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MIXED BOUNDARY-TRANSMISSION PROBLEMS OF THE GENERALIZED THERMO-ELECTRO-MAGNETO-ELASTICITY THEORY FOR PIECEWISE HOMOGENEOUS COMPOSED STRUCTURES

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Abstract. The paper is devoted to the investigation of mixed boundary-transmission problems for composed elastic structures consisting of two contacting anisotropic bodies occupying two three-dimensional adjacent regions with a common contacting interface, being a proper part of their boundaries. It is assumed that the contacting elastic bodies are subject to different mathematical models. In particular, we consider Green-Lindsay's model of generalized thermo-electro-magneto-elasticity in one elastic component, while in the other one, we considered Green-Lindsay's model of generalized thermo-elasticity. The interaction of the thermo-mechanical and electro-magnetic fields in the composed piecewise elastic structure is described by the fully coupled systems of partial differential equations of pseudo-oscillations, obtained from the corresponding dynamical models by the Laplace transform. These systems are equipped with the appropriate mixed boundary-transmission conditions which cover the conditions arising in the case of interfacial cracks. Using the potential method and the theory of pseudodifferential equations on manifolds with a boundary, the uniqueness and existence theorems in suitable function spaces are proved, the regularity of solutions is analyzed and singularities of the corresponding thermo-mechanical and electro-magnetic fields near the interfacial crack edges are characterized. The explicit expressions for the stress singularity exponents are derived and it is shown that they depend essentially on the material parameters. A special class of composed elastic structures is considered, where the so-called oscillating stress singularities do not occur.

1. Introduction

In the present paper, we consider a boundary-transmission problem for a composed elastic structure consisting of two contacting bodies occupying two three-dimensional adjacent regions $\overline{\Omega^{(1)}}$ and $\overline{\Omega^{(2)}}$ with a common contacting interface, being a proper part of the boundaries $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$ (see Figure 1). We analyze the case in which contacting elastic bodies are subject to different mathematical models. In particular, we consider *Green-Lindsay's model of generalized thermo-electro-magneto-elasticity* in $\Omega^{(1)}$ and *Green-Lindsay's model of generalized thermo-elasticity* in $\Omega^{(2)}$. Theoretical study of such problems attracts great attention due to the widespread application of modern sensing and actuating devices based on the ability to transform mechanical, electric, magnetic and thermal energies from one form to another. Therefore, the mathematical models having regard to the coupling effects between thermo-mechanical and electro-magnetic fields in elastic composites became very popular over the last decades (see, e.g., [1, 28, 29, 34], and references therein).

A remarkable feature of the generalized Green-Lindsay's model is a finite speed of heat propagation in contrast to an infinite speed of heat transfer occurring in the classical heat equation theory (see, e.g., [32]).

We investigate a general mixed boundary-transmission problem for the above described two-component elastic structure with the appropriate boundary and transmission conditions which cover the conditions arising in the case of interfacial cracks. In each region we consider the corresponding

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system of partial differential equations of pseudo-oscillations containing a complex parameter τ. These systems are obtained from the corresponding dynamical models by the Laplace transform.

Using the potential method and the theory of pseudodifferential equations on manifolds with a boundary, we study the mixed boundary-transmission problems and prove the uniqueness and existence of solutions in appropriate function spaces. Further, we analyze regularity of solutions and characterize singularities of the corresponding thermo-mechanical and electro-magnetic fields near the exceptional curves (crack edges, lines where the different type boundary conditions collide, and interface edges). In the upcoming papers, we plan to use the obtained results in the study of asymptotic properties of solutions of the corresponding dynamical problems.

Remark that in [8], we have investigated the mixed type boundary value problems of the theory of generalized thermo-electro-magneto-elasticity for homogeneous anisotropic materials with interior cracks. The interfacial crack problems for multilayered piecewise homogeneous anisotropic nested elastic structures, when all interacting components are subject to generalized thermo-electro-magneto-elasticity model with distinct material parameters in distinct elastic components, are considered in the reference [26]. The present investigation can be considered as a continuation of papers [5, 8–10, 24] and [26], but it turned out to be more difficult as far as it refers to the interaction between different dimensional physical fields (for the six-dimensional field in $\Omega^{(1)}$ and four-dimensional field in $\Omega^{(2)}$ see the problem setting in Subsection 2.4).

The paper is organized as follows. In Section 2, we describe the geometrical structure of the elastic composite body consisting of two interacting components, write down the governing pseudo-oscillation equations of Green-Lindsay's model of generalized thermo-electro-magneto-elasticity (GTEME model) and generalized thermo-elasticity (GTE model), formulate the mixed boundary-transmission problem and prove the uniqueness theorem in appropriate function spaces. In Section 3, we reduce equivalently the boundary-transmission problem to the system of boundary pseudodifferential equations, investigate the mapping properties of the corresponding pseudodifferential operator and prove the invertibility of the pseudodifferential operator in appropriate Bessel potential and Besov spaces. Further, we prove the theorem on the existence of solutions to the original mixed boundary-transmission problem, study asymptotic properties of solutions and their derivatives near the exceptional curves and evaluate explicitly the corresponding stress singularity exponents. It should be mentioned that in our analysis, we essentially use some approaches and results presented in [7] and [8]. In Section 4, we consider a particular case when an elastic solid medium occupying the region $\Omega^{(1)}$ belongs to the 422 (Tetragonal) or 622 (Hexagonal) classes of crystals or to the class of transversally isotropic materials, while the solid medium occupying the domain $\Omega^{(2)}$ is an isotropic material. These types of media includeF some key polymers and bio-materials (see [31]). For this particular problem, we analyze the asymptotic properties of solutions near the interfacial crack edges and derive explicit expressions for stress singularity exponents, playing an essential role in fracture mechanics. The stress singularity exponents essentially depend on the elastic, piezoelectric, piezomagnetic, dielectric and permeability constants. We prove that unlike the classical elasticity theory, in the case under consideration we have no oscillating stress singularities for physical fields near the interfacial crack edges. However, it should be mentioned that in comparison with the classical elasticity case, the stress singularity exponents increase and are greater than $\frac{1}{2}$, in general.

