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A COMPARISON BETWEEN BERNOULLI-COLLOCATION METHOD AND HERMITE–GALERKIN METHOD FOR SOLVING TWO-DIMENSIONAL MIXED VOLTERRA–FREDHOLM SINGULAR INTEGRAL EQUATIONS

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Abstract. In this paper, a numerical solution of two-dimensional singular integral equations is proposed. For this, two operative methods are demonstrated, Bernoulli polynomials with collocation method and Hermite polynomials through Galerkin method which is a useful technique in two-dimensional integral equations. Various numerical examples are presented to illustrate the efficiency of these two methods. Maple 17 program will be used to solve the system numerically.

1. Introduction

In the last years, there was a significant importance of multidimensional singular integral equations (MSIE). Many problems in physical, biological and applied mathematics fields reduce to a singular integral equation. Such as hydrodynamics, population genetics, elasticity, and others. In 1928, F. G. Tricomi [20] was the first who proposed an important study concerning (MSIE). He considered double singular integrals. Recently, many researchers had studied the numerical solutions of singular integral equations in several formulas. For instance, [2, 7] involved solutions of the nonlinear singular integral equations, whereas in [6,15] with Hilbert kernel. J. Obaiyst et al. [11] deal with hypersingular integral equations. E. Hashim [5], V. A. Zisis and E. G. Ladopoulus [21] presented solutions for the singularity of linear integral equations. S. Banerjea et al. [3] worked on a weak singular kernel with a water wave problem as an application. There are different methods for solving two-dimensional integral equations (see [4, 8–10] and others). M. Rahman [14] discussed the solution of linear integral equations in one-dimension using the Hermite–Galerkin method. In our paper, we work on the solution of two-dimensional singular mixed Volterra–Fredholm integral equations using the Bernoulli-collocation method and Hermite–Galerkin method. One can observe that the Hermite–Galerkin method is a novel technique in the two-dimensional integral equations.

The aim of this paper is to convert the singular integral equation to a non-singular form by repealing the singularity and then converting it into a system of algebraic equations based on orthogonal polynomials.

The next sections are arranged as follows; some definitions and properties of Bernoulli and Hermite polynomials are introduced in Section 2. The description of the collocation and Galerkin methods with two-dimensional singular mixed Volterra–Fredholm integral equations are explained in Section 3. Section 4 includes some numerical examples that illustrate the above-mentioned methods. Finally, Section 5 gives the conclusions.

We list here some of the most important advantages of the proposed methods.

- The proposed methods are easy to implement, and it is a powerful mathematical tool to obtain the numerical solution of various kind of problems with little additional works.
- By using these methods, the problem under consideration is transformed into a system of algebraic equations which can be solved via a suitable numerical method.

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2. Some Definitions and Properties

2.1. **Bernoulli Polynomials.** In many topics of mathematics, Bernoulli polynomials have a vital role, e.g., in the theory of numbers [13] and in complex differential equations [19].

The Bernoulli polynomials are expressed by the formula [19]

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k, \tag{2.1}$$

where $\binom{n}{k} = \frac{n!}{k!\ (n-k)!}$ and $B_n(x)$ is the Bernoulli polynomial of n^{th} degree.

In a special case, if $x = 0$ in (2.1), then $B_n(0) = B_n$ are called Bernoulli numbers, and $B_0 = 1$.

The Bernoulli numbers can be calculated as follows:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (1-n)\ x^n \ , \quad n = 0, 1, 2, \dots .$$

The first few Bernoulli polynomials are

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \quad B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x,$$
$$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 + \frac{1}{2}x^2 - \frac{1}{42}.$$

2.2. **Hermite Polynomials [12].** The differential equation $y'' - 2xy' + 2\lambda y = 0$ has polynomial solutions called Hermite polynomials which were introduced for the first time by Pierre–Simon Laplace in 1810. Charles Hermite defined the multidimensional polynomials. Hermite polynomials are a mutually orthogonal function with weight functions, which can be determined easily by using the Rodrigues formula

$$H_n(x) = (-1)^n\ e^{x^2}\ \frac{d^n}{dx^n}(e^{-x^2}) \ , \quad n = 0, 1, 2, \dots .$$

