,1]$ we denote by $L([0,1], \lambda_t)$ the class of λ_t-integrable functions. Then for $f \in L([0,1], \lambda_t)$, we have*

$$\lambda_t^{\infty}(\{\{x_k : k \in \mathbf{N}\} \in [0,1]^{\infty} : \lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(x_k)}{n} = \int_{[0,1]} f(x) d\lambda_t(x)\}) = 1.$$

Proof. For fixed $t \in [0,1]$, we set

$$(\Omega, S, P) = (C_t^{\infty}, F(C_t^{\infty}), \nu_t^{\infty}),$$

where

i) $F(C_t) = \{C_t \cap Y : Y \in S[0,1]\}$, where $S[0,1]$ comes from Remark 2.5.

ii) $\nu_t = \lambda_t|_{F(C_t)}$, where $\lambda_t|_{F(C_t)}$ denotes restriction of the measure λ_t to the σ-algebra $F(C_t)$.

For $k \in \mathbf{N}$ and $\{x_k : k \in \mathbf{N}\} \in C_t^{\infty}$, we put

$$\xi_k(\{x_k : k \in \mathbf{N}\}) = f(x_k).$$

Then all conditions of Lemma 2.6 are satisfied which implies that

$$\nu_t^{\infty}(\{\{x_k : k \in \mathbf{N}\} \in C_t^{\infty} : \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \xi_k\{x_k : k \in \mathbf{N}\}}{n} = \int_{C_t^{\infty}} \xi_1(\{x_i : i \in \mathbf{N}\} d\nu_t^{\infty}(\{(x_k : k \in \mathbf{N}\}) = 1,$$

equivalently,

$$\nu_t^{\infty}(\{\{x_k : k \in \mathbf{N}\} \in C_t^{\infty} : \lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(x_k)}{n} = \int_{C_t} f(x) d\nu_t(x)\}) = 1.$$

The latter relation implies that

$$\lambda_t^{\infty}(\{\{x_k : k \in \mathbf{N}\} \in [0,1]^{\infty}, \lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(x_k)}{n} = \int_{[0,1]} f(x) d\lambda_t(x)\})$$

$$\geq \nu_t^{\infty}(\{\{x_k : k \in \mathbf{N}\} \in C_t^{\infty} : \lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(x_k)}{n} = \int_{C_t} f(x) d\nu_t(x)\}) = 1. \quad \square$$

Definition 3.2. A sequence of real numbers $\{x_k : k \in \mathbf{N}\} \in [0,1]^\infty$ is said to be λ-uniformly distributed sequence (abbreviated λ-u.d.s.) if for each c, d with $0 \le c < d \le 1$, we have

$$\lim_{n\to\infty} \frac{\mathrm{card}(\{x_k : 1 \le k \le n\} \cap [c,d])}{n} = d - c.$$

We denote by S the set of all real-valued sequences from $[0,1]^\infty$ which are λ-u.d.s. It is well known that $\{\alpha n : n \in \mathbf{N}\} \in S$ for each irrational number α, where $\{\cdot\}$ denotes the fractional part of the real number (cf. [16, Exercise 1.12, p. 16]).

Definition 3.3. A sequence of real numbers $\{x_k : k \in \mathbf{N}\} \in \mathbf{R}^\infty$ is said to be uniformly distributed module 1 if the sequence of its fractional parts $\{x_k : k \in \mathbf{N}\}$ is λ-u.d.s.

Remark 3.4. It is obvious that $\{x_k : k \in \mathbf{N}\} \in [0,1]^\infty$ is uniformly distributed module 1 if and only if $\{x_k : k \in \mathbf{N}\}$ is λ-u.d.s.

Definition 3.5. A sequence of real numbers $\{x_k : k \in \mathbf{N}\} \in [0,1]^\infty$ is said to be λ_t-uniformly distributed sequence (abbreviated λ_t-u.d.s.) if for each c, d with $0c < d1$, we have

$$\lim_{n\to\infty} \frac{\mathrm{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap C_t)}{n} = d - c.$$

We denote by S_t the set of all real valued sequences from $[0,1]^\infty$ which are λ_t-u.d.s.