In Appendix, for the reader's convenience, we collected some auxiliary results used in the main text of the paper.

2. Formulation of the Mixed Boundary-Transmission Problem

2.1. **Geometrical configuration of the composite.** Let $\Omega^{(1)}$ and $\Omega^{(2)}$ be the bounded disjoint domains of the three-dimensional Euclidean space $\mathbb{R}^3$ with boundaries $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$, respectively. Moreover, let $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$ have a nonempty, simply connected intersection $\overline{\Gamma} := \partial\Omega^{(1)} \cap \partial\Omega^{(2)}$ of positive measure. From now on, Γ will be referred to as an *interface*. Throughout the paper, $n = n^{(1)}$ and $\nu = n^{(2)}$ stand for the outward unit normal vectors to $\partial\Omega^{(1)}$ and to $\partial\Omega^{(2)}$, respectively. Clearly, $n(x) = -\nu(x)$ for $x \in \Gamma$.

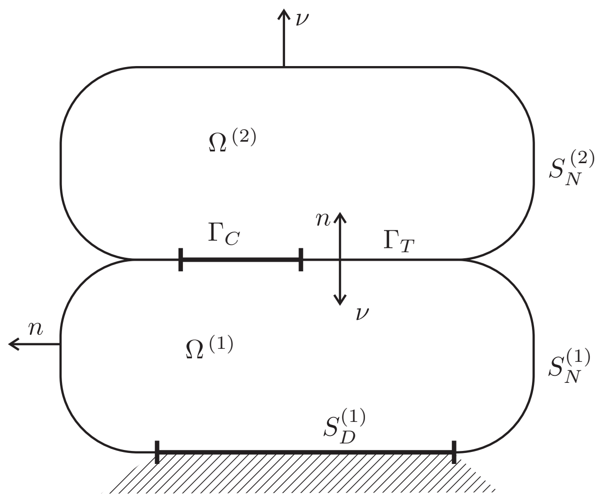

FIGURE 1. Composed body

Further, let $\overline{\Gamma} = \overline{\Gamma_T} \cup \overline{\Gamma_C}$, where Γ_C is an open, simply connected proper part of Γ. Moreover, $\Gamma_T \cap \Gamma_C = \varnothing$ and $\partial\Gamma \cap \overline{\Gamma_C} = \varnothing$.

We set $S_N^{(2)} := \partial\Omega^{(2)} \setminus \overline{\Gamma}$ and $S^{(1)} := \partial\Omega^{(1)} \setminus \overline{\Gamma}$. Further, we denote by $S_D^{(1)}$ some open, nonempty, proper sub-manifold of $S^{(1)}$ and put $S_N^{(1)} := S^{(1)} \setminus \overline{S_D^{(1)}}$. Thus, we have the following dissections of the boundary surfaces (see Figure 1):

$$\partial\Omega^{(1)} = \overline{\Gamma_T} \cup \overline{\Gamma_C} \cup \overline{S_N^{(1)}} \cup \overline{S_D^{(1)}}, \quad \partial\Omega^{(2)} = \overline{\Gamma_T} \cup \overline{\Gamma_C} \cup \overline{S_N^{(2)}}.$$

Throughout the paper, for simplicity, we assume that $\partial\Omega^{(2)}$, $\partial\Omega^{(1)}$, $\partial S_N^{(2)}$, $\partial\Gamma_T$, $\partial\Gamma_C$, $\partial S_D^{(1)}$, $\partial S_N^{(1)}$ are C^∞-smooth and $\partial\Omega^{(2)} \cap \overline{S_D^{(1)}} = \varnothing$.

Let $\Omega^{(1)}$ be occupied by an anisotropic homogeneous elastic medium revealing thermo-electro-magnetic properties described by Green-Lindsay's model of generalized thermo-electro-magneto-elasticity and $\Omega^{(2)}$ be filled by an anisotropic homogeneous elastic medium (e.g. metallic solid) with properties described by Green-Lindsay's generalized thermo-elasticity model. These two bodies interact along the interface Γ with the interfacial crack Γ_C. Moreover, it is assumed that the composed body is fixed along the sub-surface $S_D^{(1)}$ (the Dirichlet part of the boundary $\partial\Omega^{(1)}$), while on the sub-manifolds $S_N^{(2)}$ and $S_N^{(1)}$ we have the Neumann type boundary conditions.

In the domain $\Omega^{(1)}$ we have a six-dimensional physical field described by the displacement vector $u^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})^\top$, the electric potential $u_4^{(1)} = \varphi^{(1)}$, the magnetic potential $u_5^{(1)} = \psi^{(1)}$, and the temperature distribution function $u_6^{(1)} = \vartheta^{(1)}$, while in the domain $\Omega^{(2)}$ we have a four-dimensional thermoelastic field represented by the displacement vector $u^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)})^\top$ and temperature distribution function $u_4^{(2)} = \vartheta^{(2)}$. The superscript $(\cdot)^\top$ denotes transposition operation.

Throughout the paper, the summation over the repeated indices is meant from 1 to 3, unless otherwise stated.

