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BASIC BOUNDARY VALUE PROBLEMS FOR CIRCULAR RING WITH VOIDS

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Abstract. In the present paper, the static two-dimensional problems for an elastic material with voids are consider. The corresponding system of differential equations is written in a complex form and its general solution is presented with the use of two analytic functions of a complex variable and a solution of the Helmholtz equation. The boundary value problems are solved for a circular ring when the stress tensor and the equilibrated stress vector are given on the boundary.

Introduction

Material having small distributed voids may be called porous material or material with voids. The nonlinear and linear theories for the behaviour of porous solids, in which the skeletal or matrix material is elastic and the interstices are voids of the material, were developed by Nunziato and Cowin [3, 9]. The intended application of the theory of elastic material with voids may be found in many elds of science and technology (geology, biology, medicine, engineering, oil exploration industry, nanotechnology, etc.).

In this article, the plane strain for elastic materials with voids is considered. The corresponding system of differential equations is written in a complex form and its general solution is presented by using two analytic functions of a complex variable and a solution of the Helmholtz equation [4]. The boundary value problems are solved for a circular ring.

In resent years, many authors have investigated the BVPs for elastic materials with voids, using the theory elaborated by Cowin jointly with his collaborators [1, 2, 5, 7, 10, 11].

Basic (Governing)Equations of the Plane Strain

Let $x = (x_1,\ x_2)$ be a point of the Euclidean two- dimensional space E^2. Assume that the isotropic material with voids occupies the domain D. The basic system of equations of motion in the linear theory of elasticity with voids, for isotropic materials can be written as [5]

• Equations of equilibrium

$$\begin{gathered}\partial_j t_{ij} = 0, \quad i, j = 1, 2,\\ \partial_j h_j + g = 0,\end{gathered} \tag{1}$$

where t_{ij} is the symmetric stress tensor, h_j is the equilibrated stress vector, g is the intrinsic equilibrated body force.

• Constitutive equations

$$\begin{gathered}t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + \beta\phi\delta_{ij}, \quad k = 1, 2,\\ h_j = \alpha\partial_j\phi,\\ g = -\xi\phi - \beta e_{kk},\end{gathered} \tag{2}$$

where λ and μ are the Lamé constants, α, β and ξ are the constants characterizing the body porosity, δ_{ij} is the Kronecker delta, ϕ is the change of the volume fraction, e_{ij} is the strain tensor and $e_{ij} = 0.5\,(\partial_j u_i + \partial_i u_j)$, where u_i is the components of the displacement vector.

On the plane Ox_1x_2, we introduce the complex variable $z = x_1 + ix_2 = re^{i\vartheta}$, $(i^2 = -1)$ and the operators $\partial_z = 0.5(\partial_1 - i\partial_2)$, $\partial_{\bar z} = 0.5(\partial_1 + i\partial_2)$, $\bar z = x_1 - ix_2$, and $\Delta = 4\partial_z\partial_{\bar z}$.

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To write system (1) in a complex form, we multiply the second equation of this system by i and sum up with the first equation

$$\begin{gathered}\partial_z(t_{11}-t_{22}+2it_{12})+\partial_{\bar z}(t_{11}+t_{22})=0,\\ \partial_z h_+ + \partial_{\bar z}\bar h_+ + g = 0,\end{gathered}\tag{3}$$

where $h_+ = h_1 + ih_2$ and then we rewrite formulas (2) as follows:

$$\begin{gathered}t_{11}-t_{22}+2it_{12}=4\mu\partial_{\bar z}u_+,\\ t_{11}+t_{22}=2(\lambda+\mu)\theta+2\beta\phi,\\ h_+=2\alpha\partial_{\bar z}\phi,\\ g=-\xi\phi-\beta\theta,\end{gathered}\tag{4}$$

$$\theta=\partial_z u_+ + \partial_{\bar z}\bar u_+,\quad u_+=u_1+iu_2.$$

Substituting relations (4) into system (3), we have

$$\begin{gathered}2\mu\partial_{\bar z}\partial_z u_+ + (\lambda+\mu)\partial_{\bar z}\theta+\beta\partial_{\bar z}\phi=0,\\ (\alpha\Delta-\xi)\phi-\beta\theta=0.\end{gathered}\tag{5}$$