In order to construct λ_t-u.d.s. for each $t \in [0,1]$, we need the following lemma.

Lemma 3.6 ([16, Theorem 1.2, p. 3])**.** *If the sequence $\{x_n : n \in \mathbf{N}\}$ is u.d.* $\bmod 1$*, and if $\{y_n : n \in \mathbf{N}\}$ is a sequence with the property*

$$\lim_{n\to\infty} (x_n - y_n) = \alpha$$

for some real constant α , then $\{y_n : n \in \mathbf{N}\}$ is u.d. $\bmod 1$.

Theorem 3.7. *For each $t \in [0,1]$ there exists λ_t-u.d.s.*

Proof. Let us consider a sequence $\{x_k : k \in \mathbf{N}\} \in [0,1]^\infty$ which is λ-u.d.s. For each $n \in \mathbf{N}$, we choose such an element y_n from the set $C_t \cap (0, x_n)$ that

$$\|x_n - y_n\| < \frac{1}{n}.$$

This we can do because C_t is everywhere dense in $(0,1)$. Now it is obvious that

$$\lim_{n\to\infty} (x_n - y_n) = 0.$$

By Lemma 3.6, we deduce that $\{y_n : n \in \mathbf{N}\}$ is λ-u.d.s. Let us show that $\{y_n : n \in \mathbf{N}\}$ is λ_t-u.d.s. Indeed, since $y_k \in C_t$ for each $k \in \mathbf{N}$ and $\{y_n : n \in \mathbf{N}\}$ is λ-u.d.s., for each c, d with $0 \le c < d \le 1$, we have

$$\lim_{n\to\infty} \frac{\mathrm{card}(\{y_k : 1 \le k \le n\} \cap [c,d] \cap C_t)}{n} = \lim_{n\to\infty} \frac{\mathrm{card}(\{y_k : 1 \le k \le n\} \cap [c,d])}{n} = d - c. \qquad \square$$

Theorem 3.8. *For each $t \in [0,1]$ λ_t-u.d.s. is λ-u.d.s..*

Proof. Let $\{x_k : k \in \mathbf{N}\}$ be λ_t-u.d.s. On the one hand, for each c, d with $0 \le c < d \le 1$, we have

$$\underline{\lim}_{n\to\infty} \frac{\mathrm{card}(\{x_k : 1 \le k \le n\} \cap [c,d])}{n} \ge \lim_{n\to\infty} \frac{\mathrm{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap C_t)}{n} = d - c.$$

Since $\{x_k : k \in \mathbf{N}\}$ is λ_t-u.d.s., we have

$$\lim_{n\to\infty} \frac{\mathrm{card}(\{x_k : 1 \le k \le n\} \cap [0,1] \cap C_t)}{n} = 1.$$

It is obvious that

$$\lim_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap [0,1])}{n} = 1.$$

The last two conditions imply that

$$\lim_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap ([0,1] \setminus C_t))}{n}$$
$$= \lim_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap [0,1])}{n}$$
$$- \lim_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap C_t)}{n} = 1 - 1 = 0.$$

The last relation implies that for each c, d with $0 \le c < d \le 1$

$$\lim_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap ([0,1] \setminus C_t))}{n}$$
$$\le \lim_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap ([0,1] \setminus C_t))}{n} = 0.$$

Finally, for each c, d with $0 \le c < d \le 1$ we get

$$\overline{\lim}_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap [c,d])}{n}$$
$$\le \lim_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap C_t)}{n}$$
$$+ \lim_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap ([0,1] \setminus C_t))}{n}$$
$$= (d-c) + 0 = d - c.$$