2.2. **GTE Model.** In the domain $\Omega^{(2)}$ of the composed body, the system of pseudo-oscillation equations obtained from the dynamical equations of the generalized Green-Lindsay's linear model of thermoelasticity in matrix form reads as (see [7,11])

$$A^{(2)}(\partial_x, \tau)\, U^{(2)}(x, \tau) = \Phi^{(2)}(x, \tau)\,,$$

where $U^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, u_4^{(2)})^\top := (u^{(2)}, \vartheta^{(2)})^\top$ is a sought for complex-valued vector function, $\Phi^{(2)} = (\Phi_1^{(2)}, \dots, \Phi_4^{(2)})^\top$ is a given vector function, and

$$A^{(2)}(\partial_x, \tau) = \Big[A_{pq}^{(2)}(\partial_x, \tau)\Big]_{4\times 4} := \begin{bmatrix} [c_{rjkl}^{(2)}\partial_j\partial_l - \varrho^{(2)}\tau^2\delta_{rk}]_{3\times 3} & [-(1+\nu_0^{(2)}\tau)\lambda_{rj}^{(2)}\partial_j]_{3\times 1} \\ [-\tau\lambda_{kl}^{(2)}\partial_l]_{1\times 3} & \eta_{jl}^{(2)}\partial_j\partial_l - \tau d_0^{(2)} - \tau^2 h_0^{(2)} \end{bmatrix}_{4\times 4}. \tag{2.1}$$

Here, $\tau = \sigma + i\omega$ is a complex parameter, $u^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)})^\top$ is the displacement vector, $u_4^{(2)} := \vartheta^{(2)} = T^{(2)} - T_0$ is the relative temperature (temperature increment), $\varrho^{(2)}$ is the mass density, $c_{ijkl}^{(2)}$ are the elastic constants, $\varkappa_{kj}^{(2)}$ are the thermal conductivity constants, $\lambda_{rj}^{(2)}$ are the coefficients, coupling thermal, electric and magnetic fields, $\nu_0^{(2)}$ and $h_0^{(2)}$ are two relaxation times, $d_0^{(2)}$ is the constitutive coefficient; $T_0 > 0$ is the initial temperature, i.e., the temperature in the natural state in the absence of deformation and electromagnetic fields. We employ the notation $\partial = \partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$.

For an isotropic medium we have (see [22]):

$$c_{ijlk}^{(2)} = \lambda^{(2)}\,\delta_{ij}\,\delta_{lk} + \mu^{(2)}\,(\delta_{il}\,\delta_{jk} + \delta_{ik}\,\delta_{jl}), \quad \lambda_{ij}^{(2)} = \lambda^{(2)}\,\delta_{ij}, \ \ \eta_{ij}^{(2)} = \eta^{(2)}\,\delta_{ij}, \tag{2.2}$$

where $\lambda^{(2)}$ and $\mu^{(2)}$ are the Lamé constants and δ_{ij} is Kronecker's delta.

The stress operator in the generalised thermo-elasticity theory has the form

$$\mathcal{T}^{(2)}(\partial_x, \nu, \tau) = \big[\, \mathcal{T}_{pq}^{(2)}(\partial_x, \nu, \tau)\, \big]_{4\times 4} := \begin{bmatrix} [c_{rjkl}^{(2)}\,\nu_j\partial_l]_{3\times 3} & [-(1+\nu_0^{(2)}\tau)\lambda_{rj}^{(2)}\nu_j]_{3\times 1} \\ [0]_{1\times 3} & \eta_{jl}^{(2)}\nu_j\partial_l \end{bmatrix}_{4\times 4}.$$

Note that for a four-dimensional vector $U^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, u_4^{(2)})^\top$ we have

$$\mathcal{T}^{(2)}(\partial_x, \nu, \tau)U^{(2)} = (\sigma_{1j}^{(2)}\nu_j, \sigma_{2j}^{(2)}\nu_j, \sigma_{3j}^{(2)}\nu_j, -T_0^{-1}q_j^{(2)}\nu_j)^\top,$$

where $\sigma_{kj}^{(2)}$, $k,j = 1,2,3$, are components of the stress tensor, $\sigma^{(2)} = (\sigma_{1j}^{(2)}\nu_j, \sigma_{2j}^{(2)}\nu_j, \sigma_{3j}^{(2)}\nu_j)^\top$ is the mechanical stress vector and $q = q_j^{(2)}\nu_j$ is the heat flow across the surface element with normal ν (for details see [7]).

The constants involved in the above equations satisfy the following symmetry conditions:

$$c_{ijkl}^{(2)} = c_{jikl}^{(2)} = c_{klij}^{(2)}, \ \ \lambda_{ij}^{(2)} = \lambda_{ji}^{(2)}, \ \ \eta_{ij}^{(2)} = \eta_{ji}^{(2)}, \ \ i,j,k,l = 1,2,3. \tag{2.3}$$

Moreover, from physical considerations related to the positive definiteness of the potential energy, it follows that there exist positive constants c_0 and c_1 such that

$$c_{ijkl}^{(2)}\,\xi_{ij}\,\xi_{kl} \geqslant c_0\,\xi_{ij}\,\xi_{ij}, \ \ \eta_{ij}^{(2)}\xi_i\xi_j \geqslant c_1\,\xi_i\,\xi_i \quad \text{for all } \ \xi_{ij} = \xi_{ji} \in \mathbb{R}, \ \ \xi_j \in \mathbb{R}, \ i,j = 1,2,3. \tag{2.4}$$

In particular, the first inequality above implies that the density of potential energy

$$E^{(2)}\big(u^{(2)}, u^{(2)}\big) = c_{ijlk}^{(2)}\, s_{ij}^{(2)}\, s_{lk}^{(2)},$$

corresponding to the real-valued displacement vector $u^{(2)}$, is positive definite with respect to the symmetric components of the strain tensor $s_{lk}^{(2)} = s_{kl}^{(2)} = 2^{-1}(\partial_k\, u_j^{(2)} + \partial_j\, u_k^{(2)})$.

