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THE CONTACT PROBLEM FOR PIECEWISE HOMOGENEOUS VISCOELASTIC PLATE REINFORCED WITH A FINITE RIGID PATCH

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Abstract. The contact problem of the theory of viscoelasticity for a piecewise-homogeneous plate reinforced with a finite rigid patch is considered. The patch meets the interface at a right angle and is loaded with the normal forces. The problem is reduced to a singular integral equation of first kind containing a fixed singularity with respect to a characteristic function of normal contact stresses. Using the methods of the theory of analytic functions, the Riemann problem is obtained, the solution of this problem is given explicitly. The normal contact stresses along the contact line are determined and the behavior of the contact stresses in the neighborhood of singular points is established.

Introduction

Exact and approximate solutions of static contact problems for different domains reinforced with elastic thin inclusions, stringers and patches of variable rigidity were studied earlier, and the behavior of contact stresses at the ends of the contact line have been investigated in [1–3, 11, 14–18]. Such problems as the first fundamental problem for a piecewise-homogeneous plane, when a crack of finite length arrives at the interface of two bodies at the right angle [10], the similar problem for a piecewise-homogeneous plane lying under the action of symmetrical normal stresses at the crack sides [4, 19], as well as the contact problems for a piecewise-homogeneous planes with a semi-infinite and finite inclusion [5, 6], have also been solved.

1. Statement of the Problem and Reduction to the Integral Equation

Suppose the body holds a complex plane $z = x + iy$ consisting of two dissimilar isotropic half-planes with viscoelastic property and reinforced with a rigid finite patch, perpendicular to the interface of two materials. The patch is loaded by a vertical force $P\delta(x-a)H(t-t_0)$ and the plate is free from external loads. ($\delta(x)$ is the Dirac function and $H(t)$ is the unit Heaviside function, $a \in (0,1)$). The half-planes $S_1 = \{z|\operatorname{Re} z > 0,\ \ z \notin [0,1)\}$ and $S_2 = \{z|\operatorname{Re} z < 0\}$ are connected along the Oy-axis. The contact conditions along the interface have the form

$$\sigma_x^{(1)} = \sigma_x^{(2)} \quad \tau_{xy}^{(1)} = \tau_{xy}^{(2)}, \quad \frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} \quad \frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial y}. \tag{1.1}$$

On the boundary of interaction of rigid patch and half-plane S_1, the conditions

$$\begin{gathered}\sigma_y^{(1)+} - \sigma_y^{(1)-} = p(x,t), \quad \tau_{xy}^{(1)+} - \tau_{xy}^{(1)-} = 0,\\ u_1^+ - u_1^- = 0, \quad v_1^+ = v_1^- \equiv v(x,t),\end{gathered} \tag{1.2}$$

$$\frac{dv_0(x,t)}{dx} = \frac{dv(x,t)}{dx}, \quad \frac{dv_0(x,t)}{dx} = 0, \tag{1.3}$$

$$\int_0^1 [p(x,t) - P\delta(x-a)H(t-t_0)]dx = 0 \tag{1.4}$$

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are valid, where (1.2) represents the jumps of the stress and displacement components on the contact line, (1.3) are the rigid contact condition and the constancy of normal displacements $v_0(x,t)$ of the patch points, (1.4) shows the equilibrium condition of the patch.

In the theory of viscoelatisity, we have the Kolosov–Muskhelishvili type formulas [8,9]:

$$\sigma_y^{(k)} - i\tau_{xy}^{(k)} = \Phi_k(z,t) + \overline{\Phi_k(z,t)} + z\overline{\Phi_k'}(z,t) + \overline{\Psi_k(z,t)}, \tag{1.5}$$

$$(I-L)\Big[\text{æ}_k\Phi_k(z,t) - \overline{\Phi_k(z,t)} - z\overline{\Phi_k'}(z,t) - \overline{\Psi_k(z,t)}\Big] = 2\mu_k\left(u_k' + iv_k'\right), \tag{1.6}$$

where $(I-L)g_k(t) = g_k(t) - \int\limits_{t_0}^{t} E_k \frac{\partial}{\partial\tau}C_k(t,\tau)\,g_k(\tau)\,d\tau$, $2\mu_k = \frac{E_k}{1+\nu_k}$ and

$$\text{æ}_k = \begin{cases} 3-4\nu_k \\ \dfrac{3-\nu_k}{1+\nu_k} \end{cases} \qquad k=1,2.$$