This ends the proof of theorem. □

Remark 3.9. Note that the converse to the result of Theorem 3.8 is not valid. Indeed, for fixed $t \in [0,1]$, let $\{y_n : n \in \mathbf{N}\}$ be λ_t-u.d.s. which comes from Theorem 3.7. By Theorem 3.8, $\{y_n : n \in \mathbf{N}\}$ is λ-u.d.s. Let us show that $\{y_n : n \in \mathbf{N}\}$ is not λ_s-u.d.s. for each $s \in [0,1] \setminus t$. Indeed, since $y_k \in C_s$ for each $k \in \mathbf{N}$, we deduce that $y_k \notin C_s$ for each $s \in [0,1] \setminus t$. The latter relation implies that for each $s \in [0,1] \setminus t$ and for each c, d with $0 \le c < d \le 1$, we have

$$\lim_{n\to\infty} \frac{\operatorname{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap C_t)}{n} = 0 < d - c.$$

Remark 3.10. For each λ-u.d.s. $\{y_n : n \in \mathbf{N}\}$, there exists a countable subset $T \subset [0,1]$ such that $\{y_n : n \in \mathbf{N}\}$ is not λ_t-u.d.s for each $t \in [0,1] \setminus T$. Indeed, since $\{C_t : t \in [0,1]\}$ is the partition of the $[0,1]$, for each $k \in \mathbf{N}$ there exists a unique $t_k \in [0,1]$ such that $y_k \in C_{t_k}$. Now we can put

$$T = \bigcup_{k\in\mathbf{N}} \{t_k\}.$$

Theorem 3.11. *There exists λ-u.d.s which is not λ_t-u.d.s. for each $t \in [0,1]$.*

Proof. Let us consider a sequence $\{x_n : n \in \mathbf{N}\} \in (0,1)^\infty$ which is λ-u.d.s. Since $\{C_t : t \in [0,1]\}$ is the partition of the $[0,1]$, for each $k \in \mathbf{N}$, there exists a unique $t_k \in [0,1]$ such that $y_k \in C_{t_k}$. Now we can put

$$T = \bigcup_{k\in\mathbf{N}} \{t_k\}.$$

Let $S_0 = \{s_1, s_2, \dots\}$ be a countable subset of the set $[0,1] \setminus T$. For each $n \in \mathbf{N}$, we choose the element y_n from the set $C_{s_n} \cap (0, x_n)$ such that

$$|x_n - y_n| < \frac{1}{n}.$$

This can be done because C_t is every where dense in $(0,1)$ for each $t \in [0,1]$. Now it is obvious that

$$\lim_{n\to\infty}(x_n - y_n) = 0.$$

By Lemma 3.6, we deduce that $\{y_n : n \in \mathbf{N}\}$ is λ-u.d.s.. Let us show that $\{y_n : n \in \mathbf{N}\}$ is not λ_t-u.d.s. for each $t \in [0,1]$. This follows from the fact that $\text{card}(\{y_n : n \in \mathbf{N}\} \cap C_t) \le 1$ for each $t \in [0,1]$. By this reason, for each $t \in [0,1]$ and for each c, d with $0 \le c < d \le 1$, we have

$$\lim_{n\to\infty} \frac{\text{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap C_t)}{n} \le \lim_{n\to\infty} \frac{1}{n} = 0 < d - c. \qquad \square$$

Theorem 3.12. *$S_i \cap S_j = \emptyset$ for each different $i, j \in [0,1]$.*

Proof. Assume the contrary and let $\{x_k : k \in \mathbf{N}\} \in S_i \cap S_j$. On the one hand, for each c, d with $0 \le c < d \le 1$, we have

$$\lim_{n\to\infty} \frac{\text{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap C_i)}{n} = d - c.$$

On the other hand, for same c, d, we have

$$\lim_{n\to\infty} \frac{\text{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap C_j)}{n} = d - c.$$

By Theorem 3.8, we know that $\{x_k : k \in \mathbf{N}\}$ is λ-u.d.s. which implies that for same c, d, we have

$$\lim_{n\to\infty} \frac{\text{card}(\{x_k : 1 \le k \le n\} \cap [c,d])}{n} = d - c. \tag{3.1}$$