By $A^{(2,0)}(-i\xi)$ with $\xi \in \mathbb{R}^3$ we denote the principal homogeneous symbol matrix of the operator $A^{(2)}(\partial_x, \tau)$,

$$A^{(2,0)}(-i\xi) = A^{(2,0)}(i\xi) = -A^{(2,0)}(\xi) = -\begin{bmatrix} [c_{rjkl}^{(2)}\xi_j\xi_l]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \eta_{jl}^{(2)}\xi_j\xi_l \end{bmatrix}_{4\times 4}.$$

The symmetry conditions (2.3) and inequalities (2.4) imply that the matrix $A^{(2,0)}(\xi)$ is positive definite, i.e., there is a positive constant C depending only on the material parameters such that

$$\big(A^{(2,0)}(\xi)\zeta\cdot\zeta\big)=\big(-A^{(2,0)}(-i\xi)\zeta\cdot\zeta\big)=\Big(\sum_{k,j=1}^{4}A^{(2,0)}_{kj}(\xi)\zeta_j\overline{\zeta}_k\Big)\geq C|\xi|^2|\zeta|^2$$
$$\text{for all }\ \xi\in\mathbb{R}^3\ \text{ and for all }\ \zeta\in\mathbb{C}^4.$$

Here and in what follows, the central dot denotes the scalar product in the space of complex-valued vectors $\mathbb{C}^m$ and the over bar denotes complex conjugation.

2.3. **GTEME Model.** In $\Omega^{(1)}$, the thermo-mechanical and electro-magnetic fields are governed by the following pseudo-oscillation system of equations of Green-Lindsay's thermo-electro-magneto-elasticity theory (see [7]):

$$A^{(1)}(\partial_x,\tau)U^{(1)}(x,\tau)=\Phi^{(1)}(x,\tau),$$

where

$$A^{(1)}(\partial_x,\tau)=\Big[A^{(1)}_{pq}(\partial_x,\tau)\Big]_{6\times6}$$
$$:=\begin{bmatrix}[c^{(1)}_{rjkl}\partial_j\partial_l-\varrho^{(1)}\tau^2\delta_{rk}]_{3\times3} & [e^{(1)}_{lrj}\partial_j\partial_l]_{3\times1} & [q^{(1)}_{lrj}\partial_j\partial_l]_{3\times1} & [-(1+\nu^{(1)}_0\tau)\lambda^{(1)}_{rj}\partial_j]_{3\times1}\\ [-e^{(1)}_{jkl}\partial_j\partial_l]_{1\times3} & \varkappa^{(1)}_{jl}\partial_j\partial_l & a^{(1)}_{jl}\partial_j\partial_l & -(1+\nu^{(1)}_0\tau)p^{(1)}_j\partial_j\\ [-q^{(1)}_{jkl}\partial_j\partial_l]_{1\times3} & a^{(1)}_{jl}\partial_j\partial_l & \mu^{(1)}_{jl}\partial_j\partial_l & -(1+\nu^{(1)}_0\tau)m^{(1)}_j\partial_j\\ [-\tau\lambda^{(1)}_{kl}\partial_l]_{1\times3} & \tau p^{(1)}_l\partial_l & \tau m^{(1)}_l\partial_l & \eta^{(1)}_{jl}\partial_j\partial_l-\tau^2h^{(1)}_0-\tau d^{(1)}_0\end{bmatrix}_{6\times6} \tag{2.5}$$

is the differential operator associated with the pseudo-oscillation equations of the thermo-electro-magneto-elasticity theory, obtained by the Laplace transform from the corresponding dynamical equations, $U^{(1)}=(u^{(1)}_1,u^{(1)}_2,u^{(1)}_3,u^{(1)}_4,u^{(1)}_5,u^{(1)}_6)^\top:=(u^{(1)},\varphi^{(1)},\psi^{(1)},\vartheta^{(1)})^\top$ is the sought for complex-valued vector function, $u^{(1)}=(u^{(1)}_1,u^{(1)}_2,u^{(1)}_3)^\top$ denotes the displacement vector, $\varphi^{(1)}$ and $\psi^{(1)}$ stand for the electric and magnetic potentials and $\vartheta^{(1)}=T^{(1)}-T_0$ is the relative temperature (temperature increment), and $\Phi^{(1)}=\big(\Phi^{(1)}_1,\dots,\Phi^{(1)}_6\big)^\top$ is a given vector function. Here we also employ the following notation: $\varrho^{(1)}$ is the mass density, $c^{(1)}_{rjkl}$ are the elastic constants, $e^{(1)}_{jkl}$ are the piezoelectric constants, $q^{(1)}_{jkl}$ are the piezomagnetic constants, $\varkappa^{(1)}_{jk}$ are the dielectric (permittivity) constants, $\mu^{(1)}_{jk}$ are the magnetic permeability constants, $a^{(1)}_{jk}$ are the electromagnetic coupling coefficients, $p^{(1)}_j$, $m^{(1)}_j$, and $\lambda^{(1)}_{rj}$ are the coefficients, coupling thermal field with displacement, electric and magnetic fields, $\eta^{(1)}_{jk}$ are the heat conductivity coefficients, T_0 is the initial reference temperature, that is, the temperature in the natural state in the absence of deformation and electromagnetic fields, $\nu^{(1)}_0$ and $h^{(1)}_0$ are two relaxation times, $a^{(1)}_0$ and $d^{(1)}_0$ are some constitutive coefficients.

Throughout the paper, we assume that the time relaxation parameters $\nu^{(1)}_0$ and $\nu^{(2)}_0$ involved in operators (2.5) and (2.1) are the same and we set

$$\nu^{(1)}_0=\nu^{(2)}_0=\nu_0.$$

The constants involved in the above equations satisfy the following symmetry conditions:

$$c^{(1)}_{rjkl}=c^{(1)}_{jrkl}=c^{(1)}_{klrj},\ \ e^{(1)}_{klj}=e^{(1)}_{kjl},\ \ q^{(1)}_{klj}=q^{(1)}_{kjl},$$
$$\varkappa^{(1)}_{kj}=\varkappa^{(1)}_{jk},\ \ \lambda^{(1)}_{kj}=\lambda^{(1)}_{jk},\ \ \mu^{(1)}_{kj}=\mu^{(1)}_{jk},\ \ a^{(1)}_{kj}=a^{(1)}_{jk},\ \ \eta^{(1)}_{kj}=\eta^{(1)}_{jk},\ \ r,j,k,l=1,2,3. \tag{2.6}$$

