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ON THE DISCRETE PROBLEM OF WAVE DIFFRACTION BY SEMI-INFINITE RIGID CONSTRAINT

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Abstract. The discrete problem of wave diffraction by a semi-infinite rigid constraint is revised in view of its well-posedness upon wave numbers belonging to the stop-band. The problem is analyzed with the help of difference potentials and Toeplitz operators in the space of square summable sequences. We obtain the result of the unique solvability and derive the representation formula of solutions.

1. Introduction

This paper is motivated on the one hand by the discrete analogue of diffraction by a Sommerfeld 'soft' half-plane ribbon [1, 8, 9, 14] and on another hand by application areas of recent interest: metamaterials and analog circuits [7, 12, 16]. As an example of such an application, let us consider two-dimensional passive propagation media that can be used for signal processing and filtering. Assume that these media consist of a lattice of repeated cells of a single type at the fine-scale. By way of illustration, we can take a lossy host microstrip line network periodically loaded with series capacitors and shunt inductors as shown in Figure 1. This type of inductor-capacitor lattice is referred to as a negative-refractive-index transmission-line (NRI-TL) metamaterial [4, 6], or simply, left-handed 2D metamaterial. Assume that the number of unit cells in this slab is large enough to make it prohibitively expensive to solve numerically for the voltage/current at every cell in the lattice until the system reaches a steady-state. As a simplifying strategy, it can be anticipated that the limiting case, when the lattice is effectively infinite, is more amenable to analysis and provides a good approximation of the steady-state output at the exterior boundary. Thus, we suppose that monochromatic inputs are applied on the input nodes lying on the half-line, cf. Figure 1.

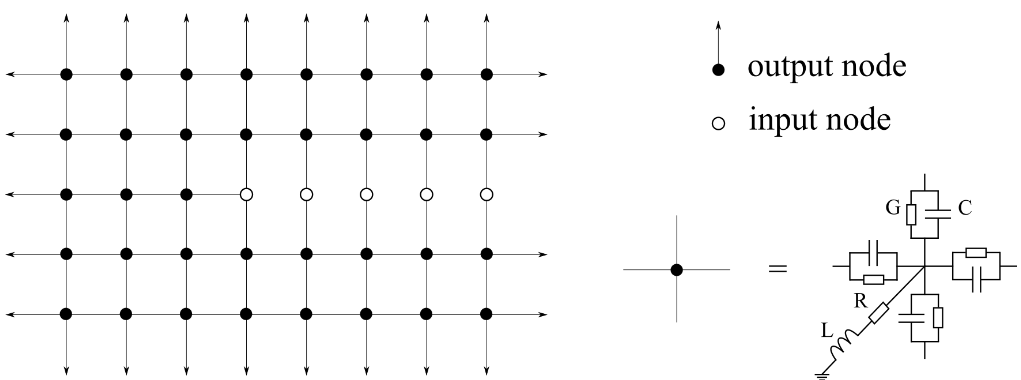

Figure 1. The unit cell of the lossy 2-D inductor-capacitor TL. The conductance G and resistance R account for the losses inherent to the series capacitor C and shunt inductor L.

Mathematical modeling of such wave diffraction problems leads us to study a discrete 2D Helmholtz equation with input data prescribed on a semi-infinite row of lattice sites. When one interested in

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an analysis of regular processes in which waves corresponding to the microstructural scales can be neglected, then the continuum limit of corresponding equations can be investigated. In this case, we arrive at the famous problem in applied mathematics, the Sommerfeld 'soft' plane problem [9, 14]. It is well known that the classical continuum model of wave diffraction can be considered only as of the slowly-varying approximation of a discrete or structured material. Therefore there is an obvious necessity to avoid continuum limits and instead analyze directly the discrete Helmholtz diffraction problems. Moreover, the development of computers and programming softwares preserves high interest in numerical discretizations of the continuum wave equation too.

