Transactions of A. Razmadze
Mathematical Institute
Vol. **175** (2021), issue 2, 205–214

HAUSDORFF MEASURE OF NONCOMPACTNESS OF CERTAIN MATRIX OPERATORS ON ABSOLUTE NÖRLUND SPACES

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Abstract. The absolute Nörlund spaces $\left|N_p^u\right|_k, k \geq 1$, have more recently been introduced and studied by Hazar and Sarıgöl [On absolute Nörlund spaces and matrix operators, Acta Math. Sin. (Engl. Ser.), 34 (5) (2018), 812-826]. In the present paper, we characterize the classes of infinite matrix and compact operators transforming from $\left|N_p^u\right|_k$ into X and obtain some identities or estimates for the Hausdorff measures of noncompactness, where X is one of the spaces ℓ_∞, c and c_0.

1. Background, Notation and Preliminaries

A linear subspace of the space w, the space of all (real-- or) complex-valued sequences, is called a *sequence space*. We write ℓ_∞, c, c_0 and ϕ for the spaces of all bounded, convergent, null sequences and the set of all finite sequences, respectively. By $e^{(n)}$ and ℓ_k $(\ell_1 = \ell)$, we denote the sequence whose only non-zero term is 1 in n-th place for each $n \in \mathbb{N}$ and the space of all k-absolutely convergent series, respectively.

Let X, Y be two sequence spaces, $A = (a_{nv})$ be an infinite matrix of complex numbers and A_n be the sequence in the n-th row of A, that is, $A_n = (a_{n\nu})_{\nu=0}^{\infty}$ for each $n \in \mathbb{N}$. Then, we write $A(x) = (A_n(x))$, the A-transform of x, if

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v$$

converges for $n \geq 0$. If $A(x) = (A_n(x)) \in Y$ for all $x = (x_v) \in X$, then A is called a matrix transformation from X into Y, denoted by $A : X \to Y$, and we also denote the class of such maps by (X, Y).

For a sequence space X, the matrix domain X_A and the β-dual of X are introduced by

$$X_A = \{x \in w : A(x) \in X\}, \tag{1.1}$$
$$X^\beta = \{\varepsilon = (\varepsilon_v) \in w : \Sigma \varepsilon_v x_v \text{ converges for all } x \in X\},$$

respectively.

If $A = (a_{nv})$ is an infinite triangle matrix, *i.e.*, $a_{nn} \neq 0$, and $a_{nv} = 0$ for $v > n$, there exists its unique inverse [30]. Throughout the paper, k^* denotes the conjugate of $k > 1$, i.e., $1/k + 1/k^* = 1$, and $1/k^* = 0$ for $k = 1$.

A sequence space X is called a BK- space if it is a Banach space with continuous coordinates $P_n : X \to \mathbb{C}$ defined by $P_n(x) = x_n$ for $n \geq 0$, where $\mathbb{C}$ denotes the complex field. Also, a BK- space $X \supset \phi$ is said to have AK if every $x = (x_\nu) \in X$ has a unique representation $x = \sum_{v=0}^{\infty} x_v e^{(\nu)}$ [2]. For example, ℓ_∞, c and c_0 are BK-spaces according to the norm $\|x\|_{\ell_\infty} = \sup_{\nu \in \mathbb{N}} |x_v|$ and ℓ_k is a BK-space according to the norm $\|x\|_{\ell_k} = \left(\sum_{v=0}^{\infty} |x_v|^k\right)^{1/k}$, $1 \leq k < \infty$. Moreover, the spaces c_0 and ℓ_k have the property AK under their natural norms [13].

2020 *Mathematics Subject Classification.* 40C05, 40F05, 46A45, 46B45, 46B50.

Key words and phrases. Absolute Nörlund spaces; Sequence spaces; Matrix operators; BK spaces; Compact operators.

If $X \supset \phi$ is a BK- space and $a = (a_\nu) \in w$, then we write

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{v=0}^{\infty} a_v x_v \right|, \tag{1.2}$$

provided the statement on the right is defined and finite, which is satisfied whenever $a \in X^\beta$, where S_X denotes the unit sphere in X, i.e., $S_X = \{x \in X : \|x\| = 1\}$ [14].

If S and H are subsets of a metric space (X, d) and $\varepsilon > 0$, then S is called an ε-net of H, if, for every $h \in H$, there exists $s \in S$ such that $d(h, s) < \varepsilon$; if S is finite, then the ε-net S of H is called a finite ε-*net* of H. By $\mathcal{M}_X$ we denote the collection of all bounded subsets of X. If $Q \in \mathcal{M}_X$, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = \inf \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

The function $\chi : \mathcal{M}_X \to [0, \infty)$ is called the Hausdorff measure of noncompactness [21].

