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ON THE COMPLEX REGIMES OF THE TAYLOR–DEAN FLOW BETWEEN TWO ROTATING POROUS CYLINDERS

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Abstract. The appearance of complex regimes in the Taylor–Dean flow between rotating porous cylinders heated up to different temperatures with radial throughflow is studied.As it has already been shown in [13] this flow takes place at the points of intersection of neutral curves corresponding to the axisymmetric and nonaxisymmetric disturbances as vortices and oscillatory modes in the form of traveling waves. The aim of the present paper is to investigate different regimes arising in a small neighborhood of the points of intersection of neutral curves corresponding to the flow instability and bifurcations.

Introduction

As is know, the flow between concentric cylinders, when the basic velocity distribution is the sum of a velocity distribution due to the rotation of the cylinders and a pumping velocity distribution due to a constant pressure gradient acting round the horizontal cylinders, is called the Taylor–Dean flow [2]. The problem of stability of the Taylor–Dean flow was first studied experimentally by Brewster and Nissan [1]. Later on, the theoretical analysis was carried out by various authors. In their works (see, e.g., [4] and references therein) we can find a theoretical analysis of the loss of stability for the Taylor–Dean flow of a viscous fluid, using approximation for close cylinders and also for cylinders with a wide gap. This problem was also studied for complex flows such as permeable cylinders with a radial flow, and heated cylinders with a temperature gradient (see, e.g., [3, 11] and references therein).

In these papers, investigations of the stability of main flows were basically studied in the linear approximation, i.e., for infinitesimal disturbances which gave a chance to study the problem of the first loss of stability of the Taylor–Dean flow and also the secondary modes bifurcating from this flow.

In our work, using the nonlinear analysis, we study bifurcation in the Taylor–Dean flow, the appearance of higher instabilities in the presence of radial flows and the case where cylinders are heated up to different temperatures.

1. Formulation of the Problem

We consider the annular space between two porous horizontal rotating cylinders partially filled with the heat-conducting liquid, maintained by a constant azimuthal pressure gradient. It is assumed that cylinders are heated up to different temperatures and the flow is subject to the action of a radial diverging and converging flows. We use the Navier–Stokes, the heat transfer, continuity and state equations in the cylindrical coordinates r, θ, z with the z-axis coinciding with that of the cylinders [10]:

$$\begin{gathered}\frac{dv'}{dt} = -\frac{1}{\rho'}\,\nabla\Pi' + \nu\Delta v', \\ \frac{\partial T'}{\partial t} = (v', \nabla T') + \chi\Delta T', \quad \frac{\partial \rho'}{\partial t} + \operatorname{div}(\rho'\,\mathrm{v}') = 0, \quad \rho' = \rho_0(1-\beta(T'-T_0)),\end{gathered} \tag{1.1}$$

and the boundary conditions

$$v'_r\Big|_{r=R_1} = Rv'_r\Big|_{r=R_2} = U_0, \quad v'_\theta = \Omega_i R_i, \quad v'_z = 0, \quad T' = T_i \ \ (i = 1, 2), \tag{1.2}$$

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where $R = \frac{R_2}{R_1}$, $v'(v'_r, v'_\theta, v'_z)$ is the velocity vector, U_0 is the radial velocity through the wall of the inner cylinder,

$$\nabla = \Big\{\frac{d}{dr}\,, \frac{1}{r}\,\frac{d}{d\theta}\,, \frac{d}{dz}\Big\}, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\,,$$
$$\frac{dv'}{dt} = \frac{\partial v'}{\partial t} + (v', \nabla)v' + \Big\{-\frac{1}{r}\,({v'_\theta}^2)\Big), \frac{1}{r}\,v'_r v'_\theta, 0\Big\}.$$