$C_k(t,\tau)$ and E_k are the creep measure and Jung's module of the material, respectively. Besides, the plate Poisson's coefficients for elastic-instant deformation $\nu_k(t)$ and creep deformation $\nu_k(t,\tau)$ are the same and constant: $\nu_k(t) = \nu_k(t,\tau) = \nu_k = \text{const}$. From relations (1.5), (1.6), we obtain the following Riemann boundary value problems:

$$\Phi_1^+(x,t) - \Phi_1^-(x,t) = \frac{1}{\text{æ}_1+1}p(x,t),$$
$$\Psi_1^+(x,t) - \Psi_1{}^-(x,t) = \frac{\text{æ}_1-1}{\text{æ}_1+1}p(x,t) - \frac{1}{\text{æ}_1+1}xp'(x,t), \quad 0<x<1.$$

The general solutions of these problems will be represented as follows [12]:

$$\Phi_1(z,t) = \frac{1}{2\pi(\text{æ}_1+1)i}\int\limits_0^1 \frac{p(x,t)dx}{x-z} + W_1(z,t) \equiv A_1(z,t) + W_1(z,t),$$

$$\Psi_1(z,t) = \frac{\text{æ}_1-1}{2\pi(\text{æ}_1+1)i}\int\limits_0^1 \frac{p(x,t)dx}{x-z} - \frac{1}{2\pi(\text{æ}_1+1)i}\int\limits_0^1 \frac{xp'(x,t)dx}{x-z}$$
$$+Q_1(z,t) \equiv B_1(z,t) + Q_1(z,t), \tag{1.7}$$

where $W_1(z,t)$ and $Q_1(z,t)$ are unknown analytic functions in the half-plane S_1, which will be defined from condition (1.1) on the interface. From relations (1.5)–(1.7), the conditions (1.1) result in

$$\text{Re}\Big[2\Phi_1(iy,t) + \overline{\omega_1(iy,t)}\Big] = \text{Re}\Big[2\Phi_2(iy,t) + \overline{\omega_2(iy,t)}\Big],$$

$$\text{Im}\Big[\overline{\omega_1(iy,t)}\Big] = \text{Im}\Big[\overline{\omega_2(iy,t)}\Big],$$

$$(1+\nu_1)\,\text{Re}\Big[\text{æ}_1\Phi_1(iy,t) - \overline{\Phi_1(iy,t)} - \overline{\omega_1(iy,t)}\Big] = (1+\nu_2)\,\text{Re}\Big[\text{æ}_2\Phi_2(iy,t) - \overline{\Phi_2(iy,t)} - \overline{\omega_2(iy,t)}\Big],$$

$$(1+\nu_1)\,\text{Im}\Big[\text{æ}_1\Phi_1(iy,t) - \overline{\Phi_1(iy,t)} - \overline{\omega_1(iy,t)}\Big] = (1+\nu_2)\,\text{Im}\Big[\text{æ}_2\Phi_2(iy,t) - \overline{\Phi_2(iy,t)} - \overline{\omega_2(iy,t)}\Big],$$

$$\omega_k(z,t) = z\Phi_k'(z,t) - \Psi_k(z,t) = \eta_k(z,t) + \Omega_k(z,t),$$

where

$$\Omega_1(z,t) = zW_1'(z,t) - Q_1(z,t), \quad \Omega_2(z,t) = z\Phi_2'(z,t) - \Psi_2(z,t)$$
$$\eta_1(z,t) = zA_1'(z,t) - B_1(z,t), \quad \eta_2(z,t) = 0.$$

Using the methods of the theory of analytic functions (particularly, using the Cauchy theorem), we obtain the following system of linear algebraic equations (with respect to the functions $W_1(z,t)$,

