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SOLITONS ON A SHALLOW FLUID OF VARIABLE DEPTH

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Abstract. The results of numerical study of evolution of the solitons of gravity and gravity-capillary waves on the surface of a shallow fluid, when the characteristic wavelength is essentially greater than the depth, $\lambda \gg H$, are presented for the cases when dispersive parameter is a function of time, and the spatial coordinates $\beta = \beta(t, x, y)$. This corresponds to the problems when the relief of the bottom is changed in time and space. We use both the one-dimensional approach (the equations of the KdV-class) and also two-dimensional description (the equations of the KP-class), in case of need.

1. Basic Equations and General Properties of Solutions

Let us consider the models of the Korteweg – de Vries (KdV) and Kadomtsev – Petviashvili (KP) equations in their application to hydrodynamics, namely, to describe the gravity waves on the surface of an ideal incompressible fluid of small (compared to wavelength) depth. In this case, the generalized density and "sound" velocity in the general set of the hydrodynamic equations [3] acquire the sense of fluid depth H, and velocity $c = \sqrt{gH}$, the term $gH^2/2$ plays the role of the pressure, this corresponds to the effective adiabatic index $\gamma = 2$ [5]. Then the Boussinesq equations take the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} + \nabla gH + \frac{gh^2}{3}\nabla\Delta H = 0\,, \tag{1}$$

$$\frac{\partial H}{\partial t} + \nabla(H\mathbf{v}) = 0\,, \tag{2}$$

($h = \text{const}$ is the depth of the fluid). It is easy to add into these equations the terms associated with the capillary effects. Assuming that the curvature of the surface is not too large and the additional pressure to the fluid caused by the surface tension is defined by the Laplace formula

$$\delta p = \sigma\left(R_1^{-1} + R_2^{-1}\right)\,,$$

where σ is the surface tension coefficient, R_1 and R_2 are the main curvature radii, we can write $\delta p = -\sigma\Delta\eta$, where $\eta(x, y, t)$ is the surface function (the value of is sufficiently small). Replacing ρgh in (1) by $\rho gH + \delta\rho$ (ρ is the fluid density), we obtain

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} + g\nabla H + \left(\frac{gh^2}{3} - \frac{\sigma}{\rho}\right)\nabla\Delta H = 0\,. \tag{3}$$

Equations (2), (3) are the Boussinesq equations having regard to the capillary effects [5]. The factor change at the dispersive term in the dispersion relation in its standard form [5] leads to the change of the dispersion equation and, instead of $\omega = c_0 k\left(1 - \frac{1}{6}H^2k^2 + \cdots\right)$, we have

$$\omega = c_0 k\left[1 - \frac{1}{6}\left(H^2 - \frac{3\sigma}{\rho g}\right)k^2 + \cdots\right]\,, \tag{4}$$

where $c_0 = \sqrt{gH}$. In this case, the dispersive factor is defined by

$$\beta = \frac{c_0}{6}\left(H^2 - \frac{3\sigma}{\rho g}\right)\,. \tag{5}$$

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Using furthermore the results of [3], we transform (1) and (2) to the form

$$\partial_t u + \alpha u \partial_x u - \beta \partial_x^3 u = -\left(c_0/2\right) \int_{-\infty}^{x} \partial_y^2 u dx\,,$$

that is, we obtain the KP equation for the gravity-capillary waves on the shallow fluid. Note that for sufficiently large $\sigma > \frac{1}{3}\rho H^2$, the dispersive parameter changes its sign that involves the qualitative change of the evolution character and the form of the solutions [1]. Consider now in more detail the following interesting case. Often there are the cases when the factor β is unusually small. As it follows from (5), $\beta = 0$ for $H = \left(3\sigma/\rho g\right)^{1/2} \approx 0.48$cm (for pure water). However, $\beta = 0$ does not mean that there is no dispersion in medium. It simply means that in this case the next terms in the Taylor expansion in k of the full dispersion relation must be taking into account. In addition, the corresponding additional terms appear in the equation. This generalization leads to the KP equation which can be written as

$$\partial_t u + \alpha u \partial_x u - \beta \partial_x^3 u + \gamma \partial_x^5 u = -\left(c_0/2\right) \int_{-\infty}^{x} \partial_y^2 u dx, \tag{6}$$

where the coefficients are

$$\alpha = \frac{3}{2}\frac{c_0}{H}, \gamma = \frac{c_0}{6}\left[H^2\left(\frac{2}{5}H^2 - \frac{\sigma}{\rho g}\right) - \frac{1}{12}\left(\frac{3\sigma}{\rho g} - H^2\right)^2\right].$$