But (3.1) is not possible because $C_i \cap C_j = \emptyset$ which implies

$$\begin{aligned} d - c &= \lim_{n\to\infty} \frac{\text{card}(\{x_k : 1 \le k \le n\} \cap [c,d])}{n} \\ &\ge \lim_{n\to\infty} \frac{\text{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap C_i)}{n} \\ &+ \lim_{n\to\infty} \frac{\text{card}(\{x_k : 1 \le k \le n\} \cap [c,d] \cap C_j)}{n} = 2(d-c). \end{aligned} \qquad \square$$

4. Historical Background for Invariant Extensions of the Haar Measure

4.1. **On the Waclaw Sierpiniski problem.** By Vitali's celebrate theorem about the existence of the linear Lebesgue non-measurable subset, it has been shown that the domain of the Lebesgue measure in $\mathbf{R}$ diers from the power set of the real axis $\mathbf{R}$. In this context, there naturally appears the following question:

"How far can we extend Lebesgue measure and what properties may preserve such an extension?"

In 1935, E. Marczewski applied Sierpinski construction of an almost invariant set, obtained a proper invariant extension of the Lebesgue measure in which the extended σ-algebra contained new sets of positive finite measures. In connection with this result, Waclaw Sierpiniski in 1936 posed the following question:

Problem. Let D_n denote the group of all isometrical transformations of the $\mathbf{R}^n$. Does there exist any maximal D_n-invariant measure?

The first result in this direction was obtained by Andrzej Hulanicki [6] as follows:

Proposition (Andrzej Hulanicki (1962))**.** *If the continuum 2^ω is not real-valued measurable cardinal then there does not exist any maximal invariant extension of the Lebesgue measure.*

Using similar methods, this result has been obtained independently by SH. Pkhakadze [23].

In 1977, A. B. Kharazishvili got the same answer in the one-dimensional case without any set-theoretical assumption (see [9, 10]).

Finally, in 1982, Krzysztof Ciesielski and Andrzej Pelc generalized Kharazishvili's result to all n-dimensional Euclidean spaces (see [1]). Following Solovay [25], if the system of axioms **ZFC** and the existing inaccessible cardinal are consistent, then the systems of axioms **ZF and DC** and every set of reals are Lebesgue measurable and also consistent. This result implies that the answer to Waclaw Sierpiniski's problem is armative. Taking Solovay's result on the one hand, and Krzysztof Ciesielski and Andrzej Pelc (or A. Hulanicki or Pkhakadze) result on the other hand, we deduce that the Waclaw Sierpiniski's question is not solvable within the theory **ZF and DC**.

4.2. **On Lebesgue measure's invariantly extension methods in ZFC.** Nowadays, there exists a vast methodology assigned for constructing invariant extensions of the Lebesgue measure in $\mathbf{R}^n$ and the Haar measure in a locally compact Hausdor topological group. Let us briefly consider the main one.

Method I (E. Marczewski). Let K be a shift-invariant σ-ideal in the n-dimensional Euclidean space $\mathbf{R}^n$ such that

$$(\forall Z)(Z \in K \Rightarrow \lambda_*(Z) = 0)$$

where λ_* denotes the inner measure defined by the n-dimensional Lebesgue measure λ. Then the functional $\bar{\lambda}$ defined by

$$\bar{\lambda}((X \cup Z') \setminus Z'') = \lambda(X)$$

where X is a Lebesgue measurable subset of $\mathbf{R}^n$ and Z' and Z'' are elements of the σ-ideal K, $\bar{\lambda}$ is an D_n-invariant extension of the Lebesgue measure λ.

Method III (J. Oxtoby and S.Kakutani). Some methods of combinatorial set theory have lately been successfully used in the measure extension problem. Among them, special mention should be made of the method of constructing a maximal (in the sense of cardinality) family of independent families of sets in arbitrary infinite base spaces. The question of the existence of a maximal (in the sense of cardinality) ω-independent family of subsets of an uncountable set E has beed considered by A. Tarski. He proved that this cardinality is equal to $2^{\mathrm{card}(E)}$.