$\Phi_2(z,t)$, $\Omega_1(z,t)$, $\Omega_2(z,t)$):

$$
\begin{gathered}
2W_1(z,t)+\Omega_1(z,t)-2\overline{\Phi_2}(-z,t)-\overline{\Omega_2}(-z,t)=-\overline{\eta_1}(-z,t)-2\overline{A_1}(-z,t),\\
\Omega_1(z,t)+\overline{\Omega_2}(-z,t)=\overline{\eta_1}(-z,t),\\
\frac{æ_1-1}{\mu_1}W_1(z,t)-\frac{1}{\mu_1}\Omega_1(z,t)-\frac{æ_2-1}{\mu_2}\overline{\Phi_2}(-z,t)+\frac{1}{\mu_2}\overline{\Omega_2}(-z,t)\\
=\frac{1+æ_1}{\mu_1}\overline{A_1}(-z,t)+\frac{1}{\mu_1}\overline{\eta_1}(-z,t),\\
\frac{æ_1+1}{\mu_1}W_1(z,t)+\frac{1}{\mu_1}\Omega_1(z,t)+\frac{æ_2+1}{\mu_2}\overline{\Phi_2}(-z,t)+\frac{1}{\mu_2}\overline{\Omega_2}(-z,t)\\
=\frac{æ_1+1}{\mu_1}\overline{A_1}(-z,t)+\frac{1}{\mu_1}\overline{\eta_1}(-z,t).
\end{gathered}
\tag{1.8}
$$

The compatibility of system (1.8) is approved, that is, the corresponding determinant is not zero, $\Delta=-4\frac{(æ_2\mu_1+\mu_2)(æ_1\mu_2+\mu_1)}{\mu_1^2\mu_2^2}\neq 0$, and the solution of this system is represented in the form

$$
W_1(z,t)=-e_1t_1\int_0^1\frac{xp'(x,t)dx}{x+z}+e_1t_1\int_0^1\frac{xp(x,t)dx}{(x+z)^2}+e_1t_1(æ_1-1)\int_0^1\frac{p(x,t)dx}{x+z},
$$

$$
\Phi_2(z,t)=h_3t_1\int_0^1\frac{p(x,t)dx}{x-z},
$$

$$
\begin{gathered}
Q_1(z,t)=-e_1t_1\int_0^1\frac{x^2p'(x,t)dx}{(x+z)^2}+m_1t_1\int_0^1\frac{p(x,t)dx}{x+z}+e_1t_1(æ_1-1)\int_0^1\frac{xp(x,t)dx}{(x+z)^2}\\
+e_1t_1z\int_0^1\frac{p(x,t)dx}{(x+z)^2}+2e_1t_1z\int_0^1\frac{xp(x,t)dx}{(x+z)^3},
\end{gathered}
$$

$$
\Psi_2(z,t)=(h_3-h_4)t_1z\int_0^1\frac{p(x,t)dx}{(x-z)^2}-h_4t_1\int_0^1\frac{xp'(x,t)dx}{x-z}+(h_4(æ_1-1)+m_1)t_1\int_0^1\frac{p(x,t)dx}{x-z},
$$

where $t_1=\frac{1}{2\pi i(æ_1+1)}$, $e_1=\frac{\mu_2-\mu_1}{æ_1\mu_2+\mu_1}$, $e_2=\frac{\mu_2-\mu_1}{æ_2\mu_1+\mu_2}$,

$$
\begin{gathered}
m_1=(æ_1+1)\,\mu_2\Big[\frac{1}{æ_2\mu_1+\mu_2}-\frac{1}{æ_1\mu_2+\mu_1}\Big]=h_2-h_4,\\
m_2=(æ_2+1)\,\mu_1\Big[\frac{1}{æ_2\mu_1+\mu_2}-\frac{1}{æ_1\mu_2+\mu_1}\Big]=h_3-h_1\\
h_1=\frac{(æ_2+1)\,\mu_1}{æ_1\mu_2+\mu_1},\quad h_2=\frac{(æ_1+1)\,\mu_2}{æ_2\mu_1+\mu_2}\quad h_3=\frac{(æ_2+1)\,\mu_1}{æ_2\mu_1+\mu_2},\quad h_4=\frac{(æ_1+1)\,\mu_2}{æ_1\mu_2+\mu_1}.
\end{gathered}
$$

Therefore, based on the conditions (1.3), (1.4) and relations (1.8), we obtain the following singular integral equation with a fixed singularity:

$$
\int_0^1\frac{p(x,t)dx}{x-\tau}+e_1æ\int_0^1\frac{p(x,t)dx}{x+\tau}=0,\quad 0<\tau<1,
\tag{1.9}
$$

by the condition

$$
\int_0^1[p(x,t)-P\delta(x-a)H(t-t_0)]dx=0.
\tag{1.10}
$$

2. The solution of problem (1.9), (1.10)

The solution of problem (1.9), (1.10) is sought in the class of functions $p \in H^*(0,1)$ [12].