Using the methods based on the implicit and explicit difference schemes [1,3], numerical integration of (6) enables us to investigate the structure of the one-dimensional (1D) and two-dimensional (2D) solitons on a shallow fluid in the case of anomalously weak dispersion. We have found that the qualitative form of the solutions depends significantly on the value of parameter $\varepsilon = \left(\beta/V\right)\left(-V/\gamma\right)^{1/2} \ll 2$, where V is the soliton's velocity in the reference frame moving along the x-axis with the phase velocity c_0. In the 1D case, for $\varepsilon = 0$, the structure of propagating solitons does not differ qualitatively from that of solitons of the usual KdV equation (see [5]), and in the 2D case − from the structure of the algebraic KP-solitons [1,3]. Such solitons on the surface of a fluid have negative polarity (the hollow solitons). When $\varepsilon > 0$, for example, in the case of the increasing fluid depth, starting from the depth $H = \left(3\sigma/\rho g\right)^{1/2}$, the structure of solitons radically changes: by remaining to decay from their maximum to zero in the transverse direction as before, now their sign varies along the direction of their propagation (in addition, the amplitude of the 2D solitons falls from the maximum to zero in the transverse direction, as before). As $\varepsilon \to 2$, the number of oscillations in the tails increases and now the solitons become similar to the 1D and 2D high-frequency trains, respectively, i.e., envelope solitons[1]. Note that a similar structure is typical also for solitons of internal gravity waves, considered in detail in [2–4]. Separately for the cases 1D and 2D, let us consider now some our results of numerical simulation of the soliton dynamics on the surface of a shallow fluid which is describes by the standard KdV and KP equations (equation (6) with $\gamma = 0$) when the factor β is a function of the space coordinates and time.

2. Structure and Evolution of 1D Solitons of Gravity and Gravity-Capillary Waves with a Varying Relief of the Bottom

First, let us consider the evolution of the 1D solitons in the framework of model (6) with $\gamma = 0$ and right-hand side being equal to zero (the KdV equation):

$$\partial_t u + \alpha u \partial_x u + \beta \partial_x^3 u = 0 \tag{7}$$

on the surface of a fluid with varying in time and space dispersive parameter $\beta = \beta\left(t, x\right)$. Such situation can take place, for example, in the problems on propagation of the gravity and gravity-capillary waves on the surface of a shallow fluid [1], when $\beta = c_0 H^2/6$ and $\beta = \left(c_0/6\right)\left[H^2 - 3\sigma/\rho g\right]$, respectively (see above). In these cases, if $H = H\left(t, x\right)$, the dispersive parameter becomes also the

[1]For the structure of the 1D solitons of the generalized KdV equation, see also [1–3].

function of the x coordinate and time. In [2, 3], it has been shown that the solutions of the KDV equation for $\beta = \text{const}$, depending on the value of β, are divided into two classes: for $|\beta| < u_0(0, x)\, l/12$ (where l is the characteristic wavelength of the initial disturbance), they have soliton character, in the opposite case, the solutions are the wave packets with asymptotes being proportional to the derivative of the Airy function (see also [5]). In these cases, the KdV equation can be integrated analytically by the inverse scattering transform (IST) method. But even in the 1D case, if $\beta = \beta(t, x)$, this approach is impossible principally, it is necessary to resort to a numerical simulation in the conforming problems.

Let us formulate the problem of numerical simulation of the KdV equation with $\beta = \beta(t, x)$ and consider some results of our numerical experiments in studying the structure and evolution of the solitary waves on the surface of a shallow fluid. To solve the initial problem for the KdV equation (7) with a variable dispersion, we have used an implicit difference scheme [3] with $O(\tau^2, h^4)$ approximation. Initial conditions were chosen in the form of the solitary disturbance

$$u(0, x) = u_0 \exp\left(-x^2/l^2\right) , \tag{8}$$

and in the form of a "smoothed step"

$$u(0, x) = \frac{c}{1 + \exp(x/l)} , \tag{9}$$

with different values of parameters u_0, l and c, defined by the convenience of numerical calculation for specific sizes of the numerical integration area. The zero conditions on the boundaries of the computation region were imposed, and simulation has been conducted for a few types of model types of function β (see Figures 1 and 2) when for $t < t_{\text{cr}}$, $\beta = \beta_0 = \text{const}$, and for $t \geq t_{\text{cr}}$,

$$1)\ \ \beta(x) = \begin{cases} \beta_0, & x \leq a; \\ \beta_0 + c, & x > a; \end{cases} \tag{10}$$