It worth mentioning that in contrast to [17], where the analogous problem was studied by Sharma using the discrete Wiener–Hopf method, we propose another method of investigation and provide rigorous analysis of the problem in view of its well-posedness upon wave numbers belonging to the stop-band. Namely, we derive the unique solvability result and solutions representation formula in the space of square-summable sequences with the help of difference potentials and Toeplitz operators.

2. Basic Notations and Formulation of the Problem

Following the customary notation in mathematics, let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of integers, positive integers, non-negative integers, real numbers and complex numbers, respectively. We denote by $e_1 = (1,0)$, $e_2 = (0,1)$ the standard base of the square lattice $\mathbb{Z}^2$ $(= \mathbb{Z}\times\mathbb{Z})$.

For any point $x = (x_1, x_2) \in \mathbb{Z}^2$ we define the 4-neighbourhood F_x^0 as the set of points $\{(x_1 - 1, x_2), (x_1 + 1, x_2), (x_1, x_2 - 1), (x_1, x_2 + 1)\}$ and the neighbourhood F_x as $F_x^0 \bigcup \{x\}$. We say that $R \subset \mathbb{Z}^2$ is a region if there exist disjoint nonempty subsets $\mathring{R}$ and ∂R of R such that

(a) $R = \mathring{R} \cup \partial R$,
(b) if $x \in \mathring{R}$ then $F_x \subset R$,
(c) if $x \in \partial R$ then there is at least one point $y \in F_x^0$ such that $y \in \mathring{R}$.

Clearly, the subsets $\mathring{R}$ and ∂R are not defined uniquely by R, but henceforth, for a given region R in $\mathbb{Z}^2$ it will always be assumed that $\mathring{R}$ and ∂R are also given and fixed. Then we say that x is an interior (boundary) point of R if $x \in \mathring{R}$ $(x \in \partial R)$. Further, a region $R \subset \mathbb{Z}^2$ is said to be connected if for any $y, z \in R$ there exists a sequence $x^{(1)}, \dots, x^{(n)} \in R$ with $x^{(1)} = y$ and $x^{(n)} = z$, such that for all $0 \le i \le n-1$, $|x^{(i)} - x^{(i+1)}| = 1$. By the definition, a region R with one interior point x is connected and coincides with F_x. Denote by S_N a region defined as a discrete square $([-N,N]^2 \cap \mathbb{Z}^2)\backslash\{(N,N), (-N,N), (-N,N), (N,-N)\}$, $N \in \mathbb{Z}^+$, where $\mathring{S}_N := [-N+1, N-1]^2 \cap \mathbb{Z}^2$ and $\partial S_N := S_N \backslash \mathring{S}_N$ will be fixed throughout the paper.

A boundary point $y \in \partial R$ is said to be

$$\begin{aligned} &\text{a left point if} && y + e_1 \in \mathring{R},\\ &\text{a right point if} && y - e_1 \in \mathring{R},\\ &\text{a top point if} && y - e_2 \in \mathring{R},\\ &\text{a bottom point if} && y + e_2 \in \mathring{R}. \end{aligned}$$

The union of all left (right, top, and bottom) points is denoted by ∂R_l (∂R_r, ∂R_t, and ∂R_b, respectively) and called a side of the boundary ∂R. Note that a boundary point y may simultaneously be a left, right, top, and bottom point. Thus, ∂R_l, ∂R_r, ∂R_t and ∂R_b may overlap each other. Clearly, ∂R is the union of its four sides, $\partial R = \partial R_l \cup \partial R_r \cup \partial R_t \cup \partial R_b$.