The first few Hermite polynomials are

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x, \quad H_4(x) = 16x^4 - 48x^2 + 12,$$
$$H_5(x) = 32x^5 - 160x^3 + 120x, \quad H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120.$$

Hermite polynomials have the generating function

$$w(x,t) = e^{2xt - x^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n, \quad |t| < \infty.$$

3. The Description of Methods

We are concerned with solving the two-dimensional singular mixed Volterra–Fredholm integral equations which have the form

$$u(x,t) = f(x,t) + \int_c^t \int_a^b (t-z)^{\alpha-1}\ \phi(x,y)\ u(y,z) dy dz, \tag{3.1}$$

where $0 < \alpha < 1$ and $(x,t) \in [a,b] \times [c,d]$, where $u(x,t)$ is an unknown function, $f(x,t)$ is a given function defined on $[a,b] \times [c,d]$ and $k(x,t,y,z) = (t-z)^{\alpha-1}\ \phi(x,y)$ is the singular kernel satisfying the discontinuity condition in the domain $([a,b] \times [c,d])^2$.

3.1. **Bernoulli-Collocation method [16,19].** This method is based on approximating the unknown function $u(x,t)$ in (3.1) on the form

$$u(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}\ B_i(x)\ B_j(t), \tag{3.2}$$

where $B_i(x)$, $B_j(t)$ are Bernoulli polynomials and a_{ij} are unknown coefficients to be determined in order to obtain the approximate solution, in the following steps:

Curtailing the infinite series (3.2), we get

$$\tilde{u}(x,t)\simeq\sum_{i=0}^{N}\sum_{j=0}^{N}a_{ij}\ B_i(x)\ B_j(t), \tag{3.3}$$

Substituting from (3.3) into (3.1) we get

$$\sum_{i=0}^{N}\sum_{j=0}^{N}a_{ij}\ [B_i(x)\ B_j(t)-\int_c^t\int_a^b(t-z)^{\alpha-1}\ \phi(x,y)\ B_i(y)\ B_j(z)dydz]=f(x,t). \tag{3.4}$$

Using the collocation points x_p, t_q of Bernoulli polynomials given by

$$x_p=a+\frac{b-a}{N}p\ ,\quad t_q=c+\frac{d-c}{N}q, \tag{3.5}$$

for $p,q=0,1,2,\dots,N$ and $x_p\in[a,b]$, $t_q\in[c,d]$,

equation (3.4) would be written as

$$\sum_{i=0}^{N}\sum_{j=0}^{N}a_{ij}\ [B_i(x_p)\ B_j(t_q)-\int_c^{t_q}\int_a^b(t_q-z)^{\alpha-1}\ \phi(x_p,y)\ B_i(y)\ B_j(z)dydz]=f(x_p,t_q),0<\alpha<1. \tag{3.6}$$

Substituting collocation points (3.5) into (3.6), we get a system of algebraic equations which contains $(N+1)^2$ of a_{ij} unknown coefficients. Solving this system to obtain a_{ij} values, we get an approximate solution $\tilde{u}(x,t)$.

The accuracy of this method is given by the formula (see [18])

$$\|u(x,t)-\tilde{u}(x,t)\|\le\gamma\ \lambda\ C\ N(2\pi)^{-N},$$

where

$$\lambda=\max_{0\le x\le b,c\le t\le d}\ |\ k(x,t,y,z)\ |,$$

λ is a positive constant, independent of N, and a bound for the partial derivative of $u(x,t),\gamma$ is a positive constant and C is the coefficient matrix.

3.2. **The Hermite–Galerkin method.** Assume that $u(x,t)$ is an approximate solution of (3.1). We use Hermite polynomials through the Galerkin method which has the form

$$\widetilde{u}(x,t)\simeq\sum_{i=0}^{N}\sum_{j=0}^{N}c_{i,j}\ H_i(x)\ H_j(t), \tag{3.7}$$

where $H_i(x)$, $H_j(t)$ are Hermite polynomials and $c_{i,j}$ are unknown Hermite coefficients to be determined in the following steps.