$$2)\ \ \beta(x, t) = \begin{cases} \beta_0, & & x \leq a; \\ \beta_0 + nc, & n = (t - t_{\text{cr}})/\tau = 1, 2, \ldots; & x > a; \end{cases} \tag{11}$$

$$\begin{aligned} 3)\ \ \beta(t) = \beta_0\left(1 + k_0 \bar{\beta} \sin \omega t\right), \quad \bar{\beta} = (\beta_{\max} - \beta_{\min})/2, \\ 0 < k_0 < 1, \quad \pi/2\tau < \omega < 2\pi/\tau; \end{aligned} \tag{12}$$

where a and c are the constants. In terms of the problem of the wave propagation on the surface of a shallow water that accordingly means that on reaching t_{cr} we have: 1) sudden "breaking up of the bottom", 2) gradual "changing of height" of the bottom area, and 3) "bottom oscillation" with time.

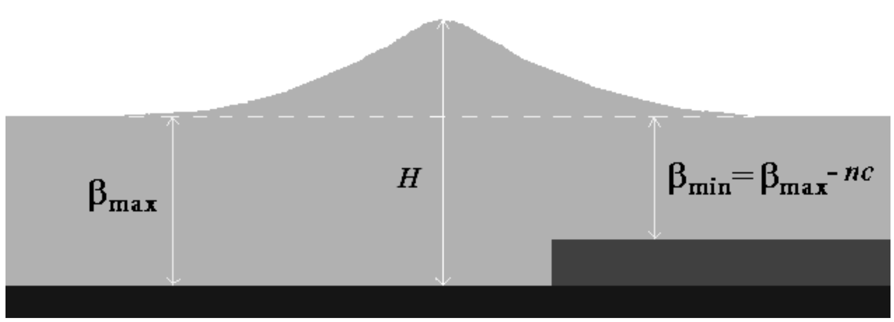

FIGURE 1. Dependence $\beta = \beta(t, x)$ of type of "step", models (10) and (11).

Consider briefly some results of numerical simulation for two types of initial conditions and different kinds of model function $\beta = \beta(t, x)$. In the first series of numerical experiments we investigate the evolution of the initial disturbance in the form of the solitary soliton-like pulse (8) for the models

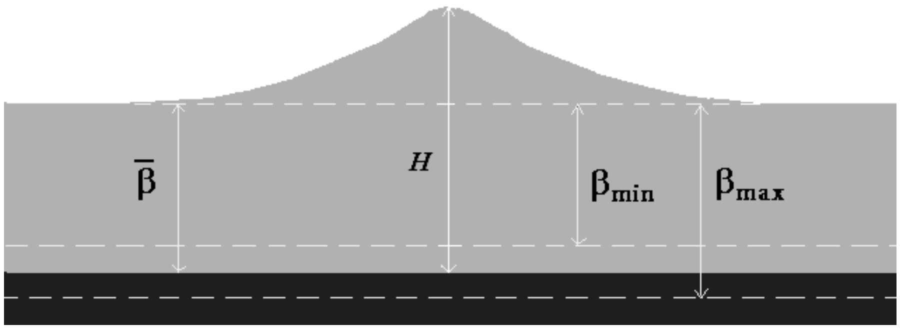

FIGURE 2. Dependence $\beta = \beta\,(t, x)$ of type of "bottom oscillations", models (12).

with spasmodic change of dispersion [models of "step" type bottom (10) and (11) with values of the parameter a corresponding (for $t = 0$) to the position of the "break" behind and ahead of soliton, and values $c < 0$ ("negative" step) and $c > 0$ ("positive" step). The obtained results show that in all cases the deformation of initial pulse occurs with time. If the step is located behind the soliton, in both cases $c < 0$ and $c > 0$, the waving tail which is not associated with the main maximum of the outgoing forward main pulse is formed, and its evolution is entirely determined by the value β in its location. In case $t = 0$, the "step" is located ahead front of the initial pulse, for $c > 0$, in the model (11) a steep front is formed quite rapidly, that leads to the overturning of the wave with time. For $c > 0$, we can observe the destruction of the soliton (Figure 3), which occurs due to the fact that in the region of localization of its front, the relative role of nonlinear effects falls due to the increase of the dispersive parameter here, and dispersive effects prevail.