If X and Y are normed spaces, the set $\mathcal{B}(X, Y)$ states the set of all bounded linear operators $L : X \to Y$ and it is also a normed space to the norm $\|L\| = \sup_{x \in S_X} \|L(x)\|_Y$, where S_X is a unit sphere in X, and we write $\mathcal{B}(X) = \mathcal{B}(X, X)$. Further, let X and Y be Banach spaces. Then a linear operator $L : X \to Y$ is said to be compact if its domain is all of X and the sequence $(L(x_n))$ has a convergent subsequence in Y for every bounded sequence $x = (x_n) \in X$. We write $\mathcal{C}(X, Y)$ for the class of such operators. Studies on the Hausdorff measure noncompactness and compact operators can be found in [11, 13, 17–21].

The following results are important tool to compute the Hausdorff measure of noncompactness.

Lemma 1.1 ([13]). *Let X and Y be Banach spaces, $L \in \mathcal{B}(X, Y)$. Then the Hausdorff measure of noncompactness of L, denoted by $\|L\|_\chi$, is defined by*

$$\|L\|_\chi = \chi(L(S_X)),$$

and L is compact, if and only if $\|L\|_\chi = 0$.

Lemma 1.2 ([21]). *Let Q be a bounded subset of the normed space X, where $X = \ell_k$ for $1 \leq k < \infty$. If $P_n : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, \ldots)$ for all $x \in X$, then*

$$\chi(Q) = \lim_{r \to \infty} \sup_{x \in Q} \|(I - P_r)(x)\|,$$

where I is the identity operator on X.

Also, we need the following known results for our investigations.

Lemma 1.3 ([13]). *Let $1 < k < \infty$ and $k^* = k/(k-1)$. Then we have $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$, $\ell_1^\beta = \ell_\infty$ and $\ell_k^\beta = \ell_{k^*}$. Furthermore, let X denote any of the spaces $\ell_\infty, c, c_0, \ell_1$ and ℓ_k. Then, we have $\|a\|_X^* = \|a\|_{X^\beta}$ for all $a \in X^\beta$, where $\|.\|_{X^\beta}$ is the natural norm on the dual space X^β.*

Lemma 1.4 ([13]). *Let X and Y be BK-spaces. Then we have $(X, Y) \subset \mathcal{B}(X, Y)$, i.e., every matrix $A \in (X, Y)$ defines a linear operator $L_A \in \mathcal{B}(X, Y)$ by $L_A(x) = A(x)$ for all $x \in X$.*

Lemma 1.5 ([7]). *s Let $X \supset \phi$ be a BK-space and Y be any of the spaces ℓ_∞, c, c_0. If $A \in (X, Y)$, then $\|L_A\| = \|A\|_{(X, \ell_\infty)} = \sup_n \|A_n\|_X^* < \infty$.*

Lemma 1.6 ([25]). *Let $1 < k < \infty$. Then $A \in (\ell_k, \ell)$ if and only if*

$$\|A\|'_{(\ell_k, \ell)} = \left\{ \sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{k^*} \right\}^{1/k^*} < \infty,$$

and there exists $1 \leq \xi \leq 4$ such that $\|A\|'_{(\ell_k, \ell)} = \xi \|A\|_{(\ell_k, \ell)}$.

Lemma 1.7 ([12]). *Let $1 \leq k < \infty$. Then $A \in (\ell, \ell_k)$ if and only if*

$$\|A\|_{(\ell, \ell_k)} = \sup_v \left\{ \sum_{n=0}^{\infty} |a_{nv}|^k \right\}^{1/k} < \infty.$$

2. Absolute Nörlund Spaces

Let Σa_v be an infinite series with the n-th partial sum s_n and (u_n) be a sequence of nonnegative terms. The series Σa_v is said to be summable $|A, u_n|_k$, $k \geq 1$, if

$$\sum_{n=0}^{\infty} u_n^{k-1} \left|\Delta A_n(s)\right|^k < \infty, \ A_{-1}(s) = 0,$$

where $\Delta A_n(s) = A_n(s) - A_{n-1}(s)$, for $n \geq 0$, $A_{-1}(s) = 0$, [22]. If we take A as a matrix of weighted mean $\left(\overline{N}, p_n\right)$ $(resp., \ u_n = P_n/p_n)$, then the summability $|A, u_n|_k$ reduces to the summability $\left|\overline{N}, p_n, u_n\right|_k$ (resp., $\left|\overline{N}, p_n\right|_k$, [5]), [29]. Further, if $u_n = n$ for $n \geq 1$ and A is the matrix of Nörlund mean (N, p_n), then it is the same as the summability $|N, p_n|_k$, $k \geq 1$, given by Borwein and Cass [6], which also includes the summability $|C, \alpha|_k$ of Flett [9]. By a Nörlund matrix $A = (a_{nv})$, we mean

$$a_{nv} = \begin{cases} p_{n-v}/P_n, & 0 \leq v \leq n, \\ 0, & v > n, \end{cases}$$

where (p_n) is a sequence of complex numbers with $P_n = p_0 + p_1 + \cdots + p_n \neq 0, \ p_0 \neq 0, \ P_{-n} = 0$ for $n \geq 1$.