From physical considerations it follows that (see, e.g., [3,27,32]):

$$c^{(1)}_{rjkl}\xi_{rj}\xi_{kl}\geq\delta_0\xi_{kl}\,\xi_{kl},\ \ \varkappa^{(1)}_{kj}\xi_k\xi_j\geq\delta^{(1)}_1|\xi|^2,\ \ \mu^{(1)}_{kj}\xi_k\xi_j\geq\delta_2|\xi|^2,\ \ \eta^{(1)}_{kj}\xi_k\xi_j\geq\delta_3|\xi|^2,$$
$$\text{for all }\ \xi_{kj}=\xi_{jk}\in\mathbb{R}\ \text{ and for all }\ \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3, \tag{2.7}$$

$$\nu_0>0,\ \ h^{(1)}_0>0,\ \ d^{(1)}_0\nu_0-h^{(1)}_0>0, \tag{2.8}$$

where δ_0, δ_1, δ_2, and δ_3 are the positive constants depending on material parameters.

Due to the symmetry conditions (2.6), with the help of (2.7), we easily derive

$$\begin{gathered} c^{(1)}_{rjkl}\zeta_{rj}\overline{\zeta_{kl}} \geq \delta_0\zeta_{kl}\overline{\zeta_{kl}}, \ \ \varkappa^{(1)}_{kj}\zeta_k\overline{\zeta_j} \geq \delta_1|\zeta|^2, \ \ \mu^{(1)}_{kj}\zeta_k\overline{\zeta_j} \geq \delta_2|\zeta|^2, \ \ \eta^{(1)}_{kj}\zeta_k\overline{\zeta_j} \geq \delta_3|\ \zeta|^2, \\ \text{for all } \ \zeta_{kj}=\zeta_{jk}\in\mathbb{C} \ \text{ and for all } \ \zeta=(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3. \end{gathered} \tag{2.9}$$

More careful analysis related to the positive definiteness of the potential energy and the thermodynamical laws insure that the following 8×8 matrix

$$M=[M_{kj}]_{8\times 8}:=\begin{bmatrix} [\varkappa^{(1)}_{jl}]_{3\times 3} & [a^{(1)}_{jl}]_{3\times 3} & [p^{(1)}_j]_{3\times 1} & [\nu_0 p^{(1)}_j]_{3\times 1} \\ [a^{(1)}_{jl}]_{3\times 3} & [\mu^{(1)}_{jl}]_{3\times 3} & [m^{(1)}_j]_{3\times 1} & [\nu_0 m_j]_{3\times 1} \\ [p^{(1)}_j]_{1\times 3} & [m^{(1)}_j]_{1\times 3} & d^{(1)}_0 & h^{(1)}_0 \\ [\nu_0 p^{(1)}_j]_{1\times 3} & [\nu_0 m^{(1)}_j]_{1\times 3} & h^{(1)}_0 & \nu_0 h^{(1)}_0 \end{bmatrix}_{8\times 8} \tag{2.10}$$

is positive definite (see [7]). Note that the positive definiteness of M remains valid if the parameters $p^{(1)}_j$ and $m^{(1)}_j$ in (2.10) are replaced by the opposite ones, $-p^{(1)}_j$ and $-m^{(1)}_j$. Moreover, it follows that the matrices

$$\Lambda^{(1)}:=\begin{bmatrix} [\varkappa^{(1)}_{kj}]_{3\times 3} & [a^{(1)}_{kj}]_{3\times 3} \\ [a^{(1)}_{kj}]_{3\times 3} & [\mu^{(1)}_{kj}]_{3\times 3} \end{bmatrix}_{6\times 6}, \quad \Lambda^{(2)}:=\begin{bmatrix} d^{(1)}_0 & h^{(1)}_0 \\ h^{(1)}_0 & \nu_0 h_0 \end{bmatrix}_{2\times 2} \tag{2.11}$$

are positive definite as well, i.e.,

$$\varkappa^{(1)}_{kj}\zeta'_k\overline{\zeta'_j}+a^{(1)}_{kj}(\zeta'_k\overline{\zeta''_j}+\overline{\zeta'_k}\zeta''_j)+\mu^{(1)}_{kj}\zeta''_k\overline{\zeta''_j}\geq \kappa^{(1)}_1\big(|\zeta'|^2+|\zeta''|^2\big) \ \ \forall\,\zeta',\zeta''\in\mathbb{C}^3, \tag{2.12}$$

$$d^{(1)}_0|z_1|^2+h_0\big(z_1\overline{z_2}+\overline{z_1}z_2\big)+\nu_0 h^{(1)}_0|z_2|^2\geq \kappa^{(1)}_2\big(|z_1|^2+|z_2|^2\big) \ \ \forall\,z_1,z_2\in\mathbb{C}, \tag{2.13}$$

with some positive constants $\kappa^{(1)}_1$ and $\kappa^{(1)}_2$ depending on the material parameters involved in (2.11) (for details see [7]).