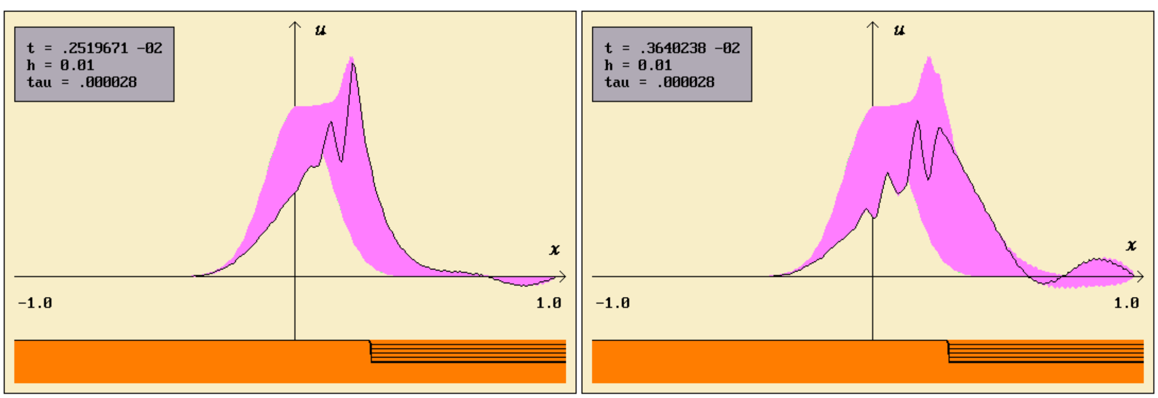

FIGURE 3. Evolution of the KdV soliton in model (11) with $c < 0$.

The second series of numerical experiments is devoted to the study of evolution of the initial disturbance of type (9) for the models of "bottom" (10), (11) for different values of parameters a and c.

Figure 4 shows the result of numerical simulation of evolution of the initial disturbance (9) for the model of "bottom" in the form of a positive step in the case if "break" is located directly under the region of the disturbance front of the fluid surface. It is seen that due to the fact that the development of perturbations occurs mainly in the region where the value of the dispersion parameter corresponds to the multi-soliton solution of the KdV equation [3, 5], the solitary disturbance propagates with the development of high-frequency oscillatory structure behind the shock front, and in the region of the soliton "tail", where dispersive effects dominate over the nonlinear ones, the high-frequency train of oscillations decays rather quickly to zero and it is limited in the region $x < 0$. Figure 5 shows the example of the results of simulation of the evolution of initial disturbance in the form of the "smoothed