Let $\Gamma := \{(x_1, 0) : x_1 \in \mathbb{N}\}$ and $\mathring{\Omega} := \mathbb{Z}^2 \backslash \Gamma$. Then we set $\partial\Omega = \Gamma$ and $\Omega = \mathring{\Omega} \cup \partial\Omega$. Describing the problem and assumptions mentioned in the Introduction, we suppose that there is an inductor connecting each node $x \in \mathring{\Omega}$ to a common ground plane, and there is a capacitor connecting each node $x \in \mathring{\Omega}$ to its four nearest neighbors $(x_1 \pm 1, x_2 \pm 1)$ (cf., Figure 1). Assume that all inductances, capacitances, resistances and all conductances are equal to L, C, R and G, respectively. Note that L, C, R, and G are positive constants. Then Kirchhoffs laws of voltage and current (while suppressing the explicit dependence on time t) imply the following second-order equation for the voltage $U(x)$

across the inductor at node x:

$$LC\frac{d^2}{dt^2}(\Delta_d U(x)) + (RC+GK)\frac{d}{dt}(\Delta_d U(x)) + RG(\Delta_d U(x)) = U(x). \tag{1}$$

Here, Δ_d denotes the discrete Laplacian defined as follows:

$$\Delta_d U(x) = \sum_{i=1}^{2}(U(x+e_i)+U(x-e_i)) - 4U(x). \tag{2}$$

We specify that (1) holds for all $x \in \mathring{\Omega}$, and we have the time-dependent boundary condition along the boundary $\partial\Omega$,

$$U(y) = f(y)e^{-\iota\omega t} \quad \text{on } \Gamma, \tag{3}$$

where ι denotes the imaginary unit, and $f \in \ell^2(\Gamma)$ is a given square-summable function (in fact, a sequence) on Γ.

We assume that at time $t = 0$ the functions $U(x)$ and all its derivatives are zero for all $x \in \mathring{\Omega}$. Then, as t increases, the boundary term causes a wave to propagate into the lattice, and the system approaches steady state. At this point the solution is given by $U(x) = u(x)e^{-\iota\omega t}$. Substituting this expression into (1) and (3), for the discrete Helmholtz equation in Ω, we obtain the following problem:

$$(\Delta_d + k^2 + \iota\varepsilon)u(x) = 0, \qquad \text{in } \mathring{\Omega}, \tag{4a}$$

$$u(y) = f(y), \quad \text{on } \Gamma, \tag{4b}$$

where k and ω are related through the formula

$$k^2 = \frac{LC\omega^2 - RG}{(LC\omega^2 - RG)^2 + (RC+GL)^2}$$

and

$$\varepsilon = \frac{RC+GL}{(LC\omega^2 - RG)^2 + (RC+GL)^2}.$$

Thus, we are interested in studying the problem of the existence and uniqueness of a square-summable function u on $\mathbb{Z}^2$, i.e., $u \in \ell^2(\mathbb{Z}^2)$ such that $u(x)$ satisfies the discrete Helmoltz equation (4a) with $\varepsilon > 0$ and the boundary condition (4b). From now on, we will refer to this problem as Problem $\mathcal{P}_{\mathrm{D}}$.

3. Green's Representation Formula

Let R be a region in $\mathbb{Z}^2$. As it has already been mentioned above, $y \in \partial R$ may be a point of intersection of several sides of ∂R. However, in our arguments presented below, it will always be clear which of the sides requires consideration. Under this condition we define the discrete derivative in the outward normal direction

$$Tu(y) = u(y) - u(y-\nu_y), \quad y \in \partial R,$$

where ν_y is $-e_1$ (e_1, e_2 or $-e_2$) if y is an element of ∂R_l (∂R_r, ∂R_t, or ∂R_b).

Let R be a finite region. Then we have discrete analogues of Green's first and second identities [10]:

$$\sum_{x\in\mathring{R}}(\nabla_d^+ u(x)\cdot\nabla_d^+ v(x) + \nabla_d^- u(x)\cdot\nabla_d^- v(x) + u(x)\Delta_d v(x)) = \sum_{y\in\partial R} u(y)Tv(y), \tag{5}$$

and

$$\sum_{x\in\mathring{R}}(u(x)\Delta_d v(x) - v(x)\Delta_d u(x)) = \sum_{y\in\partial R}(u(y)Tv(y) - v(y)Tu(y)). \tag{6}$$

