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SHARP OLSEN'S INEQUALITY FOR MULTILINEAR RIESZ POTENTIALS

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Abstract. In this note, a sharp Olsen's type inequality for multilinear Riesz potential operator $\mathcal{I}_\alpha$ is presented. The derived result yields a complete characterization of the trace inequality for $\mathcal{I}_\alpha$ in Morrey spaces. As a consequence, we have a sharp Olsen's inequality for the linear Riesz potentials I_α.

Olsen's inequality plays an important role in the study of perturbed Schrödinger equation (see [22]). For further improvement of Olsen's original inequality and its applications we refer to [26, 27].

Our aim in this note is to establish the following sharp Olsen's type inequality:

$$\Big\| g(\mathcal{I}_\alpha \overrightarrow{f}) \Big\|_{L^q_r} \le C \Big\| g \Big\|_{L^q_\ell} \prod_{j=1}^{m} \Big\| f_j \Big\|_{L^{p_j}_{s_j}}, \tag{1}$$

where $\mathcal{I}_\alpha$ is the multilinear fractional integral operator, L^q_r, L^q_ℓ, $L^{p_j}_{s_j}$, $j = 1, \dots, m$, are Morrey spaces defined on $\mathbb{R}^n$ with certain parameters. Taking $m = 1$ in (1), we get sharp Olsen's inequality for linear fractional integrals I_α.

Inequality (1) is sharp in the sense that it provides a complete characterization of the weighted inequality for a weight function V (trace inequality):

$$\|\mathcal{I}_\alpha \overrightarrow{f}\|_{L^q_r(V)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}_{s_j}}.$$

The latter result for the linear case $m = 1$ (i.e., when $\mathcal{I}_\alpha = I_\alpha$) and for the Lebesgue spaces (i.e., for $p = s$ and $q = r$) goes back to Adams [1]. It was proved in [12] for the Lebesgue spaces in the multilinear setting ($q = r$, $p_i = s_i$, $i = 1, \dots, m$), while for the linear case it was established in [3] for Morrey spaces defined with respect to measures. In the latter paper, the problem has been studied for fractional integrals defined on quasi-metric measure spaces.

Let

$$\mathcal{I}_\alpha(\vec{f})(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\dots f_m(y_m)}{(|x-y_1| + \dots + |x-y_m|)^{mn-\alpha}} d\vec{y}, \quad x \in \mathbb{R}^n,$$

be multilinear fractional integral, where $0 < \alpha < nm$, $\vec{f} := (f_1, \dots, f_m)$, $\vec{y} := (y_1, \dots, y_m)$, $d\overrightarrow{y} = dy_1 \dots dy_m$.

For $m = 1$, the operator $\mathcal{I}_\alpha$ is the linear Riesz potential operator I_α defined by the formula

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad 0 < \alpha < n, \quad x \in \mathbb{R}^n.$$

The Riesz potentials and their applications play a fundamental role in Harmonic Analysis and its applications to PDEs. For example, their role in the theory of Sobolev embeddings (see, e.g., [18]) is also worth mentioning.

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Historically, multilinear fractional integrals were introduced in [4, 5, 11]. In particular, those works deal with the operator

$$B_\alpha(f,g)(x) = \int\limits_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\alpha}} dt, \quad 0<\alpha<n.$$

In particular, if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, then B_α is bounded from $L^{p_1} \times L^{p_2}$ to L^q.

As a tool to understand B_α, the operator $\mathcal{I}_\alpha$ was studied as well.

Let $1 \le q \le r < \infty$ and let V be a weight function (i.e., V is a locally integralble a.e. positive function on $\mathbb{R}^n$). We denote by $L^q_r(V)$ a class of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^q_r(V)} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\frac{1}{q}-\frac{1}{r}}} \bigg(\int\limits_Q |f(x)|^q V(x)dx \bigg)^{1/q} < \infty,$$

where $\mathcal{Q}$ is the class of all cubes Q with sides parallel to the coordinate exes.

The weak weighted Morrey space $WL^q_r(V)$ is defined with respect to the norm

$$\|f\|_{WL^q_r(V)} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\frac{1}{q}-\frac{1}{r}}} \sup_{\lambda>0} \lambda \bigg(\int\limits_{\{x:|f(x|>\lambda\}} V(x)dx \bigg)^{1/q}.$$

It is clear that $WL^q_r(V) \hookrightarrow L^q_r(V)$.