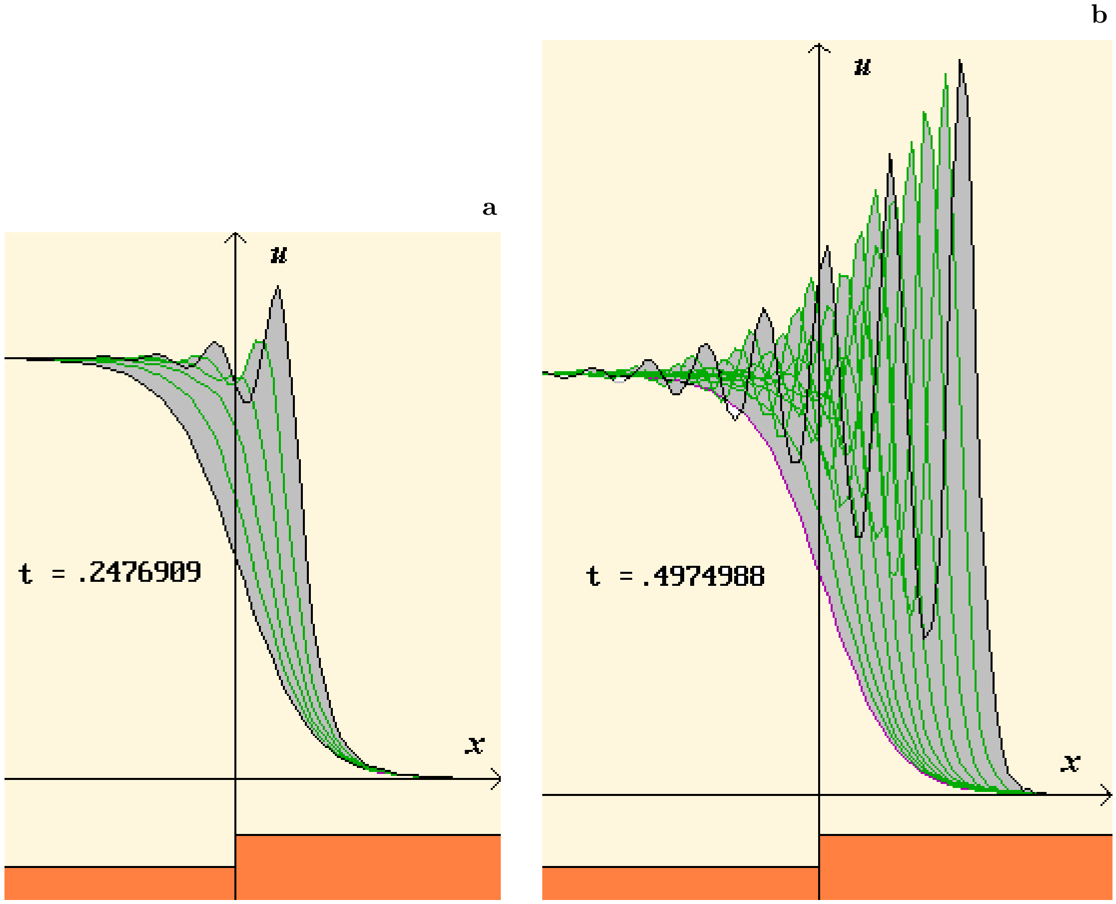

FIGURE 4. Evolution of "step" (9) in model (10) with $c > 0$: a – $t \approx 0.25$; b – $t \approx 0.5$.

step" (9) in case when the break of the "bottom" is negative and located in front of the localization region of the fluid surface disturbance. It can be seen that in this case, the front of the disturbance becomes more gentle with time, the oscillatory soliton structure in the front region is not formed, but the development of low-frequency oscillations behind the main maximum occurs. This result is easily explained within the framework of the similarity principle for the KdV equation [5]: the evolution of the "tail" of the initial disturbance occurs in the region of small values of the dispersive parameter, whereas in the front region, where the dispersion is relatively large, the formation of a shock wave does not occur.

As for the third law of change of β (harmonic oscillations of the parameter β with time on the whole x-axis), a series of numerical experiments for various $k_0 = \text{const}$ and a variable frequency ω [see law of change (12)] show that the stationary (locally) standing waves can be formed for some values of ω, in other cases, the formation of stationary periodic wave structures is possible, and in intermediate cases, a chaotic regime is usually realized.

3. Structure and Evolution of 2D Solitons of Gravity and Gravity-Capillary Waves with a Varying Relief of the Bottom

Let us now consider the problem of evolution of the 2D solitons in the framework of the standard KP equation

$$\partial_t u + \alpha u \partial_x u + \beta \partial_x^3 u = \kappa \int_{-\infty}^{x} \partial_y u \, dx \tag{13}$$

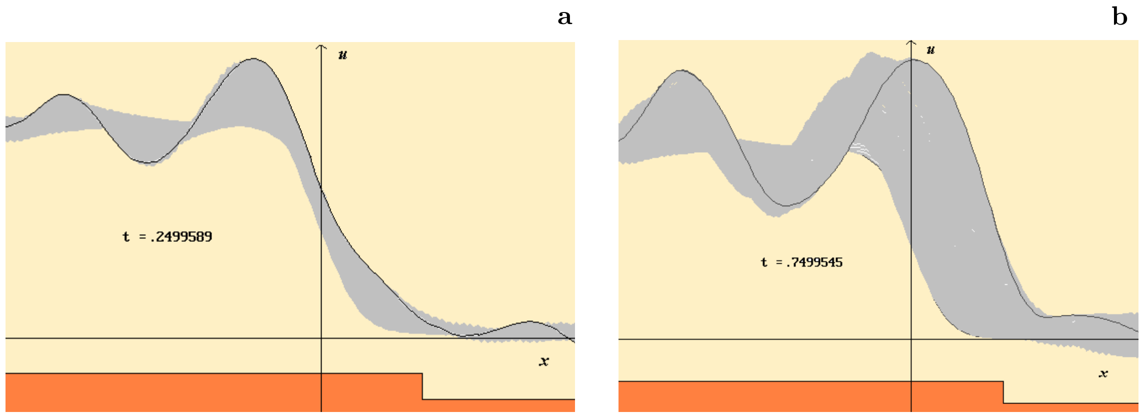

FIGURE 5. Evolution of “step” (9) in model (10) with $c < 0$: a – $t \approx 0.25$; b – $t \approx 0.75$.

with a varying in time and space dispersive parameter $\beta = \beta(t, x, y)$. This situation can take place in the problems dealing with the propagation of gravity and gravity-capillary waves on the surface of a shallow fluid [3] when the fluid depth is the function of the spatial coordinates and time $H = H(t, x, y)$.