Under the above assumptions, of system (1.1)–(1.2) we obtain the following exact solution for the velocity V_0, temperature T_0, pressure Π_0:

$$V_0 = \big\{u_0(r), v_0(r), 0\big\}, \quad T_0 = c_1 + c_2 r^{\varkappa P_r},$$
$$u_0(r) = \frac{R_1 U_0}{r}\,, \quad v_0(r) = \begin{cases} \dfrac{K}{\varkappa}\Big(a r^{\varkappa+1} + \dfrac{b}{r} - r\Big) + A r^{\varkappa+1} + \dfrac{B}{r}, & \varkappa \neq -2, \\ \dfrac{K}{2}\Big(\dfrac{a_1 \ln r + b_1}{r}\Big) + \dfrac{A_1 \ln r + B_1}{r}, & \varkappa = -2, \end{cases} \tag{1.3}$$
$$\frac{\partial \Pi_0}{\partial r} = \frac{\rho(u_0^2 + v_0^2)}{r}\,,$$

where

$$K = \frac{1}{2\rho\nu}\Big(\frac{\partial \Pi_0}{\partial \theta}\Big)_0 = \text{const}\,, \quad a = \frac{R^2-1}{(R^{\varkappa+2}-1)R_1^{\varkappa}}\,, \quad a_1 = \frac{R_1^2(R^2-1)}{\ln R}\,,$$
$$b = \frac{R_2^2(R^{\varkappa}-1)}{R^{\varkappa+2}-1}, \quad b_1 = -\frac{R_1^2 \ln R_2 - R_2^2 \ln R_1}{\ln R}\,,$$
$$A = \frac{\Omega_1(\Omega R^2-1)}{(R^{\varkappa+2}-1)R_1^{\varkappa}}\,, \quad A_1 = \frac{\Omega_1 R_1^2(\Omega R^2-1)}{\ln R}\,,$$
$$B = \frac{\Omega_1 R_2^2(R^{\varkappa}-\Omega)}{R^{\varkappa+2}-1}, \quad B_1 = -\frac{\Omega_1 R_1^2(\ln R_2 - \Omega R^2 \ln R_1)}{\ln R}\,,$$
$$c_1 = \frac{T_1 R^{\mathrm{Pr}\,\varkappa} - T_2}{R^{\varkappa\,\mathrm{Pr}}-1}\,, \quad c_2 = \frac{T_2 - T_1}{R_1^{\varkappa\,\mathrm{Pr}}(R^{\varkappa\,\mathrm{Pr}}-1)}\,, \quad \Omega = \frac{\Omega_2}{\Omega_1}\,,$$

$\varkappa = \frac{U_0 R_1}{\nu}$ is the radial Reynolds number, $P_r = \frac{\nu}{\chi}$ is the Prandtl number, ν, χ, β are, respectively, the coefficients of kinematic viscosity, thermal diffusion and thermal expansion. The radial flow is inward for $\varkappa < 0$ (converging flow) and outward for $\varkappa > 0$ (diverging flow).

The flow (1.3) with the velocity vector V_0, temperature T_0 and pressure Π_0 will be called the main stationary flow. This flow is a superposition of the heat-conducting flow in the transverse direction (maintained by a pumping fluid round the cylinders) and a distribution of angular velocities (maintained by the rotation of both cylinders).

Let the perturbed state be taken as

$$v' = V_0 + v(v_r, v_\theta, v_z), \quad T' = T_0 + T, \quad \Pi' = \Pi_0 + \Pi. \tag{1.4}$$

Taking into account the fact that the main stationary flow (1.3) involves the rotating shear flow, we denote rotation shear S by $\frac{V_m}{d}$, where V_m is an average velocity in the azimuthal direction, $d = R_2 - R_1$ is a gap width between the cylinders. The basic velocity given in (1.3) can be written as $v_0 = V_m g(r)$, where $g(r) = \frac{v_o}{V_m}$. Introducing dimensionless variables for time, length, velocity, temperature and pressure by S, R_2, SR_2, $T_2 - T_1$, $\nu\rho' S$, respectively, the nonlinear system of perturbed equations takes the form [13]:

$$\frac{\partial v}{\partial t} + Nv - \frac{1}{\mathrm{Ta}}\,Mv + \frac{1}{\mathrm{Ta}}\,\nabla_1 \Pi = -\mathcal{L}(v,v), \quad (\nabla_1, rv) = 0, \quad v\Big|_{r=1/R,1} = 0, \tag{1.5}$$