Morrey spaces introduced in 1938 by C. Morrey in relation to regularity problems of solutions of partial differential equations turned out to be a useful tool in the regularity theory of PDE's.

If V is a constant function, then we denote $L^q_r(V)$ and $WL^q_r(V)$ by L^q_r and WL^q_r, respectively. In case $q=r$, we have weighted Lebesgue spaces denoted by $L^q(V)$ and $WL^q(V)$, respectively.

In his paper [19], K. Moen gave one–weight characterization for $\mathcal{I}_\alpha$. The weighted problems were also studied in the works [2, 7, 12–14, 17, 24, 28], etc.

The weighted Morrey spaces were introduced by Komori and Shirai [15] in 2009. In their paper, the authors studied the boundedness of singular integral operators in those spaces. In the definition of weighted Morrey space introduced in [15], the weighted norm $\|\chi_Q f\|_{L^p(W)}$ is divided by $W(Q)^\lambda$, where W is the weight function. In the present note, we give weighted norm inequalities for fractional integral operators in different type weighted Morrey spaces. In our case, the weighted norm $\|\chi_Q f\|_{L^p(W)}$ is divided by $|Q|^{1/p-1/s}$. Such weighted Morrey spaces were also considered in [25]. For weighted results regarding fractional integrals I_α and corresponding fractional maximal operators in Morrey spaces we refer to the papers [20, 21, 23, 25]. The mapping properties for multilinear fractional integrals in unweighted and weighted Morrey spaces were studied in [6, 8–10, 14] (see also references cited in [14]). In [8] and [9], the Olsen-type inequalities for multilinear fractional integrals are also studied.

Our main results read as follows:

Theorem 1. *Let $1<q\le r<\infty$, $1<p_i\le s_i<\infty$, $i=1,\dots,m$, $p<q$, $0<\alpha<\frac{n}{s}$, $\frac{1}{p}-\frac{1}{q}=\frac{1}{s}-\frac{1}{r}=\frac{\alpha}{n}-\frac{1}{\ell}$, where $\frac{1}{s}=\sum\limits_{j=1}^m \frac{1}{s_j}$, $\frac{1}{p}=\sum\limits_{j=1}^m \frac{1}{p_j}$. Then there exists a positive constant C depending only on n, α, q, r, p_i, s_i, $i=1,\dots,m$, such that for all $f_j \in L^{p_j}_{s_j}$, $j=1,\dots,m$, inequality* (1) *holds.*

Theorem 2. *Let $1<q\le r<\infty$, $1<p_i\le s_i<\infty$, $i=1,\dots,m$, $p<q$, $0<\alpha<\frac{n}{s}$, $\frac{1}{p}-\frac{1}{q}=\frac{1}{s}-\frac{1}{r}$, where $\frac{1}{s}=\sum\limits_{j=1}^m \frac{1}{s_j}$, $\frac{1}{p}=\sum\limits_{j=1}^m \frac{1}{p_j}$. Suppose that V is a weight function on $\mathbb{R}^n$. Then the following statements are equivalent:*

(i) there is a positive constant C such that

$$\|\mathcal{I}_\alpha \overrightarrow{f}\|_{L^q_r(V)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}_{s_j}}.$$

(ii) the inequality

$$\|\mathcal{I}_\alpha \overrightarrow{f}\|_{WL^q_r(V)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}_{s_j}}$$

holds;

(iii) the condition

$$[V]_{\alpha,p,q} := \sup_{Q\in\mathcal{Q}} \left(\int\limits_Q V(x)(x)dx \right)^{\frac{1}{q}} |Q|^{\frac{\alpha}{n}-\frac{1}{p}} < \infty$$

is satisfied.

Moreover, norms of the operator $\|\mathcal{I}_\alpha\| \approx [V]_{\alpha,p,q}$.

The following statement gives the weighted sharp Olsen's inequality for the linear Riesz potentials I_α.

Theorem 3. *Let $1 < q \le r < \infty$, $1 < p \le s < \infty$, $p < q$ and $0 < \alpha < \frac{n}{s}$. Let $\frac{1}{p} - \frac{1}{q} = \frac{1}{s} - \frac{1}{r} = \frac{\alpha}{n} - \frac{1}{\ell}$. Then the inequality*

$$\big\|g(I_\alpha f)\big\|_{L^q_r} \le C\big\|g\big\|_{L^q_\ell}\big\|f\big\|_{L^p_s}$$

holds with the positive constant C, independent of f, g.

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