Here, we have the same situation as for the 1D model of the KdV equation described above: if analytical solutions of the KP equation are known, in case $\beta = \beta(t, \mathbf{r})$, the dispersion term of equation becomes quasi-linear and the model is not exactly integrable (the IST method is not applicable) [3]. The problem of numerical simulation of the KP equation with $\beta = \beta(t, x, y)$ is formulated analogously to the problem for the KdV equation (see previous section). To solve the initial problem for the KP equation (13) with a variable dispersion (varying relief of the bottom), we use an implicit difference scheme [1] with $O(\tau^2, h^4)$ approximation. The initial conditions are chosen in the form of the exact 2D one-soliton solution of the KP equation [3], the complete absorption conditions on the boundaries of computation region [1, 3] are imposed, and simulation is conducted for the same types of model function as for the KdV equation [see formulae (10)–(12)]. Consider the basic results of the numerical experiments on the investigation of the structure and the evolution of 2D solitary waves on the fluid surface with a variable dispersion.

The first series of numerical experiments have been aimed at the study of soliton dynamics under spasmodic character of the dispersion change (the function $\beta = \beta(t, x, y)$ has the form of the ”step”). First, we investigated evolution of the initial pulse when the spasmodic change of β for t_{cr} takes place behind the soliton [”negative” step when $c < 0$ in formulae (10), (11)]. In addition, we have studied the dependence of the spatial structure of a solution on the value of parameter a in models (10) and (11). The obtained results (see Example in Figure 6) show that in all the cases the evolution leads to the formation of waving tail which is not connected with the soliton going away and caused only by a local influence of sudden change of the “relief” $\beta = \beta(t, x)$. Consequently, the formation of oscillatory structure is connected not so much with decreasing of a role of the dispersion effects behind the soliton as with the spasmodic changing of β in space.

In the next series of numerical simulation, we considered the evolution of a 2D soliton when the sudden change of the dispersion parameter is located directly under or in front of an initial pulse (”negative” step). An example of the results of this series is given in Figure 7. Analysing the obtained results of the whole series, we can see that for such character of the relief of the function β the disturbance caused by sudden change of the dispersive parameter has also a local character, i.e., it doesn’t propagate together with the going away soliton. But, unlike the cases considered in the first series of simulation, the asymptotes of a leaving soliton become oscillating (in any case, in the time limits of numerical experiment), besides, against a background of the long-wave oscillations of the waving tail we can also see the appearance of the wave fluctuations. The effects noted may be interpreted as a result of those that for the areas of the wave surface with different values of local wave number k_x the value of the dispersive effects is different. As a result, the intensity of the phase mixing of the Fourier-harmonics within the (x, y)-region varies with the coordinates and, therefore,

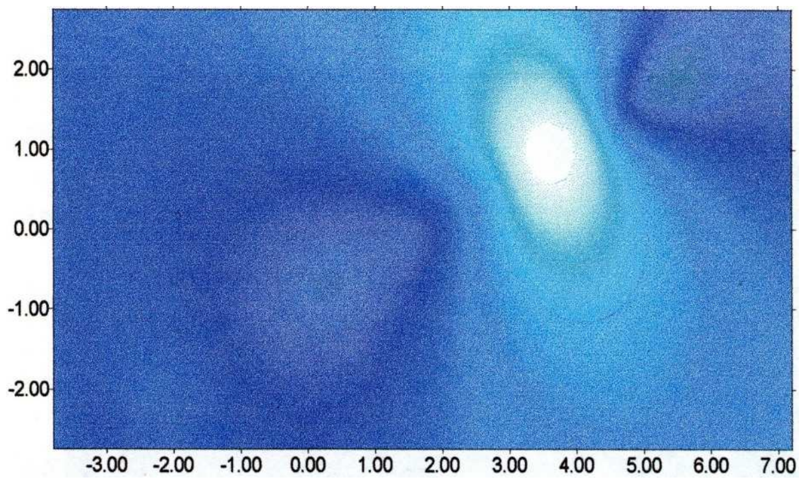

FIGURE 6. Solution of eq. (13) for the dispersion law (10) with $a = 5.0$, $c = -0.0038$ for $t = 0.6$.

it reacts differently to the nonlinear generation of the harmonics with various (in particular, large) wave-numbers k_x.