where

$$
\begin{aligned}
Mv &= \Big\{\Delta_1 v_r - \frac{1-\varkappa}{r^2}\, v_r - \frac{2}{r^2}\,\frac{\partial v_\theta}{\partial \theta}, \Delta_1 v_\theta - \frac{1+\varkappa}{r^2}\, v_\theta + \frac{2}{r^2}\,\frac{\partial v_r}{\partial \theta}, \Delta_1 v_z, \frac{1}{\mathrm{Pr}}\,\Delta_1 T\Big\},\\
Nv &= \omega_1 \frac{\partial v}{\partial \theta} + \Big\{ \mathrm{Ra}\,\omega_2\,\mathrm{Ta} - 2\,\mathrm{Ta}\,\omega_1 v_\theta, -g_1 v_r, 0, \frac{g_2}{\mathrm{Pr}} v_r, \Big\},\\
\mathcal{L}(v,v) &= \Big\{(v,\nabla_1)v_r - \frac{v_\theta v_\theta}{r},\ (v,\nabla_1)v_\theta + \frac{v_r v_\theta}{r}, (v,\nabla_1)u_z, (v,\nabla_1)T_1\Big\},
\end{aligned}
$$

$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1-\varkappa}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\,, \quad \nabla_1 = \Big\{\frac{\partial}{\partial r}, \frac{1}{r}\,\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}, 0\Big\},$$

$$\mathrm{Ta} = \frac{\Omega_1 R_2^2}{\nu} \ \text{ is Taylor number},$$

$$\mathrm{Ra} = \frac{N}{\lambda}\,, \quad N = \frac{\beta(T_2 - T_1)}{2} \ \text{ is temperature gradient parameter},$$

$$\lambda = \frac{V_m}{\Omega_1 R_2} \ \text{ is ratio of the azimuthal average velocities and rotation},$$

$$V_m = K\,\frac{R_1 R^2}{R-1}\, D(R), \quad D(R) = \frac{R^\varkappa - 1}{R^{\varkappa+2} - 1}\,\ln R - \frac{\varkappa(R^2-1)}{2R^2(\varkappa+2)}\,,$$

$$\omega_1 = \frac{v_0(r)}{r} = \lambda g(r) + g_0(r), \quad \omega_2 = \omega_1^2 r,$$

$$g(r) = \frac{d}{R_2}\,\frac{D1(R) r^{\varkappa+2} + D2(R) - r^2}{r D(R)}\,, \quad g_0(r) = D3(R) r^{\varkappa+1} + \frac{D4(R)}{r}\,,$$

$$g_1(r) = -\Big(\frac{dv_0}{dr} + \frac{v_0}{r}\Big) = -\Big(\frac{d}{R_2}\,\frac{D1(R)\,(\varkappa+2) r^\varkappa - 2}{D(R)} + (\varkappa+2) r^\varkappa D3(R)\Big),$$

$$D1(R) = \frac{(R^2-1)R^\varkappa}{R^{\varkappa+2}-1}\,, \quad D2(R) = 1 - D1(R),$$

$$D3(R) = \frac{(\Omega R^2 - 1)R^\varkappa}{R^{\varkappa+2}-1}\,, \quad D4(R) = \frac{R^\varkappa - \Omega}{R^{\varkappa+2}-1}\,,$$

$$g_2(r) = \frac{\varkappa\,\mathrm{Pr}^2\, R^{\varkappa\,\mathrm{Pr}}}{R^{\varkappa\,\mathrm{Pr}} - 1}\, r^{\varkappa\,\mathrm{Pr} - 1}.$$

Problem (1.5) is written in terms of the Boussinesq approximation [7], which is based on the assumption that the thermal expansion coefficient is small. In the sequel, it will always be assumed that the velocity, temperature and pressure components are periodic with respect to z and θ with the known periods $2\pi/\alpha$ and $2\pi/m$, respectively.