This result found an interesting application in a general topology. For ex-ample, it was proved that in an arbitrary infinite space E the cardinality of the class of all ultrafilters is equal to $2^{2^{\mathrm{card}(E)}}$ (see, e.g., [17]).

The combinatorial question of the existence of a maximal (in the sense of cardinality) strict ω-independent family of subsets of a set E with cardinality of the continuum also was investigated and it was proved that this cardinality is equal to $2^{\mathbf{c}}$. This combinatorial result found an interesting application in the Lebesgue measure theory. For example, Kakutani and Oxtoby [8] firstly constructed a family $\mathbf{A}$ of almost invariant subsets of the circle in such a way that

$$\bigcap_{n \in \mathbf{N}} A_n^{f_n}$$

has outer measure 1 for an arbitrary sequence $\{A_n : n \in \mathbf{N}\}$ of sets from $\mathbf{A}$ and arbitrary sequence $\{f_n : n \in \mathbf{N}\}$, $(f_n = 0, 1)$. After making some assumtions they obtained an extension of the Lebesgue measure on the circle to an invariant measure $\bar{\lambda}$ such that $L_2(\bar{\lambda})$ has the Hilbert space dimensional equal to $2^{\mathbf{c}}$.

Using the same combinatorial result, A.B. Kharazishvili constructed a maximal (in the sense of cardinality) mutually singular family of elementary D_n-invariant extensions of the Lebesgue measure (see [10]).

The combinatorial question of the existence of a maximal (in the sense of cardinality) strict ω-independent family of subsets of a set E with $\mathrm{card}(E^\omega) = \mathrm{card}(E)$ was investigated in [20] and it was shown that this cardinality is equal to $2^{\mathrm{card}(E)}$. Using this result, G.Pantsulaia [18] extended Kakutani and Oxtoby's [8] method to construction a maximal (in the sense of cardinality) family of orthogonal elementary H-invariant extensions of the Haar measure defined in a locally compact σ-compact topological group with $\mathrm{card}(H^\omega) = \mathrm{card}(H)$.

Method IV (K. Kodaira and S. Kakutani). Kodaira and Kakutani [15] elaborated the method extending the Lebesgue measure on the circle to an invariant measure as follows:

Let us produce a character π of the circle, i.e., a homomorphism $\varphi : T \to T$ in such a way that the outer Lebesgue measure of its graph G_φ is equal to 1 in $T \times T$. Then the extended σ-algebra S consists of sets $A_M = \{x : (x, \varphi(x)) \in M\}$, where M is a Lebesgue measurable set in $T \times T$ and the extended measure $\bar{\lambda}$ is $\bar{\lambda}(A_M) = (\lambda \times \lambda)(M)$. Note that the discontinuous character becomes

S-measurable. Later, it has been noticed [5] that one can produce $2^{\mathbf{c}}$ characters such that all of them become measurable and $L_2(\bar{\lambda})$ is of Hilbert space dimension $2^{\mathbf{c}}$.

This method has been modified for the n-dimensional Euclidean space in [20] for constructing the invariant extension $\bar{\lambda}$ of the n-dimensional Lebesgue measure such that there exists a $\bar{\lambda}$-measurable set with only one density point. This result answered positively to a certain question stated by A. B. Kharazishvili (cf. [11, Problem 9, p. 200]). Knowing this result, A. B. Kharazishvili considered similar, but originally modified method and extended previous result in [11] as follows:

there exists an invariant extension $\bar{\lambda}$ of the classical Lebesgue measure such that $\bar{\lambda}$ has the uniqueness property and there exists a $\bar{\lambda}$-measurable set with only one density point.