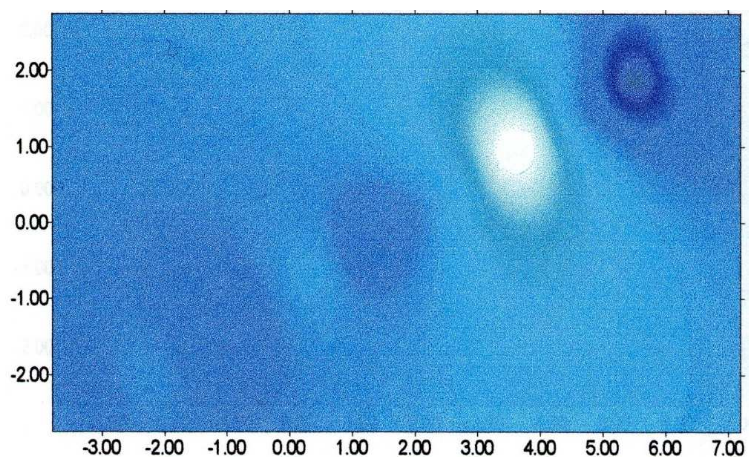

FIGURE 7. Solution of eq. (13) for the dispersion law (11) with $a = 4.0$, $c = -0.0038$ for $t = 0.6$.

In the third series of the experiments with dispersive parameter changing with the laws (10) and (11) we consider the cases of "positive" step [$c > 0$ in formulae (10) and (11)] being both in front of and behind the initial pulse for the wide diapason of values of parameter a. The examples of the most interesting results are shown in Figure 8. One can see that when "positive" step is far in front of maximum of the function $u(0, x, y)$, the soliton evolution at the initial stage does not practically differ qualitatively from that for $\beta = \text{const}$ (Figure 8a), but in the future, the evolution character is defined by the presence of the step, namely, the processes, caused by the same causes noted for the results of the second series of numerical simulation, begin to be developed (Figure 8b). As we can see in the figure, the appreciable change of the soliton structure which can lead to the wave falling is observed owing to an intensive generation of the harmonics with big k_x in the soliton front region, even for rather small height of the step (i.e., even if the value of parameter a in formulae (10), (11) is still rather small). Thus, as it follows from the results of this series, the disturbance of the propagating 2D soliton caused by sudden change in time and space of the dispersive parameter with $c > 0$ is also of local character.

As to the second law of the β change (model (12) – harmonic oscillation of the parameter β with time on the whole (x, y)-plane), the series of numerical simulations for different $k_0 = \text{const}$ and variable frequency ω [see law (12)] show that for some values of ω, the stationary (locally) standing waves can be formed, in other cases, the formation of the stationary periodical wave structures is possible, and in the intermediate cases, a chaotic regime is usually realized.

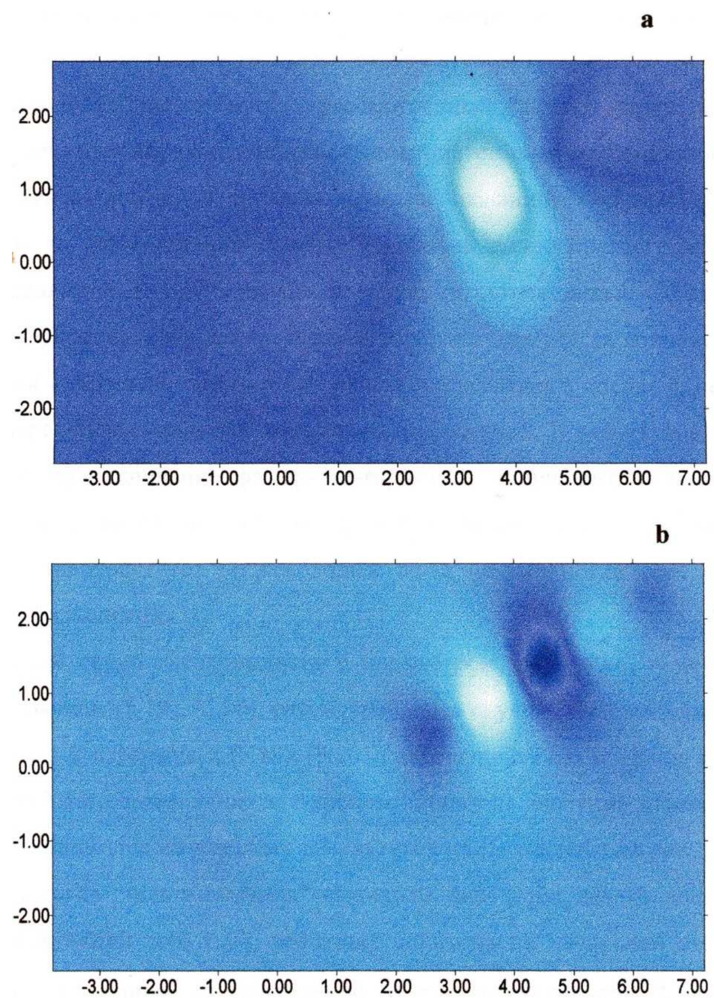

FIGURE 8. Evolution of soliton of eq. (13) for the dispersion law (11) with $a = 5.0$, $c = 0.0038$: (a) $t = 0.6$, (b) $t = 0.8$.