Substituting from (3.7) into (3.1), we get

$$\sum_{i=0}^{N}\sum_{j=0}^{N}c_{i,j}\ [H_i(x)\ H_j(t)-\int_c^t\int_a^b(t-z)^{\alpha-1}\ \phi(x,y)\ H_i(y)\ H_j(z)dydz]=f(x,t), \tag{3.8}$$

where $0<\alpha<1$, $(x,t)\in[a,b]\times[c,d]$.

Multiplying both sides in equation (3.8) by $H_p(x)\ H_q(t)$, then integrate with respect to x and y from a to b and from c to d, respectively, such that p and $q = 0, 1, 2, \ldots, N$. Hence, equation (3.8) becomes of the form

$$\sum_{i=0}^{N}\sum_{j=0}^{N} c_{i,j} \int_c^d \int_a^b G_{ij}(x,t)\ H_p(x)\ H_q(t)\ dxdt = F_{pq}, \tag{3.9}$$

where

$$F_{pq} = \int_c^d \int_a^b f(x,t)\ H_p(x)\ H_q(t)\ dxdt,$$

$$G_{ij}(x,t) = H_i(x)\ H_j(t) - \int_c^t \int_a^b (t-z)^{\alpha-1}\ \phi(x,y)\ H_i(y)\ H_j(z) dydz.$$

Substituting $p, q = 0, 1, \ldots, N$ into (3.9), we get a system of $(N+1)^2$ non-singular algebraic equations. By solving this system, we get Hermite coefficients $c_{i,j}$.

The accuracy of this method depends on reducing the error using low-degree interpolation polynomials without increasing time of calculation (see [17]). The error function is expressed by the formula

$$E(x,t) = |u(x,t) - \widetilde{u}(x,t|,$$

for $x_l \in [a,b]$ and $t_m \in [c,d]$, the error function can be written as follows:

$$E(x_l,t_m) = |u(x_l,t_m) - \widetilde{u}(x_l,t_m| \cong 0,$$

or $E(x_l,t_m) \leq 10^{-k_i}$, (k_i) is a positive integer,
if $\max(10^{-k_i}) = 10^{-k}$, k is a positive integer.

4. Numerical Examples

In this section some numerical examples of two-dimensional singular mixed Volterra–Fredholm integral equations are presented to illustrate the previous methods.

Example 1. Consider the singular VFIE [1]

$$u(x,t) = x^2t^2 - \frac{25}{156} t^{\frac{13}{5}} + \int_0^t \int_0^1 y^2 (t-z)^{-0.4}\ u(y,z) dydz, \tag{4.1}$$

where $x, t \in [0,1]$ with the exact solution $u(x,t) = x^2t^2$.

In Table 1, we give the absolute error of equation (4.1) by the Bernoulli-collocation (BC) and Hermite–Galerkin (HG) methods for different values of x, t and $N = 2, 4, 6$ according to Section 3. Figures 1, 2, and 3 clarify the exact solution of (4.1), the absolute error for $N = 6$ by BC and HG methods, respectively. Moreover, these methods are compared to the Toeplitz matrix method [1] that given for $N = 40$.

Example 2. Consider the singular VFIE [1]

$$u(x,t) = x^2t^2 - \frac{125}{336} x^2 t^{\frac{12}{5}} + \int_0^t \int_0^1 x^2\ y (t-z)^{-0.6}\ u(y,z) dydz, \tag{4.2}$$

where $x, t \in [0,1]$ with the exact solution $u(x,t) = x^2t^2$.

The absolute error of equation (4.2) for different values of x, t and $N = 2, 4, 6$ by BC and HG methods are obtained in Table 2. We plot Figures 4 and 5 to show the absolute error with $N = 6$ by our methods. Furthermore, these examples compared to Toeplitz matrix method [1] are solved for $N = 60$.