Here, we denote by $\sum_{y\in\partial R}$ the following sum:

$$\sum_{y\in\partial R} := \sum_{y\in\partial R_r} + \sum_{y\in\partial R_l} + \sum_{y\in\partial R_t} + \sum_{y\in\partial R_b},$$

and the discrete gradients ∇_d^+ and ∇_d^- are defined as follows:

$$\nabla_d^+ u(x) := \begin{pmatrix} u(x+e_1) - u(x) \\ u(x+e_2) - u(x) \end{pmatrix}$$

and

$$\nabla_d^- u(x) := \begin{pmatrix} u(x-e_1) - u(x) \\ u(x-e_2) - u(x) \end{pmatrix}.$$

The next step in deriving Green's representation formula is introduction of Green's function. Denote by $\mathcal{G}(x-y)$ the Green's function for (4a) centered at the point x and evaluated at y. Then $\mathcal{G}(x-y)$ satisfies

$$(\Delta_d + k^2 + \iota\varepsilon)\mathcal{G}(x-y) = \delta_{x,y}, \tag{7}$$

where $\delta_{x,y}$ is the Kronecker delta. The lattice Green's function $\mathcal{G}$ is quite well known (cf., e.g., [5,11,13]) and can be written in the following form

$$\mathcal{G}(x) = \mathcal{G}(x_1, x_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{e^{-\iota(x_1\xi_1 + x_2\xi_2)}}{\sigma(\xi_1, \xi_2; k, \varepsilon)} \mathrm{d}\xi_1 \mathrm{d}\xi_2, \tag{8}$$

or, equivalently, as

$$\mathcal{G}(x) = \mathcal{G}(x_1, x_2) = \frac{1}{\pi^2} \int_{0}^{\pi}\int_{0}^{\pi} \frac{\cos(x_1\xi_1)\cos(x_2\xi_2)}{\sigma(\xi; k, \varepsilon)} \mathrm{d}\xi_1 \mathrm{d}\xi_2, \tag{9}$$

where

$$\begin{aligned} \sigma(\xi; k) &= k^2 + \iota\varepsilon - 4 + 2\cos\xi_1 + 2\cos\xi_2 \\ &= k^2 + \iota\varepsilon - 4\sin^2\frac{\xi_1}{2} - 4\sin^2\frac{\xi_2}{2} \\ &= k^2 + \iota\varepsilon - 8 + 4\cos^2\frac{\xi_1}{2} + 4\cos^2\frac{\xi_2}{2} \\ &= k^2 + \iota\varepsilon - 4 + 4\cos\frac{\xi_1+\xi_2}{2}\cos\frac{\xi_1-\xi_2}{2}, \quad \xi = (\xi_1, \xi_2). \end{aligned} \tag{10}$$

Notice that if $k^2 \in \mathbb{C}\backslash[0,8]$, then $\sigma \neq 0$ and, consequently, $\mathcal{G}(x)$ in (8) is well defined. In this case, $\mathcal{G}(x)$ decays exponentially when $|x| \to \infty$. Finally, notice that $\mathcal{G}(x-y) = \mathcal{G}(y-x)$, and we denote it by $\mathcal{G}(x;y)$.

For any function $\varphi : \partial R \to \mathbb{C}$, we define the difference single- and double-layer potentials as follows:

$$V\varphi(x) = \sum_{y\in\partial R} \mathcal{G}(x;y)\varphi(y), \quad \text{for all } x \in \mathbb{Z}^2, \tag{11}$$

and

$$W\varphi(x) = \sum_{y\in\partial R} \big(T\mathcal{G}(x;y) + \delta_{x,y}\big)\varphi(y), \quad \text{for all } x \in \mathbb{Z}^2, \tag{12}$$

respectively. The role of the summand $\delta_{x,y}$ is clarified by the following result [10]: for every $x \in \mathring{R}$, we have

$$(\Delta_d + k^2 + \iota\varepsilon)V\varphi(x) = 0 \quad \text{and} \quad (\Delta_d + k^2 + \iota\varepsilon)W\varphi(x) = 0. \tag{13}$$

Theorem 3.1 (cf. [10])**.** *Let R be a finite region. If u is a solution to the discrete Helmholtz equation in $\mathring{R}$, then at any point $x \in \mathring{R}$, we have a discrete Green's representation formula*

$$u(x) = Wu(x) - V(Tu)(x). \tag{14}$$

4. Uniqueness and Existence Results

We strat this section by proving the following uniqueness result.