The stress operator $\mathcal{T}^{(1)}(\partial_x,n,\tau)$ in the generalized thermo-electro-magneto-elasticity theory reads as

$$\begin{aligned} \mathcal{T}^{(1)}(\partial_x,n,\tau) &= \big[\ \mathcal{T}^{(1)}_{pq}(\partial_x,n,\tau)\ \big]_{6\times 6} \\ &:= \begin{bmatrix} [c^{(1)}_{rjkl}n_j\partial_l]_{3\times 3} & [e^{(1)}_{lrj}n_j\partial_l]_{3\times 1} & [q^{(1)}_{lrj}n_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda^{(1)}_{rj}n_j]_{3\times 1} \\ [-e^{(1)}_{jkl}n_j\partial_l]_{1\times 3} & \varkappa^{(1)}_{jl}n_j\partial_l & a^{(1)}_{jl}n_j\partial_l & -(1+\nu_0\tau)p^{(1)}_j n_j \\ [-q^{(1)}_{jkl}n_j\partial_l]_{1\times 3} & a^{(1)}_{jl}n_j\partial_l & \mu^{(1)}_{jl}n_j\partial_l & -(1+\nu_0\tau)m^{(1)}_j n_j \\ [0]_{1\times 3} & 0 & 0 & \eta^{(1)}_{jl}n_j\partial_l \end{bmatrix}_{6\times 6}. \end{aligned}$$

Note that for a vector $U^{(1)}:=(u^{(1)},\varphi^{(1)},\psi^{(1)},\vartheta^{(1)})^\top$, the components of the corresponding generalized stress vector $\mathcal{T}^{(1)}U^{(1)}$ have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of generalized thermo-electro-magneto-elasticity, the forth and the fifth components correspond to the normal components of the electric displacement vector and the magnetic induction vector, respectively, with opposite sign, and finally, the sixth component is $(-T_0^{-1})$ times the normal component of the heat flux vector (for details see [7, Ch.2]).

Denote by $A^{(1,0)}(-i\xi)$ with $\xi\in\mathbb{R}^3$ the principal homogeneous symbol matrix of the differential operator $A^{(1)}(\partial_x,\tau)$. We have

$$A^{(1,0)}(-i\xi)=-A^{(1,0)}(\xi)=\begin{bmatrix} [-c^{(1)}_{rjkl}\xi_j\xi_l]_{3\times 3} & [-e^{(1)}_{lrj}\xi_j\xi_l]_{3\times 1} & [-q^{(1)}_{lrj}\xi_j\xi_l]_{3\times 1} & [0]_{3\times 1} \\ [e^{(1)}_{jkl}\xi_j\xi_l]_{1\times 3} & -\varkappa^{(1)}_{jl}\xi_j\xi_l & -a^{(1)}_{jl}\xi_j\xi_l & 0 \\ [q^{(1)}_{jkl}\xi_j\xi_l]_{1\times 3} & -a^{(1)}_{jl}\xi_j\xi_l & -\mu^{(1)}_{jl}\xi_j\xi_l & 0 \\ [0]_{1\times 3} & 0 & 0 & -\eta^{(1)}_{jl}\xi_j\xi_l \end{bmatrix}_{6\times 6}.$$

From the symmetry conditions (2.6), inequalities (2.7) and the positive definiteness of the matrix $\Lambda^{(1)}$ defined in (2.11) it follows that there is a positive constant C depending only on the material parameters such that

$$\mathrm{Re}\big(- A^{(1,0)}(-i\xi)\zeta \cdot \zeta\big) = \mathrm{Re}\Big(\sum_{k,j=1}^{6} A^{(1,0)}_{kj}(\xi)\zeta_j\overline{\zeta}_k\Big) \geq C|\xi|^2|\zeta|^2$$
$$\text{for all } \ \xi \in \mathbb{R}^3 \ \text{ and for all } \ \zeta \in \mathbb{C}^6.$$

Therefore, $-A^{(1)}(\partial_x, \tau)$ is a non-selfadjoint strongly elliptic differential operator.

2.4. **Formulation of the Mixed Boundary-Transmission problem.** By W^r_p, H^s_p and $B^s_{p,q}$ with $r \geqslant 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, we denote the Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively, (see, e.g., [33]). Recall that $H^r_2 = W^r_2 = B^r_{2,2}$, $H^s_2 = B^s_{2,2}$, $W^t_p = B^t_{p,p}$, and $H^k_p = W^k_p$, for any $r \geqslant 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

Let $\mathcal{M}_0$ be a smooth surface without boundary. For a proper sub-manifold $\mathcal{M} \subset \mathcal{M}_0$, we denote by $\widetilde{H}^s_p(\mathcal{M})$ and $\widetilde{B}^s_{p,q}(\mathcal{M})$ the subspaces of $H^s_p(\mathcal{M}_0)$ and $B^s_{p,q}(\mathcal{M}_0)$, respectively,

$$\widetilde{H}^s_p(\mathcal{M}) = \big\{g : \ g \in H^s_p(\mathcal{M}_0), \ \mathrm{supp}\, g \subset \overline{\mathcal{M}}\big\},$$
$$\widetilde{B}^s_{p,q}(\mathcal{M}) = \big\{g : \ g \in B^s_{p,q}(\mathcal{M}_0), \ \mathrm{supp}\, g \subset \overline{\mathcal{M}}\big\},$$

while $H^s_p(\mathcal{M})$ and $B^s_{p,q}(\mathcal{M})$ stand for the spaces of restrictions on $\mathcal{M}$ of functions from $H^s_p(\mathcal{M}_0)$ and $B^s_{p,q}(\mathcal{M}_0)$, respectively,

$$H^s_p(\mathcal{M}) = \big\{r_{\mathcal{M}} f : \ f \in H^s_p(\mathcal{M}_0)\big\}, \qquad B^s_{p,q}(\mathcal{M}) = \big\{r_{\mathcal{M}} f : \ f \in B^s_{p,q}(\mathcal{M}_0)\big\},$$

where $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$.