More recently, the space $\left|N_p^u\right|_k$ has been introduced as the set of all series, summable by the absolute Nörlund method $|N, p_n, u_n|_k$ for $k \geq 1$, *i.e.*,

$$\left|N_p^u\right|_k = \left\{a = (a_v) \in w : \sum_{n=1}^{\infty} u_n^{k-1} \left| \sum_{v=1}^{n} \left(\frac{P_{n-\nu}}{P_n} - \frac{P_{n-1-\nu}}{P_{n-1}} \right) a_\nu \right|^k < \infty \right\}, \ \left(\left|N_p^u\right|_1 = |N_p| \right).$$

Certain matrix operators on this space have been studied by Hazar and Sarıgöl [10] together with their norms, which is also generalized some known results in [6, 15, 24, 26]. Also, one can see some related works on sequences and series spaces in [1, 3, 4, 8, 11, 16, 23, 27].

Note that if the matrices $T^{(p)} = (t_{nv}^{(p)})$ and $E^{(k)} = (e_{nv}^{(k)})$, $1 \leq k < \infty$, are defined by

$$t_{nv}^{(p)} = \begin{cases} \dfrac{P_{n-v}}{P_n}, & 0 \leq v \leq n, \\ 0, & v > n, \end{cases} \tag{2.1}$$

$$e_{nv}^{(k)} = \begin{cases} -u_n^{1/k^*}, & v = n-1, \\ u_n^{1/k^*}, & v = n, \\ 0, & v \neq n, n-1, \end{cases} \tag{2.2}$$

respectively, then we may restate $\left|N_p^u\right|_k = (\ell_k)_{E^{(k)} o T^{(p)}}$ in view of the identity (1.1), where $1/k^* = 0$ for $k = 1$ [10]. Further, there exists the inverse matrix $S^{(p)}$ of $T^{(p)}$, since $T^{(p)}$ is triangle matrix. To obtain the matrix $S^{(p)}$, take p_0 as a non-zero. Then there exists a sequence (C_n) such that

$$\sum_{v=0}^{n} P_{n-v} C_v = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1, \end{cases} \tag{2.3}$$

which gives that

$$y_n = \frac{1}{P_n} \sum_{v=0}^{n} P_{n-v} x_v \quad \text{if and only if} \quad x_n = \sum_{v=0}^{n} C_{n-v} P_v y_v,$$

where $P_n = p_0 + p_1 + \cdots + p_n \neq 0$ for $n \geq 1$, and so, $S^{(p)} = (s_{nv}^{(p)})$ is defined by

$$s_{nv}^{(p)} = \begin{cases} C_{n-v} P_v, & 0 \leq v \leq n, \\ 0, & v > n. \end{cases} \tag{2.4}$$

Throughout the paper, for any sequence $x = (x_v) \in \left|N_p^u\right|_k$, we associate the sequence $z = (z_v)$ by $z = E^{(k)} o T^{(p)}(x)$. If we say that $T^{(p)}(x) = y$, then

$$z_n = u_n^{1/k^*}\Delta y_n = u_n^{1/k^*}\sum_{v=1}^{n}\left(\frac{P_{n-\nu}}{P_n}-\frac{P_{n-1-\nu}}{P_{n-1}}\right)x_\nu \tag{2.5}$$

for $n \geq 1$, $y_{-1}=0$. So, it is trivial that $x \in \left|N_p^u\right|_k$, if and only if $z = E^{(k)}oT^{(p)}(x) \in \ell_k$, and $x \in S_{\left|N_p^u\right|_k}$ if and only if $z \in S_{\ell_k}$. In other words, $E^{(k)}oT^{(p)} : \left|N_p^u\right|_k \to \ell_k$ is a bijective linear map preserving norm [10].

Further, we recall that $\left|N_p^u\right|_k$ is a BK-space (see [10]) with respect to the norm

$$\|x\|_{\left|N_p^u\right|_k} = \left\|E^{(k)}oT^{(p)}(x)\right\|_{\ell_k}. \tag{2.6}$$

We require the following notations and lemmas.

$$G_{nv} = \sum_{r=v}^{n} P_r C_{n-r}\ ; v,n \geq 0,$$

$$D_1 = \left\{\varepsilon = (\varepsilon_v) \in w : \lim_m \sum_{v=r}^{m}\varepsilon_v G_{vr} \text{ exists}\right\},$$

$$D_2 = \left\{\varepsilon = (\varepsilon_v) \in w : \sup_{m,r}\left|\sum_{v=r}^{m}\varepsilon_v G_{vr}\right| < \infty\right\},$$

$$D_3 = \left\{\varepsilon = (\varepsilon_v) \in w : \sup_m \sum_{r=0}^{m}\left|u_r^{-1/k^*}\sum_{v=r}^{m}\varepsilon_v G_{vr}\right|^{k^*} < \infty\right\}.$$

Lemma 2.1.

a) $A \in (\ell, c) \Leftrightarrow$ (i) $\lim_n a_{nv}$ *exists,* $v \geq 0$, (ii) $\sup_{n,v}|a_{nv}| < \infty$.