The general solution of system (5) is represented as

$$\begin{gathered}2\mu u_+=\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}-\frac{4\alpha\beta\mu}{\xi(\lambda+2\mu)-\beta^2}\partial_{\bar z}\chi(z,\bar z),\\ \phi=\chi(z,\bar z)-\frac{\beta}{\xi(\lambda+\mu)-\beta^2}(\varphi'(z)+\overline{\varphi'(z)}),\end{gathered}$$

where $\kappa=\frac{\xi(\lambda+3\mu)-\beta^2}{\xi(\lambda+\mu)-\beta^2}$, $\varphi(z)$ and $\psi(z)$ are arbitrary analytic functions of a complex variable z, and $\chi(z,\bar z)$ is an arbitrary solution of the Helmholtz equation

$$\Delta\chi(z,\bar z)-\gamma^2\chi(z,\bar z)=0,$$

where $\gamma^2=\frac{\xi(\lambda+2\mu)-\beta^2}{\alpha(\lambda+2\mu)}$ and $\gamma>0$ [6].

From (4), we have

$$\begin{gathered}t_{11}-t_{22}+2it_{12}=-2z\overline{\varphi''(z)}-2\overline{\psi'(z)}-\frac{8\alpha\beta\mu}{\xi(\lambda+2\mu)-\beta^2}\partial_{\bar z}\partial_{\bar z}\chi(z,\bar z),\\ t_{11}+t_{22}=2\left(\varphi'(z)+\overline{\varphi'(z)}\right)+\frac{2\mu\beta}{\lambda+2\mu}\chi(z,\bar z),\\ h_+=2\alpha\partial_{\bar z}\chi(z,\bar z)-\frac{2\alpha\beta}{\xi(\lambda+\mu)-\beta^2}\overline{\varphi''(z)},\\ g=\left(\frac{\beta^2}{\lambda+2\mu}-\xi\right)\chi(z,\bar z).\end{gathered}\tag{6}$$

Assume that mutually perpendicular unit vectors $\mathbf{l}$ and $\mathbf{s}$ are such that

$$\mathbf{l}\times\mathbf{s}=\mathbf{e}_3,$$

where $\mathbf{e}_3$ is the unit vector directed along the x_3-axis. The vector $\mathbf{l}$ forms the angle ϑ with the positive direction of the x_1-axis. Then the displacement components $u_l=\mathbf{u}\cdot\mathbf{l}$, $u_s=\mathbf{u}\cdot\mathbf{s}$ as well as the stress and moment stress components acting on an arbitrarily oriented area are expressed by the formulas

$$\begin{gathered}u_l+iu_s=e^{-i\vartheta}u_+,\\ t_{ll}+it_{ls}=\frac{1}{2}\left[t_{11}+t_{22}+(t_{11}-t_{22}+2it_{12})e^{-2i\vartheta}\right],\\ h_l=\frac{1}{2}\left[h_+e^{-i\vartheta}+\bar h_+e^{i\vartheta}\right].\end{gathered}\tag{7}$$

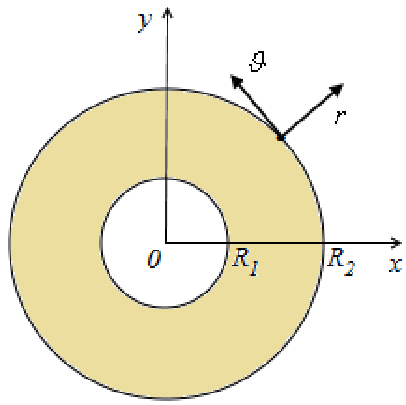

FIGURE 1. The circular ring

THE PROBLEM FOR A CIRCULAR RING

We consider the boundary value problem for a concentric circular ring of radii R_1 and R_2 (Figure 1). On the circumference, we consider the following boundary value problem [8]:

$$t_{rr}+it_{r\vartheta}=\begin{cases}\sum\limits_{-\infty}^{+\infty}A'_n e^{in\vartheta}, & |z|=R_1,\\ \sum\limits_{-\infty}^{+\infty}A''_n e^{in\vartheta}, & |z|=R_2,\end{cases}\qquad h_r=\begin{cases}\sum\limits_{-\infty}^{+\infty}B'_n e^{in\vartheta}, & |z|=R_1,\\ \sum\limits_{-\infty}^{+\infty}B''_n e^{in\vartheta}, & |z|=R_2.\end{cases}\tag{8}$$

By substituting formulas (6) into (7), the boundary conditions (8) may then be written as follows:

$$\begin{gathered}\varphi'(z)+\overline{\varphi'(z)}+\frac{\beta\mu}{\lambda+2\mu}\chi(z,\bar z)-\left[z\overline{\varphi''(z)}+\overline{\psi'(z)}+\frac{4\alpha\beta\mu}{\xi(\lambda+2\mu)-\beta^2}\partial_{\bar z}\partial_{\bar z}\chi(z,\bar z)\right]e^{-2i\vartheta}\\ =\begin{cases}\sum\limits_{-\infty}^{+\infty}A'_n e^{in\vartheta}, & |z|=R_1,\\ \sum\limits_{-\infty}^{+\infty}A''_n e^{in\vartheta}, & |z|=R_2,\end{cases}\\ \left[\alpha\partial_{\bar z}\chi(z,\bar z)-\frac{\alpha\beta}{\xi(\lambda+\mu)-\beta^2}\overline{\varphi''(z)}\right]e^{-i\vartheta}+\left[\alpha\partial_{z}\chi(z,\bar z)-\frac{\alpha\beta}{\xi(\lambda+\mu)-\beta^2}\varphi''(z)\right]e^{i\vartheta}\\ =\begin{cases}\sum\limits_{-\infty}^{+\infty}B'_n e^{in\vartheta}, & |z|=R_1,\\ \sum\limits_{-\infty}^{+\infty}B''_n e^{in\vartheta}, & |z|=R_2.\end{cases}\end{gathered}\tag{9}$$

The analytic functions $\varphi'(z)$, $\psi'(z)$ and the metaharmonic function $\chi(z,\bar z)$ are represented in the form of the following series [8]

$$\begin{gathered}\varphi'(z)=\delta\ln z+\sum_{-\infty}^{+\infty}a_n z^n,\quad \psi'(z)=\sum_{-\infty}^{+\infty}b_n z^n,\\ \chi(z,\bar z)=\sum_{-\infty}^{+\infty}\left(\alpha_n I_n(\gamma r)+\beta_n K_n(\gamma r)\right)e^{in\vartheta},\end{gathered}\tag{10}$$

where $I_n(\gamma r)$ and $K_n(\gamma r)$ are the modified Bessel functions of n-th order.

We use the condition of single-valuedness of displacements which in the present case is expressed as

$$\delta = 0, \quad \kappa a_{-1} + \bar{b}_{-1} = 0. \tag{11}$$

Substituting the formulas (10) in (9), one finds that

$$\sum_{-\infty}^{+\infty} a_n r^n e^{in\vartheta} + \sum_{-\infty}^{+\infty} (1-n)\bar{a}_n r^n e^{-in\vartheta} - \sum_{-\infty}^{+\infty} \bar{b}_{n-2} r^{n-2} e^{-in\vartheta}$$

$$+\frac{\mu\beta}{\lambda+2\mu}\sum_{-\infty}^{+\infty}\frac{2n+2}{\gamma r}\left(I_{n+1}(\gamma r)\alpha_n - K_{n+1}(\gamma r)\beta_n\right)e^{in\vartheta} = \begin{cases}\sum\limits_{-\infty}^{+\infty} A'_n e^{in\vartheta}, & |z| = R_1,\\ \sum\limits_{-\infty}^{+\infty} A''_n e^{in\vartheta}, & |z| = R_2,\end{cases}$$