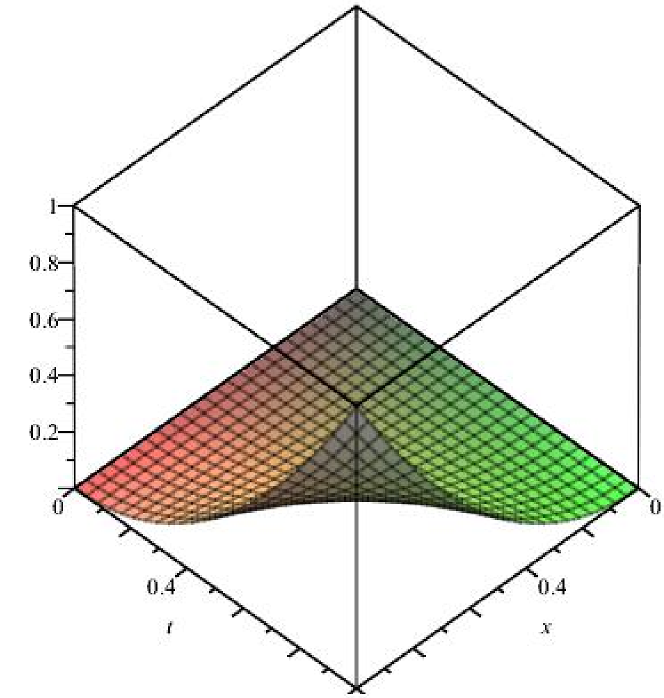

FIGURE 1. Exact solution of Examples 1 and 2.

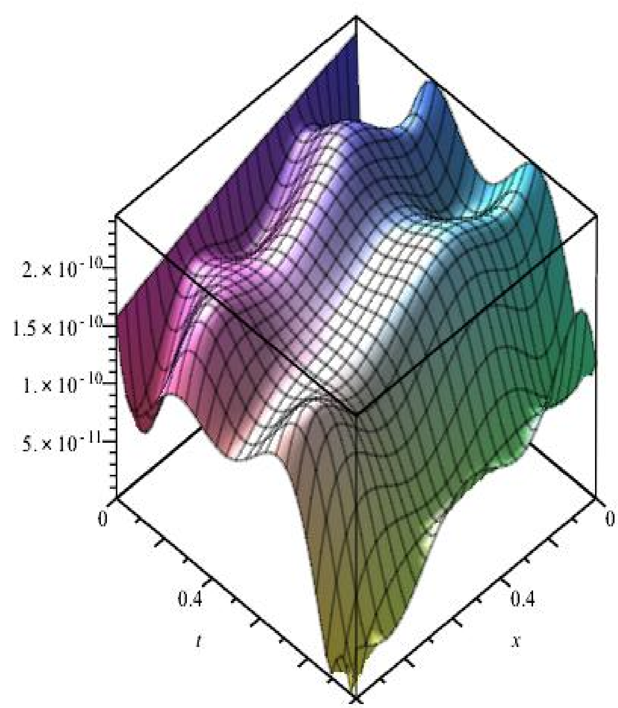

FIGURE 2. Absolute error of Example 1, $N = 6$ by BC method.

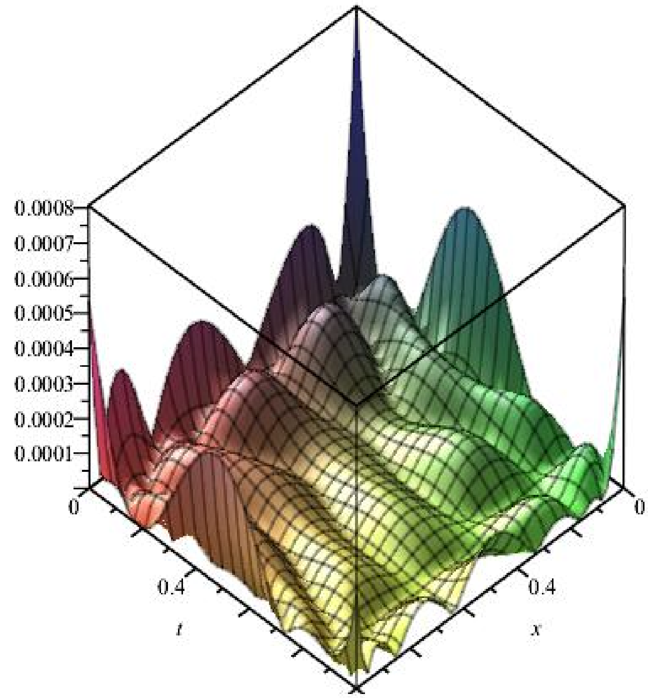

FIGURE 3. Absolute error of Example 1, $N = 6$ by HG method.