Theorem 4.1. *The $\mathcal{P}_{\mathrm{D}}$ has at most one solution.*

Proof. Let N be a sufficiently large positive number and set $\mathring{\Omega}_N = \mathring{S}_N \cap \mathring{\Omega}$. Let u be a solution of the homogeneous problem. Then the first Green's identity (5) for u and its complex conjugate $\bar{u}$ in the region $\mathring{\Omega}_N$, together with zero boundary conditions on $\partial\mathring{\Omega}_N$, yields

$$\sum_{x\in\mathring{\Omega}_N} \left(|\nabla_d^+ u(x)|^2 + |\nabla_d^- u(x)|^2 - (k^2 + \iota\varepsilon)|u(x)|^2\right) = \sum_{y\in\partial\mathring{S}_N} u(y)Tu(y). \tag{15}$$

From the real and imaginary parts of the last identity, we obtain

$$-\varepsilon \sum_{x\in\mathring{\Omega}_N} |u(x)|^2 = \Im\mathrm{m} \sum_{y\in\partial\mathring{S}_N} u(y)Tu(y).$$

Note, since $u \in \ell^2(\mathbb{Z}^2)$, there is a monotonic sequence of positive numbers $\{N_j\}$ such that $N_j \to \infty$ as $j \to \infty$ and

$$\lim_{j\to\infty} \sum_{y\in\partial\mathring{S}_{N_j}} u(y)Tu(y) = 0. \tag{16}$$

Indeed, due to $u \in \ell^2(\mathbb{Z}^2)$, we find that the sum

$$\sum_{N=0}^{\infty} \sum_{y\in\partial\mathring{S}_{N_j}} |u(y)|^2$$

is finite. This fact in particular implies that there exists a monotonic sequence of positive numbers $\{N_j\}$ such that $N_j \to \infty$ as $j \to \infty$ and

$$\sum_{y\in\partial\mathring{S}_{N_j}} |u(y)|^2 = \bar{o}(N_j^{-1})$$

and consequently,

$$\sum_{y\in\partial\mathring{S}_{N_j}} |Tu(y)|^2 = \bar{o}(N_j^{-1}).$$

Further, applying the Cauchy-Schwarz inequality for every N_j, we get

$$\Bigg|\sum_{y\in\partial\mathring{S}_{N_j}} u(y)Tu(y)\Bigg| \le \Bigg(\sum_{y\in\partial\mathring{S}_{N_j}} |u(y)|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{y\in\partial\mathring{S}_{N_j}} |Tu(y)|^2\Bigg)^{\frac{1}{2}} = \bar{o}(N_j^{-1}) \text{ as } j \to \infty,$$

and therefore we obtain (16).

Since the expressions under the sum on the left-hand side of the equalities in (15) are non-negative, have that this sum is monotonic with respect to N. This observation together with (16) implies

$$\sum_{x\in\mathbb{Z}^2} |u(x)|^2 = \lim_{N\to\infty} \sum_{x\in\mathring{R}_N} |u(x)|^2 = 0.$$