b) $A \in (\ell, \ell_\infty) \Leftrightarrow$ (ii) *holds.*

c) *If* $1 < k < \infty$, *then* $A \in (\ell_k, c) \Leftrightarrow$ (i) *holds,* (iii) $\sup_n \sum_{v=0}^{\infty}|a_{nv}|^{k^*} < \infty$.

d) *If* $1 < k < \infty$, *then* $A \in (\ell_k, \ell_\infty) \Leftrightarrow$ (iii) *holds.*

e) *If* $1 < k < \infty$, *then* $A \in (\ell_k, c_0) \Leftrightarrow$ (iii) *holds,* (iv) $\lim_n a_{nv} = 0$, $v \geq 0$.

f) $A \in (\ell, c_0) \Leftrightarrow$ (ii) *and* (iv) *holds* [28].

Lemma 2.2. *Let* $1 \leq k < \infty$. *If* $a = (a_\nu) \in \left|N_p^u\right|_k^\beta$, *then* $\tilde{a} = (\tilde{a}_\nu) \in \ell_{k^*}$ *for* $k > 1$, *and* $\tilde{a} \in \ell_\infty$ *for* $k = 1$. *Moreover,*

$$\sum_{v=1}^{\infty} a_v x_v = \sum_{v=1}^{\infty}\tilde{a}_\nu z_v \tag{2.7}$$

holds for every $x = (x_k) \in \left|N_p^u\right|_k$, *where* $z = E^{(k)}oT^{(p)}(x)$ *is the associated sequence defined by* (2.5) *and*

$$\tilde{a}_\nu = u_v^{-1/k^*}\sum_{r=v}^{\infty} a_r G_{rv}. \tag{2.8}$$

Also, the following result is immediate by Lemma 2.2.

Lemma 2.3. *Let* (u_n) *be a sequence of nonnegative numbers. Then* $\left|N_p^u\right|_k^\beta = D_1 \cap D_3$ *for* $1 < k < \infty$ *and* $|N_p|_1^\beta = D_1 \cap D_2$ *for* $k = 1$ [10].

Lemma 2.4. *Let* $\tilde{a} = (\tilde{a}_\nu)$ *be defined as in* (2.8). *Then* $\|a\|^*_{\left|N_p^u\right|_k} = \|\tilde{a}\|_{\ell_{k^*}}$ *for* $1 < k < \infty$ *and* $\|a\|^*_{|N_p|} = \|\tilde{a}\|_{\ell_\infty}$ *for* $k = 1$, *where* $a \in \left|N_p^u\right|_k^\beta$.

Proof. Let $1 < k < \infty$ and $a \in \left|N_p^u\right|_k^\beta$. Then by Lemma 2.2, we get $\tilde{a} = (\tilde{a}_\nu) \in \ell_{k^*}$, and equality (2.7) holds, and also, by (2.6), $x \in S_{\left|N_p^u\right|_k}$, if and only if $z \in S_{\ell_k}$. So, it follows from (1.2) and (2.7) that

$$\|a\|^*_{\left|N_p^u\right|_k} = \sup_{x \in S_{\left|N_p^u\right|_k}} \left|\sum_{v=1}^{\infty} a_v x_v\right| = \sup_{z \in S_{\ell_k}} \left|\sum_{v=1}^{\infty} \tilde{a}_\nu z_v\right| = \|\tilde{a}\|^*_{\ell_k}$$

and, since $\tilde{a} = (\tilde{a}_\nu) \in \ell_{k^*}$, by Lemma 1.3,

$$\|a\|^*_{\left|N_p^u\right|_k} = \|\tilde{a}\|^*_{\ell_k} = \|\tilde{a}\|_{\ell_{k^*}}.$$

This concludes the proof. □

The proof for $k = 1$ is similar to the above, so it is omitted.

Lemma 2.5. *Let V be a sequence space and (u_n) be a sequence of nonnegative numbers. If $A \in \left(\left|N_p^u\right|_k, V\right)$, then $F^{(k)} \in (\ell_k, V)$, where the matrix $F^{(k)} = \left(f_{nv}^{(k)}\right)$ is defined by*

$$f_{nv}^{(k)} = u_v^{-1/k^*} \sum_{r=v}^{\infty} a_{nr} G_{rv}. \tag{2.9}$$

Proof. The proof is seen at once by Lemma 2.2. □

Now, we give some lemmas on the operator norms.