In the experiments, carried out for different values of the parameter k_0 and $\omega = \text{const}$, we have found that the stable (in any case, in the limits of the numerical computation time) solutions can be derived only for $k_0 \leq \beta_0$ in formula (12), and the solutions are unstable in the other cases. An example of evolution of the 2D soliton, when its structure along the $x--$ and $y-$axes acquires the wave character and the amplitude of its maximum decreases with time, is given in Figure 9.

Summing up the above, one can note that the numerical simulation of evolution of the 2D solitons describing by the model of the KP equation with $\beta = \beta(t, x, y)$ enable us to find different types of stable and unstable solutions including those of the mixed "soliton – non-soliton" type for various character of dispersion changes in time and space.

The obtained results open the new perspectives in the investigation of a number of applied problems of dynamics of the non-one-dimensional nonlinear waves in the specific physical media, including upper atmosphere (ionosphere), magneto-sphere and in a plasma [1–3, 5].

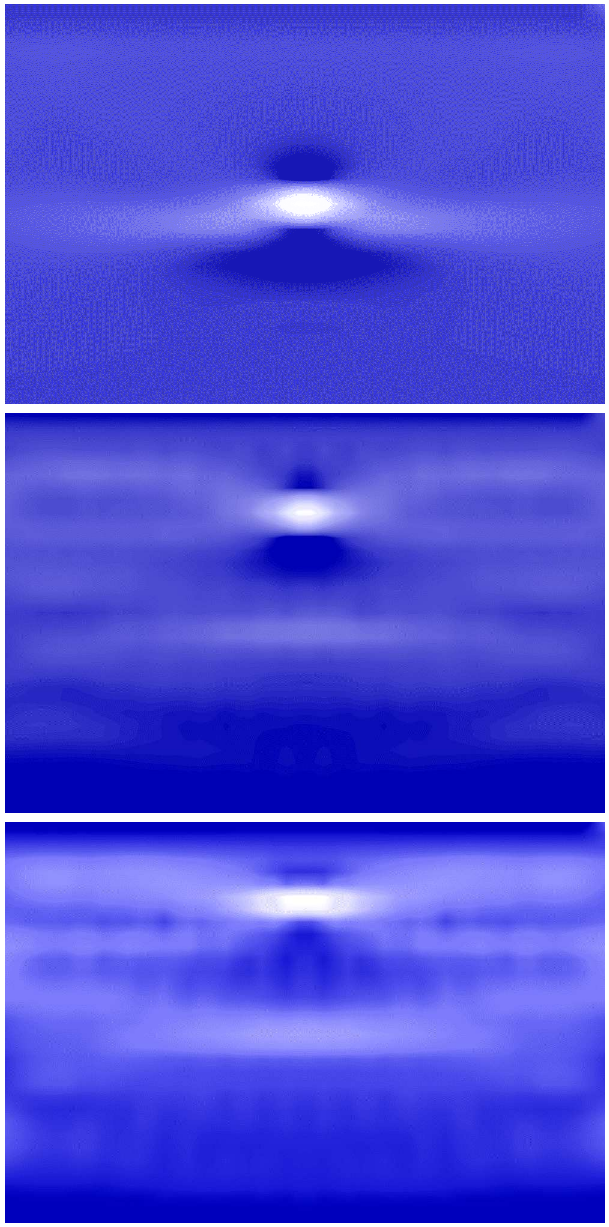

FIGURE 9. Evolution of 2D soliton of eq. (13) for $t = 0.4,\ 1.2,\ 2.0$.

4. Conclusion

In the paper, the results of numerical study of evolution of the solitons of gravity and gravity-capillary waves on the surface of a shallow fluid when the characteristic wavelength is essentially greater then depth, $\lambda \gg H$, were presented for the cases, when dispersive parameter is a function of time and spatial coordinates, $\beta = \beta\,(t, x, y)$. This corresponds to the problems when the relief of the bottom is changed in time and space. We have considered three cases of variable dispersion when the sudden "breaking up of the bottom", the gradual "changing of height" of the bottom area, and the "bottom oscillation" with time take place. To solve the problem, we have used both the 1D approach (the equations of the KdV-class) and also the 2D description (the equations of the KP-class). For all cases, numerical solutions of the problem in 1D and 2D geometry were presented. It was noted that the realized approach can be useful also in other applications of the nonlinear wave theory such as dynamics of 1D and multidimensional solitary waves in other specific physical media, including upper atmosphere (ionosphere), magnetosphere and in a plasma.

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