Introducing the notation $\varphi(x,t) = \int\limits_0^x p(y,t)dy$, problem (1.9), (1.10) takes the form

$$\int\limits_0^1 \frac{\varphi'(x,t)dx}{x-\tau} + e_1 æ_1 \int\limits_0^1 \frac{\varphi'(x,t)dx}{x+\tau} = 0, \quad 0<\tau<1, \tag{2.1}$$

$$\varphi(1,t) = PH(t-t_0) \tag{2.2}$$

and by the change of variables $x = e^{\xi}$, $\tau = e^{\zeta}$, from (2.1), (2.2), we obtain

$$\int\limits_{-\infty}^0 \frac{\varphi_0'(\zeta,t)d\zeta}{1-e^{-(\xi-\zeta)}} - e_1 æ_1 \int\limits_{-\infty}^0 \frac{\varphi_0'(\zeta,t)d\zeta}{1+e^{-(\xi-\zeta)}} = 0, \quad \xi<0,$$
$$\varphi_0(0,t) = PH(t-t_0), \tag{2.3}$$

where $\varphi_0(\xi,t) = \varphi(e^{\xi},t)$.

Applying Fourier's transform [7] with the variable $s = s_0 + i\varepsilon$ to both parts of equation (2.3) and using the convolution theorem, we obtain the following boundary condition of the Riemann problem [13]

$$\Psi^+(s,t) = sG(s)\Phi^-(s,t) + \frac{1}{\sqrt{2\pi}} iPH(t-t_0)G(s), \quad |s_0|<\infty, \tag{2.4}$$

where

$$\Phi^-(s,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^0 \varphi^-(\zeta,t)e^{is\zeta}d\zeta, \quad G(s) = cth\pi s - \frac{e_1 æ_1}{sh\pi s},$$

$$\Psi^+(s,t) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty \psi^+(\zeta,t)e^{is\zeta}d\zeta,$$

$$\varphi^-(\xi,t) = \begin{cases} \varphi_0(\xi,t) & \xi<0 \\ 0, & \xi>0 \end{cases}, \qquad \psi^+(\xi,t) = \begin{cases} 0, & \xi<0, \\ \int\limits_{-\infty}^0 \frac{\varphi_0'(\zeta,t)d\zeta}{1-e^{-(\xi-\zeta)}} - e_1 æ_1 \int\limits_{-\infty}^0 \frac{\varphi_0'(\zeta,t)d\zeta}{1+e^{-(\xi-\zeta)}}, & \xi>0. \end{cases}$$

The problem can be formulated as follows: Find the function $\Psi^+(z,t)$, holomorphic in the $\operatorname{Im} z>0$ half-plane and the function $\Phi^-(z,t)$, holomorphic in the $\operatorname{Im} z<1$ half-plane (with the exception of finite number roots of the function $G(z)$), which vanish at infinity and satisfy condition (2.4). The boundary condition (2.4) is represented in the form

$$\frac{\Psi^+(s,t)}{\sqrt{s+i}} = \frac{sG(s)}{\sqrt{1+s^2}}\Phi^-(s,t)\sqrt{s-i} + \frac{iPH(t-t_0)G(s)}{\sqrt{2\pi}\sqrt{s+i}}. \tag{2.5}$$

By $\sqrt{z+i}$ and $\sqrt{z-i}$ we mean the branches that are analytic in the planes with cuts along the rays, drawn from the points $z=-i$ and $z=i$, respectively, in the Ox direction and taking positive and negative values, respectively, on the upper side of the cut. With this choice of branches, the function $\sqrt{z^2+1}$ is analytic in the strip $-1<\operatorname{Im} z<1$ and takes a positive value on the real axis. The function $G_1(s) \equiv sG(s)(1+s^2)^{-1/2}$ satisfies the conditions

$$\operatorname{Re} G_1(s)>0, \quad G_1(\infty) = G_1(-\infty) = 1, \quad \operatorname{Ind} G_1(s) = 0.$$