Lemma 2.6. *Let (u_n) be a sequence of nonnegative numbers and define the matrix $F^{(k)} = \left(f_{nv}^{(k)}\right)$ by (2.9). If A is in any of the classes $\left(\left|N_p^u\right|_k, c_0\right)$, $\left(\left|N_p^u\right|_k, c\right)$ and $\left(\left|N_p^u\right|_k, \ell_\infty\right)$, then for $1 < k < \infty$,*

$$\|L_A\| = \|A\|_{\left(\left|N_p^u\right|_k, \ell_\infty\right)} = \sup_n \left\|F_n^{(k)}\right\|_{\ell_{k^*}},$$

and for $k = 1$,

$$\|L_A\| = \|A\|_{(|N_p|, \ell_\infty)} = \sup_n \left\|F_n^{(1)}\right\|_{\ell_\infty}.$$

Proof. It follows immediately by combining Lemmas 1.4, 1.5 and 2.4. □

Lemma 2.7. *Let (u_n) be a sequence of nonnegative numbers and the matrix $F^{(k)} = \left(f_{nv}^{(k)}\right)$ be given by (2.9).*

a) *If $A \in (|N_p|, \ell_k)$, then for $k \geq 1$,*

$$\|L_A\| = \|A\|_{(|N_p|, \ell_k)} = \left\|F^{(1)}\right\|_{(\ell, \ell_k)}.$$

b) *If $A \in \left(\left|N_p^u\right|_k, \ell\right)$, then for $1 < k < \infty$, there exists $1 \leq \xi \leq 4$ such that*

$$\|L_A\| = \|A\|_{\left(\left|N_p^u\right|_k, \ell\right)} = \left\|F^{(k)}\right\|_{(\ell_k, \ell)} = \frac{1}{\xi} \left\|F^{(k)}\right\|'_{(\ell_k, \ell)}.$$

Proof. It follows by combining Lemmas 1.4, 1.6, 1.7 and 2.5. □

3. Compact Operators on Absolute Nörlund Spaces

In this section, we characterize the classes $\left(\left|N_p^u\right|_k, X\right)$ and $\mathcal{C}\left(\left|N_p^u\right|_k, X\right)$, and also obtain some identities or estimates for the Hausdorff measures of noncompactness in these classes, where X is one of the spaces ℓ_∞, c and c_0.

Theorem 3.1. *Let (u_n) be a sequence of nonnegative numbers and let $F^{(1)}=\left(f_{nv}^{(1)}\right)$ be given by*

$$f_{nv}^{(1)}=\lim_m \sum_{j=\nu}^m a_{nj}G_{j\nu},\ n,v\geq 0.$$

a) *$A\in(|N_p|,\ell_\infty)$, if and only if*

$$\lim_m \sum_{j=\nu}^m a_{nj}G_{j\nu} \text{ exists for each } n,v\geq 0, \tag{3.1}$$

$$\sup_{m,v}\left|\sum_{r=v}^m a_{jr}G_{rv}\right|<\infty \text{ for each } j, \tag{3.2}$$

$$\sup_{n,j}\left|f_{nj}^{(1)}\right|<\infty. \tag{3.3}$$

b) *$A\in(|N_p|,c)$, if and only if (3.1), (3.2), (3.3) hold and*

$$\lim_n f_{nj}^{(1)} \text{ exists for each } j.$$

c) *$A\in(|N_p|,c_0)$, if and only if (3.1), (3.2), (3.3) hold and*

$$\lim_n f_{nj}^{(1)}=0 \text{ for each } j.$$

Proof. **a)** $A\in(|N_p|,\ell_\infty)$, if and only if $(a_{nv})_{v=0}^\infty\in|N_p|^\beta$ for each n, and $A(x)\in\ell_\infty$ for every $x\in|N_p|$. Also, by Lemma 2.3, it is seen that $(a_{nj})_{j=0}^\infty\in D_1\cap D_2$, i.e., (3.1) and (3.2) hold for each n. To prove the necessity and sufficiency of (3.3), let $x\in|N_p|$. Consider the composite operator $E^{(1)}oT^{(p)}:|N_p|\to\ell$ defined by (2.1) and (2.2). Then it is easy to see that $E^{(1)}oT^{(p)}$ is a bijective linear operator, since $T^{(p)}$ and $E^{(1)}$ are bijective linear operators (see, [10]). Now, we write $z\in\ell$, where $T^{(p)}(x)=y$ and $z=\left(E^{(1)}oT^{(p)}\right)(x)$, i.e., $z_n=\Delta y_n$ for $n\geq 0$, $y_{-1}=0$, and also $y_n=\sum_{j=0}^n z_j$. Then, it follows from (2.3) and (2.4) that

$$\sum_{v=0}^m a_{nv}x_v=\sum_{j=0}^m\left(P_j\sum_{v=j}^m a_{nv}C_{v-j}\right)y_j=\sum_{j=0}^m \tilde{f}_{mj}^{(1)}z_j,$$

where

$$\tilde{f}_{mj}^{(1)}=\begin{cases}\sum_{v=j}^m a_{nv}G_{vj}, & 0\leq j\leq m,\\ 0, & j>m.\end{cases}$$