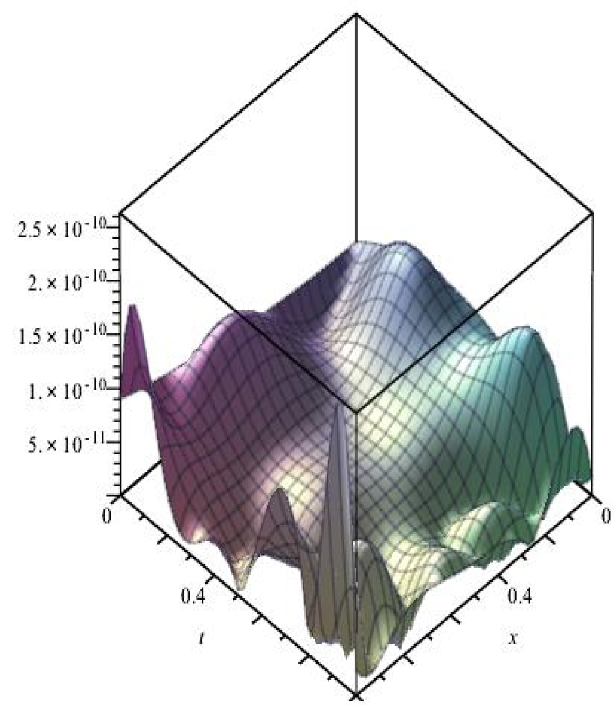

FIGURE 4. Absolute error of Example 2, $N = 6$ by BC method.

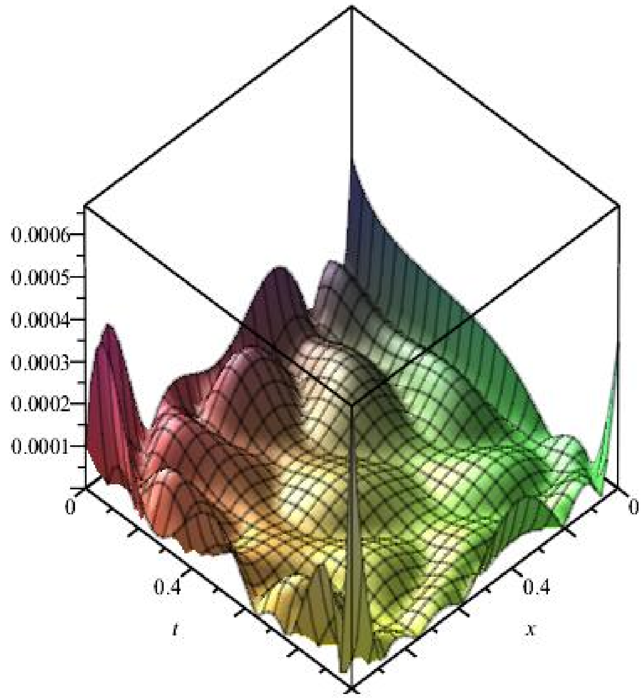

FIGURE 5. Absolute error of Example 2, $N = 6$ by HG method.

TABLE 1. Absolute Error of Example 1 by BC and HG methods for $N = 2, 4, 6$.