$$\alpha\gamma\sum_{-\infty}^{+\infty}\left(I'_n(\gamma r)\alpha_n - K'_n(\gamma r)\beta_n\right)e^{in\vartheta} - \frac{\alpha\beta}{\xi(\lambda+\mu)-\beta^2}$$

$$\times\sum_{-\infty}^{+\infty} n\left(a_n r^{n-1} e^{in\vartheta} + \bar{a}_n r^{n-1} e^{-in\vartheta}\right) = \begin{cases}\sum\limits_{-\infty}^{+\infty} B'_n e^{in\vartheta}, & |z| = R_1,\\ \sum\limits_{-\infty}^{+\infty} B''_n e^{in\vartheta}, & |z| = R_2.\end{cases}$$

As a conclusion of the previous relations, we use the following well-known formulas

$$I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x} I_n(x), \quad I_{n-1}(x) + I_{n+1}(x) = 2I'_n(x),$$

$$K_{n-1}(x) - K_{n+1}(x) = -\frac{2n}{x} K_n(x), \quad K_{n-1}(x) + K_{n+1}(x) = -2K'_n(x).$$

Comparison of terms independent of ϑ gives

$$\begin{aligned}
&2a_0 - R_1^{-2}\bar{b}_{-2} + \frac{2\mu\beta}{(\lambda+2\mu)\gamma R_1}\left(I_1(\gamma R_1)\alpha_0 - K_1(\gamma R_1)\beta_0\right) = A'_0,\\
&2a_0 - R_2^{-2}\bar{b}_{-2} + \frac{2\mu\beta}{(\lambda+2\mu)\gamma R_2}\left(I_1(\gamma R_2)\alpha_0 - K_1(\gamma R_2)\beta_0\right) = A''_0,\\
&I_1(\gamma R_1)\alpha_0 - K_1(\gamma R_1)\beta_0 = \frac{B'_0}{\alpha\gamma},\\
&I_1(\gamma R_2)\alpha_0 - K_1(\gamma R_2)\beta_0 = \frac{B''_0}{\alpha\gamma}.
\end{aligned} \tag{12}$$

Here, the assumption has been made that $a_0 = \bar{a}_0$ i.e., that a_0 is real, since any constant imaginary part of $\varphi'(z)$ does not affect the stress distribution [5].

Comparison of terms involving $e^{in\vartheta}$ for $n = \pm1, \pm2, \dots$ gives

$$\begin{aligned}
&R_1^n a_n + (1+n)R_1^{-n}\bar{a}_{-n} - R_1^{-n-2}\bar{b}_{-n-2} + \frac{(2n+2)\mu\beta}{(\lambda+2\mu)\gamma R_1}\\
&\qquad\times\left(I_{n+1}(\gamma R_1)\alpha_n - K_{n+1}(\gamma R_1)\beta_n\right) = A'_n,\\
&R_2^n a_n + (1+n)R_2^{-n}\bar{a}_{-n} - R_2^{-n-2}\bar{b}_{-n-2} + \frac{(2n+2)\mu\beta}{(\lambda+2\mu)\gamma R_2}\\
&\qquad\times\left(I_{n+1}(\gamma R_2)\alpha_n - K_{n+1}(\gamma R_2)\beta_n\right) = A''_n,
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\alpha\gamma\left(I'_n(\gamma R_1)\alpha_n + K'_n(\gamma R_1)\beta_n\right) - \frac{\alpha\beta n}{\xi(\lambda+\mu)-\beta^2}\left(R_1^{n-1}a_n - R_1^{-n-1}\bar{a}_{-n}\right) = B'_n,\\
&\alpha\gamma\left(I'_n(\gamma R_2)\alpha_n + K'_n(\gamma R_2)\beta_n\right) - \frac{\alpha\beta n}{\xi(\lambda+\mu)-\beta^2}\left(R_2^{n-1}a_n - R_2^{-n-1}\bar{a}_{-n}\right) = B''_n.
\end{aligned} \tag{14}$$