2. The Amplitude System and Transitions

The stability of the main flow (1.3) depends on the following dimensional parameters: on the Taylor Ta, radial Reynolds $\varkappa$ and Prandtl Pr numbers, on the ratio of the cylinders radii R, and angular velocities Ω, radial axial, azimuthal wave numbers α, m, respectively, and also on the ratio of the parameter of temperature gradient N and the parameter $\lambda = \frac{V_m}{\Omega_1 R_2}$.

As is know from [13] for the definite parameters Ta, $\varkappa$, Pr, R, Ω, α, m, N and λ, after the loss of stability of the main flow, the neutral curves, dividing the stability and instability regions, have the crossing points corresponding to vortices and oscillatory modes in the form of traveling waves. This indicates that at those points after the loss of stability of the main flow in a small neighborhood of the points of intersection, we may expect the appearance of complex regimes.

Let $(\mathrm{Ra}_0, \mathrm{Ta}_0)$ be the points lying on the plane of parameters $(\mathrm{Ra}, \mathrm{Ta})$ and corresponding to the intersection of the neutral curves corresponding to the convective symmetric and non-axisymmetric three-dimensional loss of stability of the main stationary flow (1.3).

To investigate the secondary flows and the appearance of high instabilities in the flow (1.3), we use the nonlinear theory of bifurcation of hydrodynamic flows with cylindrical symmetry (see [5,6,9,14]).

This theory allows one to find various liquid motion regimes existing in the vicinity of the points of intersection of neutral curves corresponding to the two types of secondary instability-vortices and azimuthal waves in the flow. This theory has been applied to a rather wide class of problems, where the object of this investigation is a nonlinear dynamical system of six-dimensional amplitude equations, which are a generalization of Landau's amplitude equations. Numerical studies will be carried out by the methods adopted for investigation of flow instability in various fluids (see [12] and references therein; a detailed description is given in [8]).

The $SO(2) * O(2)$ symmetry enables one to reduce the six-dimensional amplitude system to the four-dimensional motor subsystem for the modules of amplitudes with free parameters σ, μ_r (the damping decrements of the monotonic and oscillatory perturbations, respectively).

To the equilibria of this system, lying on the invariant subspaces, there correspond the motions of a fluid having a concrete physical nature: the main stationary flow; vortex flows, i.e., a secondary stationary axisymmetric flow; purely azimuthal waves, i.e., secondary oscillatory modes; spiral waves, i.e., secondary autooscillatory modes; mixed azimuthal waves, i.e., three-frequency regimes; equilibria not lying on the invariant subspaces, i.e., equilibria of a general state, each corresponding to a quasiperiodic two-frequency solution of the amplitude system.

As our calculations show, the motor system depending on the parameter values of the problem, may have no equilibria or may have equilibria of the above-mentioned types. It is found that transition schemes may turn out to be either rather complicated or absolutely trivial.

In Figures 1–4, we present the scheme of equilibria bifurcations of the motor subsystem, which we consider the most interesting and allowing us to judge about the transition characteristics of the system under consideration.

The single lines show symmetric equilibria, the double lines indicate connected pair of equilibria. Stable equilibria are drawn by solid lines and unstable equilibria by dotted lines. The circles are the points at which the motor subsystem limit cycles bifurcate.

We present here several of our results obtained for $R = 2$ (the radius of the outer cylinder is two times greater than that of the inner one), $P_r = 7$ (the working medium is water), $m = 0, 1$, for different values of α and small absolute values of $\varkappa$.

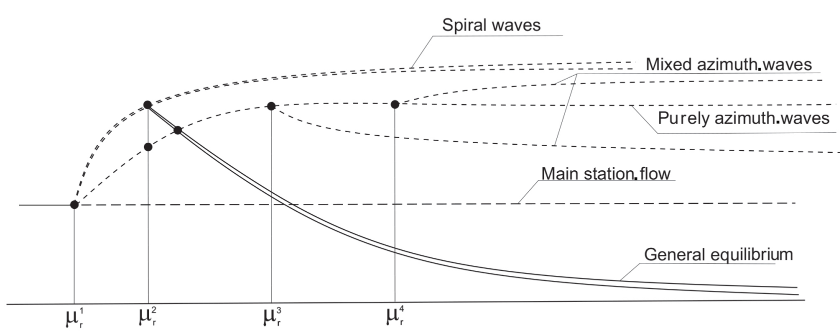

FIGURE 1. $\sigma = -10$, $\lambda = 1$, $\Omega = -0.2$, $\varkappa = -1.9$, $\mathrm{Ra}_0 = -3.12$, $\alpha = 5$, $\mathrm{Ta} = 61.734$, $c_0 = 2.55$. Bifurcation values: $\mu_r^1 = 0$, $\mu_r^2 = 0.2057$, $\mu_r^3 = 0.38$, $\mu_r^4 = 0.612$.