Now we formulate the mixed boundary-transmission problem: *Find vector functions*

$$U^{(1)} = (u^{(1)}, \varphi^{(1)}, \psi^{(1)}, \vartheta^{(1)})^\top = (u^{(1)}_1, \dots, u^{(1)}_6)^\top \ : \ \Omega^{(1)} \to \mathbb{C}^6,$$
$$U^{(2)} = (u^{(2)}, \vartheta^{(2)})^\top = (u^{(2)}_1, \dots, u^{(2)}_4)^\top \ : \ \Omega^{(2)} \to \mathbb{C}^4,$$

belonging, respectively, to the spaces $[W^1_p(\Omega^{(2)})]^4$ *and* $[W^1_p(\Omega^{(1)})]^6$ *with* $1 < p < \infty$ *and satisfying*
(i) *the systems of partial differential equations*:

$$A^{(1)}(\partial_x, \tau)\, U^{(1)} = 0 \ \text{ in } \ \Omega^{(1)}, \tag{2.14}$$
$$A^{(2)}(\partial_x, \tau)\, U^{(2)} = 0 \ \text{ in } \ \Omega^{(2)}, \tag{2.15}$$

(ii) *the boundary conditions*:

$$\big\{\mathcal{T}^{(1)}(\partial_x, n, \tau)\, U^{(1)}\big\}^+ = Q^{(1)} \ \text{ on } \ S^{(1)}_N, \tag{2.16}$$
$$\big\{\mathcal{T}^{(2)}(\partial_x, \nu, \tau)\, U^{(2)}\big\}^+ = Q^{(2)} \ \text{ on } \ S^{(2)}_N, \tag{2.17}$$
$$\big\{U^{(1)}\big\}^+ = f^{(1)} \ \text{ on } \ S^{(1)}_D, \tag{2.18}$$
$$\{u^{(1)}_4\}^+ = f_4 \ \text{ on } \ \Gamma_T, \tag{2.19}$$
$$\{u^{(1)}_5\}^+ = f_5 \ \text{ on } \ \Gamma_T, \tag{2.20}$$

(iii) *the transmission conditions on* Γ_T *:*

$$\Big\{u^{(1)}_j\Big\}^+ - \big\{u^{(2)}_j\big\}^+ = f_j \quad \text{on } \ \Gamma_T, \qquad j = 1, 2, 3, \tag{2.21}$$
$$\Big\{u^{(1)}_6\Big\}^+ - \big\{u^{(2)}_4\big\}^+ = f_6 \quad \text{ on } \ \Gamma_T, \tag{2.22}$$
$$\big\{\big[\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big]_j\big\}^+ + \big\{[\mathcal{T}^{(2)}(\partial_x, \nu, \tau)U^{(2)}]_j\big\}^+ = F_j, \quad \text{on } \ \Gamma_T, \qquad j = 1, 2, 3, \tag{2.23}$$
$$\big\{\big[\mathcal{T}^{(1)}(\partial_x, n, \tau)U^{(1)}\big]_6\big\}^+ + \big\{[\mathcal{T}^{(2)}(\partial_x, \nu, \tau)U^{(2)}]_4\big\}^+ = F_4, \quad \text{on } \ \Gamma_T, \tag{2.24}$$

(iv) *the interfacial crack conditions on* Γ_C:

$$\big\{\mathcal{T}^{(1)}(\partial_x, n, \tau)\, U^{(1)}\big\}^+ = \widetilde{Q}^{(1)} \quad \text{on } \ \Gamma_C, \tag{2.25}$$

$$\big\{\mathcal{T}^{(2)}(\partial_x, \nu, \tau)\, U^{(2)}\big\}^+ = \widetilde{Q}^{(2)} \quad \text{on } \ \Gamma_C, \tag{2.26}$$

where $n = -\nu$ *on* Γ,

$$\begin{aligned}
Q^{(1)} &= (Q_1^{(1)}, Q_2^{(1)}, Q_3^{(1)}, Q_4^{(1)}, Q_5^{(1)}, Q_6^{(1)})^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(S_N^{(1)})\,\big]^6,\\
\widetilde{Q}^{(1)} &= (\widetilde{Q}_1^{(1)}, \widetilde{Q}_2^{(1)}, \widetilde{Q}_3^{(1)}, \widetilde{Q}_4^{(1)}, \widetilde{Q}_5^{(1)}, \widetilde{Q}_6^{(1)})^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(\Gamma_C)\,\big]^6,\\
Q^{(2)} &= \big(Q_1^{(2)}, Q_2^{(2)}, Q_3^{(2)}, Q_4^{(2)}\big)^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(S_N^{(2)})\,\big]^4,\\
\widetilde{Q}^{(2)} &= \big(\widetilde{Q}_1^{(2)}, \widetilde{Q}_2^{(2)}, \widetilde{Q}_3^{(2)}, \widetilde{Q}_4^{(2)}\big)^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(\Gamma_C)\,\big]^4,\\
f^{(1)} &= (f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, f_4^{(1)}, f_5^{(1)}, f_6^{(1)})^\top \in \big[\, B_{p,p}^{1-\frac{1}{p}}(S_D^{(1)})\,\big]^6,\\
f &= \big(f_1, f_2, f_3, f_4, f_5, f_6\big)^\top \in \big[\, B_{p,p}^{1-\frac{1}{p}}(\Gamma_T)\,\big]^6,\\
F &= \big(F_1, F_2, F_3, F_4\big)^\top \in \big[\, B_{p,p}^{-\frac{1}{p}}(\Gamma_T)\,\big]^4.
\end{aligned} \tag{2.27}$$

Note that, in addition, the functions F_j, $Q_j^{(1)}$, $\widetilde{Q}_j^{(1)}$, $\widetilde{Q}_j^{(2)}$ and $Q_j^{(2)}$ have to satisfy some evident compatibility conditions (see Subsection 3.1, inclusion (3.22), (3.23)).
We have the following uniqueness theorem for $p = 2$.

Theorem 2.1. *Let* $\Omega^{(1)}$ *and* $\Omega^{(2)}$ *be the Lipschitz domains and either* $\tau = \sigma + i\omega$ *with* $\sigma > 0$ *or* $\tau = 0$. *Then the mixed boundary transmission problem* (2.14)–(2.26) *has at most one solution pair* $(U^{(1)}, U^{(2)})$ *in the space* $[W_2^1(\Omega^{(1)})]^6 \times [W_2^1(\Omega^{(2)})]^4$, *provided* $\operatorname{mes} S_D^{(1)} > 0$.

Proof. Proof of the theorem is quite similar to that of Theorem 1.1 in reference [6]. □

Later we will prove the uniqueness theorem for $p \neq 2$.