The solution of this problem can be represented in the form [13]

$$\begin{gathered}\Phi^-(z,t)=\frac{\tilde{X}(z,t)}{\sqrt{z-i}},\quad \operatorname{Im} z\le 0;\quad \Psi^+(z,t)=\tilde{X}(z,t)\sqrt{z+i},\quad \operatorname{Im} z>0,\\ \Phi^-(z,t)=\frac{\Psi^+(z,t)-(2\pi)^{-1/2}iPH(t-t_0)G(z)}{zG(z)},\quad 0<\operatorname{Im} z<1,\end{gathered}\tag{2.6}$$

where

$$\tilde{X}(z,t)=X(z)\left\{\frac{PH(t-t_0)}{2\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{G(y)}{X^+(y)\sqrt{y+i}(y-z)}dy\right\},\quad X(z)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln G_1(y)dy}{y-z}\right\}.$$

It can be shown that $\Phi^-(s+i0)=\Phi^-(s-i0)$, and consequently, the function $\Phi^-(z)$ is holomorphic in the half-plane $\operatorname{Im} z<1$, except the points, being the zeros of the function $G(z)$ in the strip $0<\operatorname{Im} z<1$.

The boundary value of the function $K(z,t)=\frac{PH(t-t_0)}{\sqrt{2\pi}}-iz\Phi^-(z,t)$ is the Fourier transform of the function $\varphi'(e^\xi,t)$. By virtue of (2.6), the function $K(z,t)$ has the representation

$$\begin{aligned}K(z,t)&=\frac{PH(t-t_0)}{\sqrt{2\pi}}-\frac{izX(z)PH(t-t_0)}{2\pi\sqrt{2\pi}\sqrt{z-i}}\int_{-\infty}^{\infty}\frac{G(y)dy}{X^+(y)\sqrt{y+i}(y-z)}\\ &=\frac{PH(t-t_0)}{\sqrt{2\pi}}-\frac{izX(z)PH(t-t_0)}{2\pi\sqrt{2\pi}\sqrt{z-i}}\int_{-\infty}^{\infty}\frac{[G(y)-\operatorname{sgn} y]dy}{X^+(y)\sqrt{y+i}(y-z)}\\ &\quad-\frac{izX(z)PH(t-t_0)}{2\pi\sqrt{2\pi}\sqrt{z-i}}\int_{-\infty}^{\infty}\frac{\operatorname{sgn} y dy}{X^+(y)\sqrt{y+i}(y-z)}\\ &\equiv\frac{PH(t-t_0)}{\sqrt{2\pi}}+K_1(z,t)+K_2(z,t),\qquad \operatorname{Im} z<0.\end{aligned}$$

Now let us study the behavior of the function $K(z,t)$ at infinity. The function

$$K_1(z,t)=-\frac{izX(z)PH(t-t_0)}{2\pi\sqrt{2\pi}\sqrt{z-i}}\int_{-\infty}^{\infty}\frac{[G(y)-\operatorname{sgn} y]dy}{X^+(y)\sqrt{y+i}(y-z)}$$

tends to zero at infinity and applying the well-known Cauchy theorem, we can represent the function $K_2(z,t)$ in the form

$$\begin{aligned}K_2(z,t)&=-\frac{izX(z)PH(t-t_0)}{2\pi\sqrt{2\pi}\sqrt{z-i}}\int_{-\infty}^{\infty}\frac{\text{sgny}\, dy}{X^+(y)\sqrt{y+i}(y-z)}\\ &=\frac{izX(z)PH(t-t_0)}{\pi\sqrt{2\pi}\sqrt{z-i}}\int_{-\infty}^{0}\frac{dy}{X^+(y)\sqrt{y+i}(y-z)}.\end{aligned}$$