Moreover, if any matrix $R=(r_{nv})\in(\ell,c)$, then the series $R_n(x)=\Sigma_v r_{nv}x_v$ converges uniformly in n, since, by Lemma 2.1, the remaining term tends to zero uniformly in n, that is,

$$\left|\sum_{v=m}^\infty r_{nv}x_v\right|\leq \sup_{n,v}|r_{nv}|\sum_{v=m}^\infty|x_\nu|\to 0 \text{ as } m\to\infty,$$

and so we get

$$\lim_n R_n(x)=\sum_{v=0}^\infty \lim_n r_{nv}x_v. \tag{3.4}$$

Hence, it is easily seen from (3.1) and (3.2) that $\tilde{F}^{(1)}=\left(\tilde{f}_{mj}^{(1)}\right)\in(\ell,c)$, and so, by (3.4), we have

$$A_n(x)=\sum_{j=0}^\infty\left(\lim_m \tilde{f}_{mj}^{(1)}\right)z_j=\sum_{j=0}^\infty f_{nj}^{(1)}z_j=F_n^{(1)}(z),$$

where $f_{nj}^{(1)}=\lim_m \tilde{f}_{mj}^{(1)}$. This results in $A(x)\in\ell_\infty$ for every $x\in|N_p|$, if and only if $F^{(1)}(z)\in\ell_\infty$ for every $z\in\ell$, which implies that $A\in(|N_p|,\ell_\infty)$ if and only if (3.1) and (3.2) hold, and $F^{(1)}\in(\ell,\ell_\infty)$.

Also, it follows from Lemma 2.1 that $F^{(1)} \in (\ell, \ell_\infty)$, if and only if (3.3) is satisfied. This concludes the proof of the part of a).

The parts b) and c) can be proved similarly, so we omit the detail. $\square$

Theorem 3.2. *Let $k > 1$, (u_n) be a sequence of nonnegative numbers. Define the matrix $F^{(k)} = \left(f_{nv}^{(k)}\right)$ by*

$$f_{nv}^{(k)} = u_v^{-1/k^*} \sum_{j=\nu}^{\infty} a_{nj} G_{j\nu}, \;\; n, v \geq 0.$$

Then

a) $A \in \left(\left| N_p^u \right|_k, \ell_\infty \right)$ *if and only if* (3.1) *holds, and*

$$\sup_m \sum_{v=0}^{m} \left| u_v^{-1/k^*} \sum_{r=v}^{m} a_{nr} G_{rv} \right|^{k^*} < \infty, \tag{3.5}$$

$$\sup_n \sum_{\nu=0}^{\infty} \left| f_{nv}^{(k)} \right|^{k^*} < \infty. \tag{3.6}$$

b) $A \in \left(\left| N_p^u \right|_k, c \right)$, *if and only if* (3.1), (3.5), (3.6) *hold, and*

$$\lim_n f_{nv}^{(k)} \;\; \text{exists for each} \;\; \nu.$$

c) $A \in \left(\left| N_p^u \right|_k, c_0 \right)$ *if and only if* (3.1), (3.5), (3.6) *hold, and*

$$\lim_n f_{nv}^{(k)} = 0, \;\; \text{for each} \;\; \nu.$$

Proof. **a)** Let $A \in \left(\left| N_p^u \right|_k, \ell_\infty \right)$. Then, equivalently, $(a_{nj})_{j=0}^\infty \in \left(\left| N_p^u \right|_k \right)^\beta$ and $A(x) \in \ell_\infty$ for every $x \in \left| N_p^u \right|_k$. Also, by Lemma 2.3, it is seen that $(a_{nj})_{j=0}^\infty \in \left(\left| N_p^u \right|_k \right)^\beta$, if and only if $(a_{nj})_{j=0}^\infty \in D_1 \cap D_3$ for each n, which is the same as (3.1) and (3.5). To prove the necessity and sufficiency of (3.6), by considering (2.1) and (2.2), we define the operator $E^{(k)} o T^{(p)} : \left| N_p^u \right|_k \to \ell_k$ by

$$\left(E^{(k)} o T^{(p)} \right)_n (x) = u_n^{1/k^*} \Delta T_n^{(p)} (x).$$

It is easy to see that a composite function $E^{(k)} o T^{(p)}$ is a bijective linear operator, since $T^{(p)}$ and $E^{(k)}$ are bijective linear operators (see, [10]). Given $x \in \left| N_p^u \right|_k$. If we say that $T^{(p)}(x) = y$ and $z = \left(E^{(k)} o T^{(p)} \right)(x)$, *i.e.*,$z_n = u_n^{1/k^*} \Delta y_n$ for $n \geq 0$, $y_{-1} = 0$, then we have $z \in \ell_k$, and since the space $\left| N_p^u \right|_k$ is isomorphic to ℓ_k, it follows that $x \in \left| N_p^u \right|_k$, if and only if $z \in \ell_k$. Further, $y_n = \sum_{j=0}^{n} u_j^{-1/k^*} z_j$. So, considering (2.3), as in the proof of Theorem 3.1, we obtain

$$\sum_{j=0}^{m} a_{nj} x_j = \sum_{j=0}^{m} \tilde{f}_{mj}^{(k)} z_j,$$

where

$$\tilde{f}_{mj}^{(k)} = \begin{cases} u_j^{-1/k^*} \sum\limits_{r=j}^{m} a_{nr} G_{rj}, & 0 \leq j \leq m, \\ 0, & j > m. \end{cases}$$