Figures 1–4 show the schemes of transitions of short–wave perturbations, when axial number $\alpha = 4, 5$ for small absolute values of radial Raynolds number $\varkappa$, angular velocities ratios Ω and the frequency of neutral azimuthal waves c_0.

For the opposite rotating cylinders, for example (Figure 1), $\Omega = -0.2$, and $\varkappa = -1.9$ (converging flow) the main stationary flow exists for any value of parameter μ_r. It is stable for $\mu_r < 0$ and unstable for $\mu_r > 0$. For $\mu_r = 0$, there simultaneously bifurcate unstable spiral and purely azimuthal waves. For $\mu_r = \mu_r^2$, from spiral waves branches off a stable connected pair of general equilibria (not lying on the invariant planes), and for $\mu_r = \mu_r^3$, from purely azimuthal waves branch off mixed azimuthal waves.

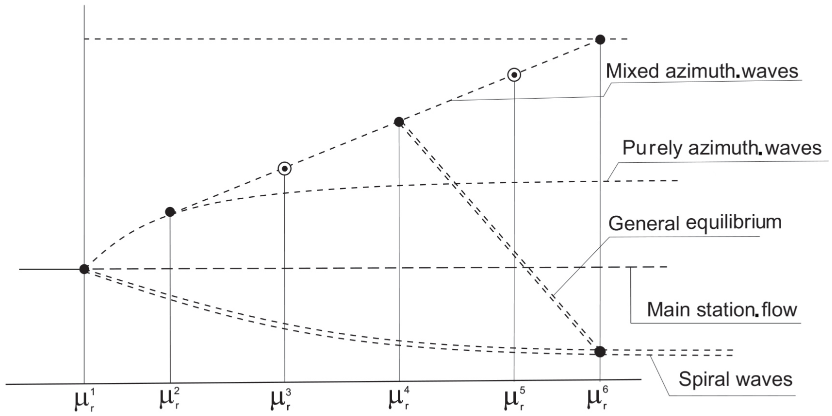

FIGURE 2. $\sigma = -10$, $\lambda = 1$, $\Omega = 0.1$, $\varkappa = -1.9$, $\alpha = 5$, $c_0 = 2.6678$, $\mathrm{Ra}_0 = -1.9809$, $\mathrm{Ta}_0 = 55.138$. Bifurcation values: $\mu_r^1 = 0$, $\mu_r^2 = 0.11297$, $\mu_r^3 = 0.12018$, $\mu_r^4 = 0.13419$, $\mu_r^5 = 0.2837$, $\mu_r^6 = 1.1$.

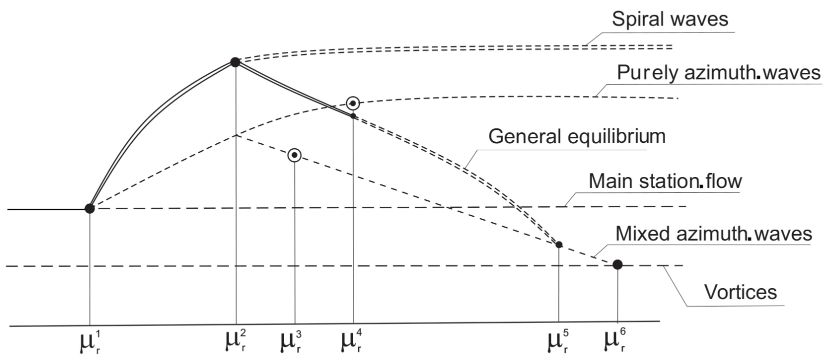

FIGURE 3. $\lambda = 1$, $\sigma = -10$, $\Omega = 0$, $\varkappa = -1.9$, $\mathrm{Ra}_0 = -2.2118$, $\mathrm{Ta}_0 = 59.377$, $\alpha = 5$, $c_0 = 2.12$. Bifurcation values: $\mu_r^1 = 0$, $\mu_r^2 = 0.28$, $\mu_r^3 = 0.31$, $\mu_r^4 = 0.346$, $\mu_r^5 = 0.587$, $\mu_r^6 = 0.8504$.