Thus it follows from the last identity that $u \equiv 0$ in $\mathbb{Z}^2$. □

Now we are ready to show the existence result. Let us look for a solution to Problem $\mathcal{P}$ in the following form

$$u(x) = V\varphi(x) = \sum_{y\in\Gamma} \mathcal{G}(x;y)\varphi(y), \tag{17}$$

where $\varphi(y) \in \ell^2(\Gamma)$ is an unknown function. Then, due to the discrete Youngs inequality (cf. [15, Lemma 3.3.30]), together with a fact $\varphi(y)\delta_{x,y} \in \ell^2(\mathbb{Z}^2)$, we conclude that $u \in \ell^2(\mathbb{Z}^2)$. Moreover, (13) implies that u is a solution to the discrete Helmoltz equation (4a). Further, we need to satisfy the boundary condition (4b) which yields

$$\sum_{y_1=0}^{\infty} \mathcal{G}(x_1 - y_1, 0)\varphi(y_1) = f(x_1), \quad x_1 \in \mathbb{N},$$

or equivalently,

$$A\Phi = F, \tag{18}$$

where

$$A = \begin{pmatrix} \mathcal{G}(0,0) & \mathcal{G}(-1,0) & \mathcal{G}(-2,0) & \dots \\ \mathcal{G}(1,0) & \mathcal{G}(0,0) & \mathcal{G}(-1,0) & \dots \\ \mathcal{G}(2,0) & \mathcal{G}(1,0) & \mathcal{G}(0,0) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

is a Toeplitz matrix, and

$$\Phi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \vdots \end{pmatrix}, \qquad F = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix}.$$

Now it remains to show that the Toeplitz operator A is invertible in $\ell^2(\mathbb{N})$ spaces.

Theorem 4.2. *The operator $A : \ell^2(\Gamma) \to \ell^2(\Gamma)$ is invertible.*

Proof. Notice that

$$\mathcal{G}(x_1, 0) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{-\iota x_1 \xi_1}}{\alpha(\xi_1) + 2\cos\xi_2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{a}(\xi_1) e^{-\iota x_1 \xi_1} \mathrm{d}\xi_1,$$

where

$$\alpha(\xi) = \alpha(\xi; k) = k^2 + \iota\varepsilon - 4 + 2\cos\xi_1$$

and

$$\tilde{a}(\xi_1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\xi_2}{\alpha(\xi_1; k) + 2\cos\xi_2} = \frac{1}{2\pi\iota} \oint_C \frac{dz}{z^2 + \alpha(\xi_1)z + 1}.$$

Here, C is the positively oriented complex unit circle. From the residue theory, we derive

$$\tilde{a}(\xi_1) = \frac{1}{2\pi\iota} \oint_C \frac{dz}{(z - z_{int}(\xi_1))(z - z_{ext}(\xi_1))} = \frac{1}{z_{int}(\xi_1) - z_{ext}(\xi_1)},$$

where $z_{int}(\xi_1)$ and $z_{ext}(\xi_1)$ are the roots of the polynomial $z^2 + \alpha(\xi_1)z + 1$ such that $|z_{int}(\xi_1)| < 1$, $|z_{ext}(\xi_1)| > 1$ and $z_{int}(\xi_1) z_{ext}(\xi_1) = 1$.

Then the symbol of the Toeplitz matrix $A = T(a)$ is given by

$$a(z) = a(e^{\iota\xi_1}) = \tilde{a}(\xi_1) = \tilde{a}(\mathrm{Arg}(z)), \quad z = e^{\iota\xi_1}.$$

Let us show that $z_{int}(\xi_1) - z_{ext}(\xi_1) \notin \mathbb{R}$. Indeed, if $z_{int}(\xi_1) - z_{ext}(\xi_1) \in \mathbb{R}$, then

$$\begin{aligned} (z_{int} - z_{ext})^2 &= (z_{int} + z_{ext})^2 - 4 z_{int} z_{ext} = \alpha^2 - 4 \\ &= (k^2 - 4 + 2\cos\xi_1)^2 - \varepsilon^2 - 4 + 2\iota\varepsilon(k^2 - 4 + 2\cos\xi_1) \end{aligned}$$

implies

$$k^2 - 4 + 2\cos\xi_1 = 0.$$

In this case we have $\alpha = \iota\varepsilon$ and the roots of the quadratic equation are pure imaginary

$$z_{int} = \iota \frac{\sqrt{\varepsilon^2 + 4} - \varepsilon}{2}, \quad z_{ext} = \iota \frac{-\sqrt{\varepsilon^2 + 4} - \varepsilon}{2},$$

and consequently $z_{int} - z_{ext} \notin \mathbb{R}$.