Furthermore, if any matrix $R = (r_{nv}) \in (\ell_k, c)$, then the series $R_n(x) = \Sigma_v r_{nv} x_v$ converges uniformly in n, by Lemma 2.1. In fact, applying Hölder's inequality to the remaining term, we get

$$\left| \sum_{v=m}^{\infty} r_{nv} x_v \right| \leq \left(\sum_{v=m}^{\infty} \left| r_{nv} \right|^{k^*} \right)^{1/k^*} \left(\sum_{v=m}^{\infty} \left| x_\nu \right|^k \right)^{1/k}$$

and the right-hand side of this inequality tends to zero as $m \to \infty$, since $x \in \ell_k$. This means that the remaining term tends to zero uniformly in n, and so, $R_n(x) = \Sigma_v r_{nv} x_v$ converges uniformly in n, which implies

$$\lim_n R_n(x) = \sum_{v=0}^{\infty} \lim_n r_{nv} x_v. \tag{3.7}$$

Thus, it is easily seen from (3.1) and (3.5) that $\tilde{F}^{(k)} = \left(\tilde{f}_{mj}^{(k)}\right) \in (\ell_k, c)$, and so, by (3.7),

$$A_n(x) = \sum_{v=0}^{\infty} \left(\lim_m \tilde{f}_{mv}^{(k)}\right) z_v = \sum_{v=0}^{\infty} f_{nv}^{(k)} z_v = F_n^{(k)}(z),$$

where $\lim_m \tilde{f}_{mv}^{(k)} = f_{nv}^{(k)}$. This gives that $A(x) \in \ell_\infty$ for every $x \in \left|N_p^u\right|_k$, if and only if $F^{(k)}(z) \in \ell_\infty$ for every $z \in \ell_k$, which implies that $F^{(k)} \in (\ell_k, \ell_\infty)$, and so, it follows by applying Lemma 2.1 to the matrix $F^{(k)}$ for $k > 1$ that $F^{(k)} \in (\ell_k, \ell_\infty)$, if and only if (3.6) holds. This concludes the proof of the part of a).

Since b) and c) can be proved similarly, so we omit the details.

The following lemma is required to characterize a subclass of compact operators $\mathcal{K}\left(\left|N_p^u\right|_k, X\right)$, where X is one of the spaces ℓ_∞, c_0 and c. □

Lemma 3.3 ([19])**.** *Let $X \supset \phi$ be a BK-space. Then we have:*

a) *If $A \in (X, \ell_\infty)$, then*

$$0 \leq \|L_A\|_\chi \leq \lim_{n\to\infty} \sup \|A_n\|_X^* .$$

b) *If $A \in (X, c_0)$, then*

$$\|L_A\|_\chi = \lim_{n\to\infty} \sup \|A_n\|_X^* .$$

c) *If X has AK or $X = \ell_\infty$ and $A \in (X, c)$, then*

$$\frac{1}{2} \lim_{n\to\infty} \sup \|A_n - \alpha\|_X^* \leq \|L_A\|_\chi \leq \lim_{n\to\infty} \sup \|A_n - \alpha\|_X^* ,$$

where $\alpha = (\alpha_v)$ is given by $\alpha_v = \lim_{n\to\infty} a_{n\nu}$, for all $\nu \in \mathbb{N}$.

By using Lemma 3.3, we establish the following result.

Theorem 3.4. *Let $k \geq 1$ and (u_n) be a sequence of nonnegative numbers. Also, define the matrix $F^{(k)} = \left(f_{nv}^{(k)}\right)$ by (2.9).*

Then we have:

a) *If $A \in \left(\left|N_p^u\right|_k, \ell_\infty\right)$, then*

$$0 \leq \|L_A\|_\chi \leq \lim_{n\to\infty} \sup \left\|F_n^{(k)}\right\|_{\ell_k}^* \tag{3.8}$$

and

$$L_A \text{ is compact if } \lim_{n\to\infty} \left\|F_n^{(k)}\right\|_{\ell_k}^* = 0. \tag{3.9}$$

b) *If $A \in \left(\left|N_p^u\right|_k, c_0\right)$, then*

$$\|L_A\|_\chi = \lim_{n\to\infty} \sup \left\|F_n^{(k)}\right\|_{\ell_k}^* , \tag{3.10}$$

$$L_A \text{ is compact, if and only if } \lim_{n\to\infty} \left\|F_n^{(k)}\right\|_{\ell_k}^* = 0. \tag{3.11}$$

c) *If $A \in \left(\left|N_p^u\right|_k, c\right)$, then*

$$\frac{1}{2} \lim_{n\to\infty} \sup \left\|F_n^{(k)} - \tilde{\alpha}\right\|_{\ell_k}^* \leq \|L_A\|_\chi \leq \lim_{n\to\infty} \sup \left\|F_n^{(k)} - \tilde{\alpha}\right\|_{\ell_k}^* , \tag{3.12}$$

$$L_A \text{ is compact, if and only if } \lim_{n\to\infty} \left\|F_n^{(k)} - \tilde{\alpha}\right\|_{\ell_k}^* = 0, \tag{3.13}$$

where $\tilde{\alpha} = (\tilde{\alpha}_v)$ *is given by* $\tilde{\alpha}_v = \lim_{n\to\infty} f_{nv}^{(k)}$, *for all* $\nu \in \mathbb{N}$.