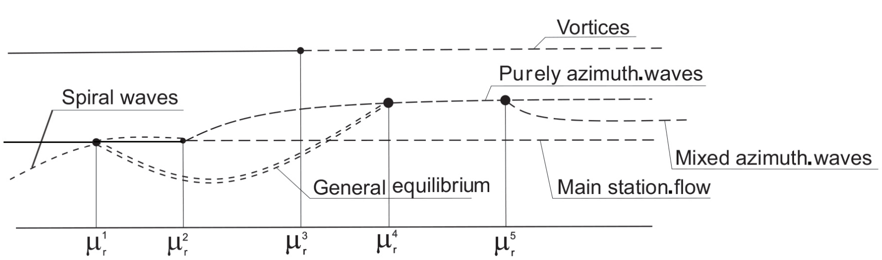

FIGURE 4. $\sigma = 10$, $\lambda = 1$, $\Omega = -0.2$, $\mathrm{Ra}_0 = -0.0256$, $\varkappa = 0.5$, $\alpha = 4$, $\mathrm{Ta}_0 = 87.46$, $c_0 = 2.43416$. Bifurcation values: $\mu_r^1 = -0.841$, $\mu_r^2 = 0$, $\mu_r^3 = 0.35$, $\mu_r^4 = 0.959$, $\mu_r^5 = 2.513$.

In case $\Omega = -0.2$ and $\varkappa = 0.5$, for a diverging flow (Figure 4), we have a subcritical bifurcation in $\mu_r < 0$. The stable main flow loses its stability and spiral waves branch off into the subcritical region. For $\mu_r = \mu_r^1$, from spiral waves branch off an unstable connected pair of general equilibria.

For $\mu_r > 0$, there are in general unstable equilibria. In this case, after the loss of stability of the main flow, there arise rather complex fluid motions.

If the cylinders rotate in the same direction or rotates only the inner cylinder and the outer is at rest (Figures 2, 3), then we have a somewhat different picture. As (Figure 2) shows, for $\Omega = 0.1$, $\varkappa = -1.9$ (converging flow), we have found the unstable limit cycles branching off from the mixed azimuthal waves for $\mu_r = \mu_r^3$ and $\mu_r = \mu_r^5$. In case $\Omega = 0$, $\varkappa = -1.9$ (Figure 3), from mixed azimuthal and purely azimuthal waves bifurcate unstable limit cycles for $\mu_r = \mu_r^3$ and $\mu_r = \mu_r^4$. So, there are several stable equilibria in the range $\mu_r^1 < \mu_r = \mu_r^4$.

3. Conclusion

We have presented the results of the numerical analysis of different regimes arising in the Taylor–Dean flow between rotating porous cylinders heated up to different temperatures with radial throughflow. When the liquid pumping and the inner cylinder rotate in the same direction, we have found different regimes, appearing in a small neighborhood of the points of intersection of neutral curves corresponding to the flow instability.

Depending on the direction of rotation of cylinders, after the loss of stability of the main flow, there arise different modes of a fluid motion. Our calculations have shown that under certain parameters, in the case of differently rotating cylinders, we have generally unstable equilibria of the motor system. Therefore, experimentally, in this case one may expect complex regimes of fluid motion.

When cylinders rotate in same directions or rotates only the inner cylinder, we have found that for certain parameters of the problem, unstable limit cycles branch off from some equilibria (they correspond to three-frequency periodic modes of motion). In some cases, when the outer cylinder is at rest there are simultaneously several stable equilibria, and therefore the hysteresis phenomena can be observed in real experiments [8].

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