Dividing the first equation of (13) by R_1^{-n-2}, and the second by R_2^{-n-2}, and then subtracting, one obtains the first of the following formulae:

$$\begin{aligned}(R_2^{2n+2} - R_1^{2n+2})a_n + (1+n)(R_2^2 - R_1^2)\bar{a}_{-n} + \frac{(2n+2)\mu\beta}{(\lambda+2\mu)\gamma}\big[(R_2^{n+1}I_{n+1}(\gamma R_2)\\ -R_1^{n+1}I_{n+1}(\gamma R_1))\alpha_n - (R_2^{n+1}K_{n+1}(\gamma R_2) - R_1^{n+1}K_{n+1}(\gamma R_1))\beta_n\big] = B_n,\\ (1-n)(R_2^2 - R_1^2)a_n + (R_2^{-2n+2} - R_1^{-2n+2})\bar{a}_{-n} + \frac{(-2n+2)\mu\beta}{(\lambda+2\mu)\gamma}\big[(R_2^{-n+1}I_{n-1}(\gamma R_2)\\ -R_1^{-n+1}I_{n-1}(\gamma R_1))\alpha_n - (R_2^{-n+1}K_{n-1}(\gamma R_2) - R_1^{-n+1}K_{n-1}(\gamma R_1))\beta_n\big] = \bar{B}_{-n},\end{aligned} \tag{15}$$

where $B_n = R_2^{n+2}A''_n - R_1^{n+2}A'_n$.

The coefficients a_n, b_n, α_n and β_n are found by solving (11)–(15).

It is easy to prove the absolute and uniform convergence of the series obtained in the circular ring (including the contours) when the functions set on the boundaries have sufficient smoothness.

The procedure of solving the boundary value problem remains the same in the case where the stresses and the change of the volume fraction, or the displacement vector and the change of the volume fraction, or the change of the volume fraction and the equilibrated stress vector are prescribed on the boundary of the domain.

References

1. L. Bitsadze, Explicit solutions of the BVPs of the theory of thermoelasticity for an elastic circle with voids and microtemperatures. *ZAMM Z. Angew. Math. Mech.* **100** (2020), no. 10, e201800303, 13 pp.
2. M. Ciarletta, A. Scalia, M. Svanadze, Fundamental solution in the theory of micropolar thermoelasticity for materials with voids. *J. Thermal Stresses* **30** (2007), no. 3, 213–229.
3. S. C. Cowin, J. W. Nunziato, Linear elastic materials with voids. *J. Elasticity* **13** (1983), no. 2, 125-147.
4. B. Gulua, R. Janjgava, On construction of general solutions of equations of elastostatic problems for the elastic bodies with voids. *PAMM* **18** (2018) no. 1, e201800306.
5. B. Gulua, R. Janjgava, Boundary value problems of the plane theory of elasticity for materials with voids. *Applications of mathematics and informatics in natural sciences and engineering*, 227–236, Springer Proc. Math. Stat., 334, *Springer, Cham*, 2020.
6. B. Gulua, T. Kasrashvili, Some basic problems of the plane theory of elasticity for materials with voids. *Seminar of I. Vekua Institute of Applied Mathematics, REPORTS* **46** (2020), 27–36.
7. R. Janjgava, B. Gulua, S. Tsotniashvili, Some boundary value problems for a micropolar porous elastic body. *Arch. Mech. (Arch. Mech. Stos.)* **72** (2020), no. 6, 485–509.
8. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity.* Nauka, Moscow, 1966.
9. J. W. Nunziato, S. C. Cowin, A nonlinear theory of elastic materials with voids. *Arch. Rational Mech. Anal.* **72** (1979/80), no. 2, 175–201.
10. M. M. Svanadze, Potential method in the theory of thermoviscoelasticity for materials with voids. *Journal of Thermal Stressesthis* **37** (2014), no. 8, 905–927.
11. I. Tsagareli, Explicit solution of elastostatic boundary value problems for the elastic circle with voids. *Adv. Math. Phys.* **2018**, Art. ID 6275432, 6 pp.

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