Proof. First, by Lemma 1.1, we point out that (3.9), (3.11) and (3.13) are obtained from (3.8), (3.10) and (3.12), respectively. Also, since $\left|N_p^u\right|_k, k \geq 1$ is a BK-space, using parts a) and b) of Lemma 3.3 with Lemma 2.4, we get (3.8) and (3.10), respectively.

Finally, we see that (3.12) holds. In fact, if $A \in \left(\left|N_p^u\right|_k, c\right)$, we write $F^{(k)} \in (\ell_k, c)$ by using Lemma 2.5, where $A(x) = F^{(k)}(z)$ for all $x \in \left|N_p^u\right|_k$ and $z \in \ell_k$. So, since ℓ_k has AK, from part c) of Lemma 3.3, we get

$$\frac{1}{2} \lim_{n\to\infty} \sup \left\| F_n^{(k)} - \tilde{\alpha} \right\|_{\ell_k}^* \leq \left\| L_{F^{(k)}} \right\|_\chi \leq \lim_{n\to\infty} \sup \left\| F_n^{(k)} - \tilde{\alpha} \right\|_{\ell_k}^*, \tag{3.14}$$

where $\tilde{\alpha} = (\tilde{\alpha}_v)$ is given by $\tilde{\alpha}_v = \lim_{n\to\infty} f_{nv}^{(k)}$ for all $\nu \in \mathbb{N}$.

On the other hand, $x \in S_{\left|N_p^u\right|_k}$, if and only if $z \in S_{\ell_k}$ by (2.6). So, it follows from Lemmas 1.1, 1.4 and 2.5 that

$$\left\| L_A \right\|_\chi = \chi\left(AS_{\left|N_p^u\right|_k}\right) = \chi\left(F^{(k)} S_{\ell_k}\right) = \left\| L_{F^{(k)}} \right\|_\chi. \tag{3.15}$$

Hence (3.12) is obtained by (3.14) and (3.15), which completes the proof. □

Theorem 3.5. *Let* $F^{(k)} = \left(f_{nv}^{(k)}\right)$ *be defined as in* (2.9) *and* (u_n) *be a sequence of nonnegative numbers. Then we have:*

a) *If* $A \in (|N_p|, \ell_k)$, *then for* $1 \leq k < \infty$,

$$\left\| L_A \right\|_\chi = \lim_{r\to\infty} \sup_v \left(\sum_{n=r+1}^{\infty} \left| f_{nv}^{(1)} \right|^k \right)^{1/k} \tag{3.16}$$

and

$$L_A \textit{ is compact, if and only if } \lim_{r\to\infty} \sup_v \sum_{n=r+1}^{\infty} \left| f_{nv}^{(1)} \right|^k = 0. \tag{3.17}$$

b) *If* $A \in \left(\left|N_p^u\right|_k, \ell\right)$, *then for* $k > 1$, *there exists* $1 \leq \xi \leq 4$ *such that*

$$\left\| L_A \right\|_\chi = \frac{1}{\xi} \lim_{r\to\infty} \left(\sum_{v=1}^{\infty} \left(\sum_{n=r+1}^{\infty} \left| f_{nv}^{(k)} \right| \right)^{k^*} \right)^{1/k^*}$$

and

$$L_A \textit{ is compact iff } \lim_{r\to\infty} \sum_{v=1}^{\infty} \left(\sum_{n=r+1}^{\infty} \left| f_{nv}^{(k)} \right| \right)^{k^*} = 0.$$

Proof. **a)** Let $S_X = \{x \in X : \|x\| = 1\}$. Now, from (2.6), we can write that $x \in S_{|N_p|}$, if and only if $z \in S_\ell$ for all $x \in |N_p|$ and $z \in \ell$. For brevity, we write $S_{|N_p|} = S$ and $S_\ell = \bar{S}$. By Lemmas 1.1, 1.2, 1.4 and 1.7, we have

$$\begin{aligned} \left\| L_A \right\|_\chi &= \chi(AS) = \chi\left(F^{(1)} \bar{S}\right) \\ &= \lim_{r\to\infty} \sup_{z\in\bar{S}} \left\| (I - P_r) F^{(1)}(z) \right\|_{\ell_k} = \lim_{r\to\infty} \sup_v \left(\sum_{n=r+1}^{\infty} \left| f_{nv}^{(1)} \right|^k \right)^{1/k} \end{aligned}$$

where $P_r : \ell_k \to \ell_k$ is defined by $P_r(z) = (z_0, z_1, \ldots, z_r, 0, \ldots)$, which completes our assertions.

Also, (3.17) is derived from (3.16) by using Lemma 1.1.

Since part b) is proved easily as in part a) using Lemma 1.6 instead of 1.7, so, we omit the details. □

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(Received 05.11.2019)

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