Method $\star$. More lately, Kodaira and Kakutani's method has been modified for an uncountable locally compact σ-compact topological group H with $\mathrm{card}(H^{\omega}) = \mathrm{card}(H)$ in [21] as follows: Let E be a set with $2 \le \mathrm{card}(E) \le \mathrm{card}(H)$ and let μ be a probability measure in E such that each $X \in \mathrm{dom}(\mu)$ for which $\mathrm{card}(X) < \mathrm{card}(E)$. Let us find a function $f : H \to E$ in such a way that the following two conditions:

(1) $(\forall e)(\forall F)(e \in E$ and $(F$ is a closed subset of the H with $\lambda(F) > 0) \Rightarrow \mathrm{card}(f^{-1}(e) \cap F) = \mathrm{card}(E))$;

(2) $(\forall E)(\forall g)(E \subseteq E$ and $g \in H \Rightarrow \mathrm{card}(g(\cup_{e\in E} f^{-1}(e)) \triangle (\cup_{e \in E} f^{-1}(e)) < \mathrm{card}(H)$ hold true.

Then the extended σ-algebra S consists of the sets $A_M = \{x : (x, f(x)) \in M\}$, where $M \in \mathrm{dom}(\lambda) \times \mathrm{dom}(\mu)$ and the extended measure λ_μ is defined by $\lambda_\mu(A_M) = (\lambda \times \mu)(M)$. Note that λ_μ is a non-elementary invariant extension of the measure λ, if and only if the measure μ is diffused. It has been noticed that one can produce $2^{card(H)}$ functions such that they all become measurable and $L_2(\lambda_\mu)$ is of Hilbert space dimension $2^{\mathrm{card}(H)}$.

Method V (A. Kharazishvili). This approach, as usual, can be used for uncountable commutative groups and is based on the purely algebraic properties of those groups, which are not assumed to be endowed with any topology, but only are equipped with a nonzero σ-finite invariant measures. Here, essentially is used Kulikov's well known theorem about covering of any commutative group by increasing (in the sense of inclusion) countable sequence of subgroups of G which are direct sum of cyclic groups (finite or infinite) (see, e.g., [13, 14]).

Definition 4.1. Let E be a base space, G be a group of transformations of E and let X be a subset of the space E. X is called a G-absolutely negligible set if for any G-invariant σ-finite measure μ, there exists its G - invariant extension $\bar{\mu}$ such that $X \in \mathrm{dom}(\bar{\mu})$ and $\bar{\mu}(X) = 0$.

A geometrical characterization of absolutely negligible subsets, due to A. Kharazishvili, is presented in the next proposition.

Theorem 4.2. *Let $(G, +)$ be a commutative group and Y be a subset of G. The following two assertions are equivalent:*

1) *Y is G-absolutely negligible in G;*

2) *for any countable family $\{g_n : n < \omega\}$ of elements from G, there exists a countable family $\{f_m : m < \omega\}$ of elements from G such that*

$$\bigcap_{m<\omega} (f_m + \bigcup_{n<\omega} (g_n + Y)) = \emptyset.$$

For the proof of the above-mentioned theorem, see [10].

It is of interest that the class of all countable G-configurations of the fixed G-absolutely negligible subset constitutes a G-invariant σ-ideal such that the inner measure of each element of this class is zero with respect to any σ-finite G-invariant measure in E. Hence, by using the natural modification of Method I, one can obtain G-invariant extension of an arbitrary σ-finite G-invariant measure in E.

In 1977, A. B. Kharazishvili constructed the partition of the real axis $\mathbf{R}$ into a countable family of D_1-absolutely negligible sets and got the negative answer to the question of Waclaw Sierpiniski in the one-dimensional case without any set-theoretical assumption (see [9]).

Finally, in 1982, Krzysztof Ciesielski and Andrzej Pelc generalized Kharazishvili's result to all n-dimensional Euclidean spaces, more precisely, they constructed the partition of the Euclidean space $\mathbf{R}^n$

into the countable family of D_n-absolutely negligible sets and got the negative answer to the question of Waclaw Sierpiniski in the n-dimensional case without any set-theoretical assumption (see [1]).

By using the method of absolutely negligible sets elaborated by A. Kharazishvili [10], P. Zakrzewski [28] answered positively to a question of Ciesielski asking whether an isometrically invariant σ-finite countably additive measure on $\mathbf{R}^n$ admits a strong countably additive isometrically invariant extension. It is obvious that this question is a generalization of the above-mentioned W. Sierpiniski's problem.

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