As a result of the change of variables $z=-\frac{1}{\xi}$, $y=-\frac{1}{y_0}$, we have

$$K_2^*(\xi,t)=\frac{-PH(t-t_0)X^*(\xi)\sqrt{\xi}}{\pi\sqrt{2\pi}\sqrt{1+i\xi}}\int_0^{\infty}\frac{dy_0}{X^{+*}(y_0)\sqrt{y_0(iy_0-1)}(y_0-\xi)},$$

where $K_2^*(\xi,t)=K_2(z,t)$, $X^*(\xi)=X(z)$. Hence, applying N. Muskhelishvili's formulas [13] on the behavior of the Cauchy-type integral in a neighborhood of a point $\xi=0$, we get

$$K_2^*(\xi,t)=O(1),\quad \xi\to 0.$$

Getting back to the original variable z, we conclude that the function $K_2(z,t)$ is bounded at infinity and $K_2(\infty,t)=\frac{PH(t-t_0)}{\sqrt{2\pi}}$. Therefore the function $\tilde{K}(z,t)=K(z,t)-\frac{2PH(t-t_0)}{\sqrt{2\pi}}$, $\operatorname{Im} z<0$ is holomorphic in the half-plane $\operatorname{Im} z<0$, vanishes at infinity by the order $O(|z|^{-(1/2-\delta)})$, $0<\delta<\frac{1}{2}$. Its boundary value $\widetilde{K}^-(y,t)$ is the Fourier transform of the function $\varphi_0'(\xi,t)$, which is continuous on the semi-axis $\xi\le 0$, except the point $\xi=0$, at which it may have a discontinuity of the second kind. Therefore

$$p(x,t)=\varphi'(x,t)=\frac{1}{\sqrt{2\pi}x}\lim_{\rho\to 0}\int\limits_{-\infty}^{\infty}\tilde{K}^-(y,t)e^{-\rho|y|}e^{-iy\ln x}dy. \tag{2.7}$$

Hence, by the inverse Fourier transform, we obtain the expression

$$\varphi_0'(\xi,t)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\tilde{K}^-(y,t)e^{-iy\xi}dy=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\left[\tilde{K}^-(y,t)-\frac{c(t)}{(\varepsilon+iy)^{1/2-\delta}}\right]e^{-iy\xi}dy$$

$$+\frac{c(t)}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{e^{-iy\xi}dy}{(\varepsilon+iy)^{1/2-\delta}}=\varphi_1(\xi,t)+\frac{c(t)e^{\varepsilon\xi}}{\xi^{1/2\,-\delta}},\quad \xi<0,$$

($\varphi_1(\xi,t)$ is a continuous function on the semi-axis $\xi\le 0$, ε is an arbitrary small positive number) and the behavior of the normal contact stresses in the neighborhood of the point $x=1$ has the form

$$p(x,t)=O(1-x)^{-1/2+\delta},\quad x\to 1-. \tag{2.8}$$

Now let us study the behavior of the function $p(x,t)$ in the neighborhood of the point $x=0$. By similar reasoning, from (2.6) we conclude that the boundary value of the function

$$K_0(z,t)=\frac{PH(t-t_0)}{\sqrt{2\pi}}-iz\Big(\frac{\Psi^+(z,t)-(2\pi)^{-1/2}\,iPH(t-t_0)G(z)}{zG(z)}\Big)=-i\Psi^+(z,t)G^{-1}(z)$$

is the Fourier transform of the function $\varphi_0'(\xi,t)$ and the function $\tilde{K}_0(z,t)=K_0(z,t)+\frac{PH(t-t_0)}{\sqrt{2\pi}}$ is holomorphic in the strip $D_0=\{z:0<\operatorname{Im} z<1\}$, except of the points that are the zeros of the function $G(z)$ in this strip and vanishes at infinity with order $|z|^{-(1/2-\delta_1)}$, $0<\delta_1<1/2$.

The function $G(z)$ has zero $z_0=iy_0$, $y_0=\frac{1}{\pi}\operatorname{arc}\cos(e_1æ_1)$ in the strip $0<\operatorname{Im} z<1$, thus using Cauchy's theorem on the residue to the function $e^{-i\xi z}\widetilde{K}_0(z,t)$, we obtain

$$p(x,t)=O(x^{y_0-1}),\quad x\to 0+. \tag{2.9}$$

Theorem. *Problem* (2.1)*,* (2.2) *has the solution, which is represented effectively by formula* (2.7) *and admits estimates* (2.8)*,* (2.9)*.*

Remark. For the estimate (2.9), the following conclusions are valid:

a) If $e_1>0$, $(\mu_2>\mu_1)$, then $0<y_0<1/2$;
b) If $e_1<0$, $(\mu_2<\mu_1)$, then $1/2<y_0<1$;
c) If $e_1=0$, $(\mu_2=\mu_1)$, then $y_0=1/2$.