	$n=2$		$n=4$		$n=6$	
(x, y)	BC method	HG method	BC method	HG method	BC method	HG method
$(0,0)$	1.2×10^{-10}	1.15×10^{-8}	1.0×10^{-10}	8.6399×10^{-6}	1×10^{-10}	3.1208×10^{-5}
$(0.1,0.1)$	1.2×10^{-10}	3.6077×10^{-9}	6.970×10^{-11}	6.7453×10^{-7}	3.294×10^{-11}	2.9815×10^{-7}
$(0.2,0.2)$	1.2×10^{-10}	4.545×10^{-10}	8.882×10^{-11}	1.8246×10^{-6}	1.255×10^{-10}	2.2109×10^{-6}
$(0.3,0.3)$	1.2×10^{-10}	7.659×10^{-12}	1.388×10^{-10}	1.8611×10^{-7}	1.774×10^{-10}	1.1138×10^{-5}
$(0.4,0.4)$	1.2×10^{-10}	7.046×10^{-10}	2.048×10^{-10}	5.9285×10^{-7}	2.531×10^{-10}	9.7149×10^{-6}
$(0.5,0.5)$	1.2×10^{-10}	1.4533×10^{-9}	2.702×10^{-10}	1.6656×10^{-6}	3.322×10^{-10}	1.8397×10^{-6}
$(0.6,0.6)$	1.2×10^{-10}	1.6322×10^{-9}	3.480×10^{-10}	8.6798×10^{-7}	4.016×10^{-10}	5.3769×10^{-6}
$(0.7,0.7)$	1.2×10^{-10}	1.0899×10^{-9}	4.788×10^{-10}	6.5042×10^{-8}	5.385×10^{-10}	1.6465×10^{-5}
$(0.8,0.8)$	1.2×10^{-10}	1.455×10^{-10}	6.838×10^{-10}	1.2693×10^{-6}	8.329×10^{-10}	1.0652×10^{-5}
$(0.9,0.9)$	1.2×10^{-10}	4.114×10^{-10}	9.297×10^{-10}	6.7722×10^{-7}	1.0599×10^{-9}	1.2718×10^{-5}
$(1,1)$	1.2×10^{-10}	6.791×10^{-10}	1.2191×10^{-9}	6.2889×10^{-6}	1.0661×10^{-9}	3.9371×10^{-5}

Table 2. Absolute Error of Example 2 by BC and HG methods for $N = 2, 4, 6$.

	$n=2$		$n=4$		$n=6$	
(x, y)	BC method	HG method	BC method	HG method	BC method	HG method
$(0, 0)$	8.10×10^{-11}	1.24×10^{-8}	1.100×10^{-10}	4.6954×10^{-6}	2×10^{-10}	1.0264×10^{-4}
$(0.1, 0.1)$	6.9×10^{-11}	2.9013×10^{-9}	6.996×10^{-11}	2.3239×10^{-8}	9.8772×10^{-11}	1.7232×10^{-5}
$(0.2, 0.2)$	5.6×10^{-11}	4.559×10^{-11}	1.181×10^{-10}	68951×10^{-7}	1.8759×10^{-10}	1.1683×10^{-5}
$(0.3, 0.3)$	4.1×10^{-11}	4.518×10^{-10}	2.444×10^{-10}	3.1453×10^{-7}	1.5683×10^{-10}	7.8709×10^{-6}
$(0.4, 0.4)$	2.4×10^{-11}	1.6951×10^{-9}	3.745×10^{-10}	1.5990×10^{-7}	1.3114×10^{-10}	9.7149×10^{-6}
$(0.5, 0.5)$	5×10^{-12}	2.3067×10^{-9}	5.165×10^{-10}	5.5695×10^{-8}	4.0122×10^{-12}	1.6494×10^{-5}
$(0.6, 0.6)$	1.6×10^{-11}	1.7741×10^{-9}	7.210×10^{-10}	2.9392×10^{-7}	4.6349×10^{-11}	9.1906×10^{-6}
$(0.7, 0.7)$	3.9×10^{-11}	5.408×10^{-10}	1.0063×10^{-9}	8.5861×10^{-8}	6.43557×10^{-11}	1.9329×10^{-6}
$(0.8, 0.8)$	6.4×10^{-11}	6.763×10^{-12}	1.3374×10^{-9}	1.2693×10^{-7}	$1.73078. \times 10^{-10}$	4.1436×10^{-6}
$(0.9, 0.9)$	9.1×10^{-11}	2.5278×10^{-9}	1.6814×10^{-9}	6.5965×10^{-8}	5.42449×10^{-10}	4.2443×10^{-6}
$(1, 1)$	1.2×10^{-10}	1.1416×10^{-8}	2.0952×10^{-9}	2.6867×10^{-6}	7.8985×10^{-10}	6.3423×10^{-5}

5. Conclusions and Discussions

In this paper, two methods are presented to solve the two-dimensional singular mixed Volterra–Fredholm integral equations by the results of the given two examples we compare between the two methods and establish the following deductions

1. The two methods are better and more effective than Toeplitz matrix method [1], numerically.
2. The Bernoulli-collocation method is more effective than the Hermite–Galerkin method in application.
3. The proposed computational methods could be further applied to the non-linear Volterra–Fredholm integral equations.

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