This observation, in particular, implies $\Im\mathrm{m}\, a(z) \neq 0$ and therefore $\Re\mathrm{e}\, \iota a(z) \neq 0$. The latter shows that there is $\epsilon > 0$ such that $\Re\mathrm{e}\, \iota a(z) \geq \epsilon$ a.e. on the complex unit circle, thus $a(z)$ is sectorial. Since a is sectorial, due to the Brown-Halmos theorem, the operator $A = T(a)$ is invertible (cf. [3, Theorem 2.17], [2]). □

From the direct combination of the results obtained above we have the main conclusion for the problems under consideration.

Theorem 4.3. *Problem $\mathcal{P}$ is uniquely solvable. Its solution can be represented as a difference potential* (17)*, where $\varphi \in \ell^2(\Gamma)$ is a unique solution of equation* (18)*.*

References

1. J. D. Achenbach, *Wave Propagation in Elastic Solids.* North-Holland, Amsterdam, 1984.
2. A. Böttcher, B. Silbermann, *Introduction to Large Truncated Toeplitz Matrices.* Universitext. Springer-Verlag, New York, 1999.
3. A. Böttcher, B. Silbermann, *Analysis of Toeplitz Operators.* Second edition. Prepared jointly with Alexei Karlovich. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006.
4. C. Caloz, T. Itoh, *Electromagnetic Metamaterials: Transmission Line Theory and Microwave Applications the Engineering Approach.* John Wiley & Sons, Inc. Hoboken, New Jersey, 2006.
5. E. N. Economou, *Green's Functions in Quantum Physics.* Third edition. Springer Series in Solid-State Sciences, 7. Springer-Verlag, Berlin, 2006.
6. G. V. Eleftheriades, K. G. Balmain, *Negative-Refraction Metamaterials: Fundamental Principles and Applications.* Wiley-IEEE Press, New Jersey, 2005.
7. N. Engheta, R. W. Ziolkowski, *Metamaterials: Physics and Engineering Explorations.* Wiley-IEEE Press, Piscataway, New Jersey, 2006.
8. J. G. Harris, *Linear Elastic Waves.* Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001.
9. D. S. Jones, A simplifying technique in the solution of a class of diffraction problems. *Quart. J. Math. Oxford Ser. (2)* **3** (1952), 189–196.
10. D. Kapanadze, Exterior diffraction problems for two-dimensional square lattice. *Z. Angew. Math. Phys.* **69** (2018), no. 5, Paper No. 123, 17 pp.
11. S. Katsura, S. Inawashiro, Lattice Green's Functions for the Rectangular and the Square Lattices at Arbitrary Points, *J. Math. Phys.* **12** (1971), 1622–1630.
12. R. Marqus, F. Martín, M. Sorolla, *Metamaterials with Negative Parameters: Theory, Design, and Microwave Applications.* Wiley, Hoboken, New Jersey, 2008.
13. P. A. Martin, Discrete scattering theory: Green's function for a square lattice. *Wave Motion* **43** (2006), no. 7, 619–629.
14. B. Noble, *Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations.* International Series of Monographs on Pure and Applied Mathematics, vol. 7 Pergamon Press, New York-London-Paris-Los Angeles 1958.
15. M. V. Ruzhansky, V. Turunen, *Pseudo-Differential Operators and Symmetries.* Background analysis and advanced topics. Pseudo-Differential Operators. Theory and Applications, 2. Birkhuser Verlag, Basel, 2010.
16. A. K. Sarychev, V. M. Shalaev, *Electrodynamics of Metamaterials.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.
17. B. L. Sharma, Diffraction of waves on square lattice by semi-infinite rigid constraint. *Wave Motion* **59** (2015), 52–68.

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