References

1. V. M. Aleksandrov, S. M. Mkhitaryan, *Contact Problems for Bodies with thin Coverings and Layers.* (Rusian) Nauka, Moscow, 1983.
2. Yu. A. Antipov, N. Kh. Arutynyan, A contact problem for an elastic layer with cover plates in the presence of friction and adhesion. (Russian) *translated from Prikl. Mat. Mekh.* **57** (1993), no. 1, 137–147 *J. Appl. Math. Mech.* **57** (1993), no. 1, 159–170.
3. R. Ballarini, A rigid line inclusion at a bimaterial interface. *Eng. Fract. Mech.* **37** (1990), no. 1, 1–5.
4. R. Bantsuri, The first fundamental problem for a piecewise homogeneous orthotropic plane with a cut perpendicular to the line of interface. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **91** (1978), no. 3, 569–572.
5. R. Bantsuri, N. Shavlakadze, The contact problem for a piecewise-homogeneous plane with a semi-infinite inclusion. (Russian) *translated from Prikl. Mat. Mekh.* **73** (2009), no. 4, 655–662 *J. Appl. Math. Mech.* **73** (2009), no. 4, 471–477.

6. R. Bantsuri, N. Shavlakadze, The contact problem for a piecewise-homogeneous plane with a finite inclusion. (Russian) *translated from Prikl. Mat. Mekh.* **75** (2011), no. 1, 133–139 *J. Appl. Math. Mech.* **75** (2011), no. 1, 93–97.
7. F. D. Gakhov, Yu. I. Cherskii, *Equations of Convolution Type.* Nauka, Moscow, 1978.
8. Ts. Jamaspishvili, Complex representation in the plane theory of viscoelasticity and its applications. *Trans. A. Razmadze Math. Inst.* **174** (2020), no. 3, 351–357.
9. Ts. Jamaspishvili, N. Shavlakadze, The boundary value problems for piecewise-homogeneous viscoelastic plate. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.* **34** (2020), 34–37.
10. A. Khrapkov, The first fundamental problem for a piecewise-homogeneous plane with a cut perpendicular to a straight section. *Prikl. Mat. I Mekh.* **32** (1968), no. 4, 647–669.
11. Q. Li, T. C. T. Ting, Line inclusions in anisotropic elastic solids. *J. Appl. Mech.* **56** (1989), no. 3, 556–563.
12. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity.* Fundamental equations, plane theory of elasticity, torsion and bending. Translated by J. R. M. Radok. P. Noordhoff, Ltd., Groningen, 1953.
13. N. I. Muskhelishvili, *Singular Integral Equations.* Boundary Problems of Function Theory and their Application to Mathematical Physics. (Russian) 2nd. Fizmatgiz, Moscow, 1962; English translation: Wolters-Noordhoff Publishing, Groningen, 1967.
14. B. Nuller, The deformation of an elastic wedge- shaped plate supported by a rod of variable stiffness and a method of solving mixed problems. (Russian) *Prikl. Math. Mekh.* **40** (1976), no.2, 306–316.
15. G. Ja. Popov, *Concentration of Elastic Stresses Near Punches, Cuts, thin Inclusions and Supports.* (Russian) Nauka, Moscow, 1982.
16. N. Shavlakadze, On singularities of contact stress upon tension and bending of plates with elastic inclusions. *Proc. A. Razmadze Math. Inst.* **120** (1999), 135–147.
17. N. Shavlakadze, The contact problems of the mathematical theory of elasticity for plates with an elastic inclusion. *Acta Appl. Math.* **99** (2007), no. 1, 29–51.
18. T. C. T. Ting, P. Schiavone, Uniform antiplane shear stress inside an anisotropic elastic inclusion of arbitrary shape with perfect or imperfect interface bonding. *Internat. J. Engrg. Sci.* **48** (2010), no. 1, 67–77.
19. A. Ungiadze, The first fundamental problem for a piecewise homogeneous plane that contains a semi-infinite crack crossing the boundary at right angles. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR* **81** (1986), 79–86.

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