To prove the existence of solutions to the above formulated mixed boundary-transmission problem, we use the potential method and the theory of pseudodifferential equations. To this end, we introduce the following single layer potentials:

$$V_\tau^{(1)}(h^{(1)})(x) = \int\limits_{\partial\Omega^{(1)}} \Gamma^{(1)}(x-y, \tau)\, h^{(1)}(y)\, d_y S,$$

$$V_\tau^{(2)}(h^{(2)})(x) = \int\limits_{\partial\Omega^{(2)}} \Gamma^{(2)}(x-y, \tau)\, h^{(2)}(y)\, d_y S,$$

where $\Gamma^{(1)}(x,\tau)$ and $\Gamma^{(2)}(x,\tau)$ are the fundamental matrices of the differential operators $A^{(1)}(\partial_x, \tau)$ and $A^{(2)}(\partial_x, \tau)$, respectively, $h^{(1)} = (h_1^{(1)}, \dots, h_6^{(1)})^\top$ and $h^{(2)} = (h_1^{(2)}, \dots, h_4^{(2)})^\top$ are the density vector functions. The explicit expressions of the fundamental matrices $\Gamma^{(1)}(x,\tau)$ and $\Gamma^{(2)}(x,\tau)$ and their properties can be found in references [7] and [8].

We introduce also the following boundary integral operators generated by the single layer potentials

$$\mathcal{H}_\tau^{(1)}(h^{(1)})(z) = \int\limits_{\partial\Omega^{(1)}} \Gamma^{(1)}(z-y, \tau)\, h^{(1)}(y)\, d_y S, \quad z \in \partial\Omega^{(1)}, \tag{2.28}$$

$$\mathcal{K}_\tau^{(1)}(h^{(1)})(z) = \int\limits_{\partial\Omega^{(1)}} \mathcal{T}^{(1)}(\partial_z, n(z), \tau)\Gamma^{(1)}(z-y, \tau)\, h^{(1)}(y)\, d_y S, \quad z \in \partial\Omega^{(1)}, \tag{2.29}$$

$$\mathcal{H}_\tau^{(2)}(h^{(2)})(z) = \int\limits_{\partial\Omega^{(2)}} \Gamma^{(2)}(z-y, \tau)\, h^{(2)}(y)\, d_y S, \quad z \in \partial\Omega^{(2)}, \tag{2.30}$$

$$\mathcal{K}_\tau^{(2)}(h^{(2)})(z) = \int\limits_{\partial\Omega^{(2)}} \mathcal{T}^{(2)}(\partial_z, n(z), \tau)\Gamma^{(2)}(z-y,\tau)\, h^{(2)}(y)\, d_yS, \quad z \in \partial\Omega^{(2)}. \tag{2.31}$$

Note that $\mathcal{H}_\tau^{(1)}$ and $\mathcal{H}_\tau^{(2)}$ are pseudodifferential operators of order -1, while $\mathcal{K}_\tau^{(1)}$ and $\mathcal{K}_\tau^{(2)}$ are pseudodifferential operators of order 0, i.e., singular integral operators (for details see Appendix).

Now, we formulate several auxiliary lemmas proved in reference [8].

Lemma 2.2. *Let* $\operatorname{Re}\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution vector* $U^{(2)} \in [W_p^1(\Omega^{(2)})]^4$ *to the homogeneous equation* $A^{(2)}(\partial,\tau)\,U^{(2)} = 0$ *in* $\Omega^{(2)}$, *can be uniquely represented by the single layer potential*

$$U^{(2)} \;=\; V_\tau^{(2)}\Big(\big[P_\tau^{(2)}\big]^{-1}\chi^{(2)}\Big) \;\; in \;\; \Omega^{(2)},$$

where

$$P_\tau^{(2)} := -2^{-1}\, I_4 + \mathcal{K}_\tau^{(2)}, \quad \chi^{(2)} = \big\{\mathcal{T}^{(2)}U^{(2)}\big\}^+ \in \big[\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(2)})\big]^4, \tag{2.32}$$

and $\mathcal{K}_\tau^{(2)}$ *is defined by* (2.31).

For the mapping properties and invertibility of the operator $P_\tau^{(2)}$ in appropriate function spaces see Theorem 5.4.

Lemma 2.3. *Let* $\operatorname{Re}\tau = \sigma > 0$ *and*

$$P_\tau^{(1)} := -2^{-1}\, I_6 + \mathcal{K}_\tau^{(1)} + \beta\,\mathcal{H}_\tau^{(1)}, \tag{2.33}$$

where $\mathcal{K}_\tau^{(1)}$ *and* $\mathcal{H}_\tau^{(1)}$ *are defined by* (2.29) *and* (2.28), *respectively, and* β *is a smooth real-valued scalar function on* $S^{(1)}$, *not vanishing identically and satisfying the conditions*

$$\beta \geqslant 0, \qquad \operatorname{supp}\,\beta \subset S_D^{(1)}. \tag{2.34}$$

Then the operators

$$P_\tau^{(1)} \;:\; \big[\,H_p^s(\partial\Omega^{(1)})\big]^6 \to \big[\,H_p^s(\partial\Omega^{(1)})\big]^6,$$
$$P_\tau^{(1)} \;:\; \big[\,B_{p,q}^s(\partial\Omega^{(1)})\big]^6 \to \big[\,B_{p,q}^s(\partial\Omega^{(1)})\big]^6$$

are invertible for all $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, *and* $s \in \mathbb{R}$.

As a consequence, we have the following

Lemma 2.4. *Let* $\operatorname{Re}\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution* $U^{(1)} \in [W_p^1(\Omega^{(1)})]^6$ *to the homogeneous equation* $A^{(1)}(\partial_x,\tau)U^{(1)} = 0$ *in* $\Omega^{(1)}$ *can be uniquely represented by the single layer potential*

$$U^{(1)} = V_\tau^{(1)}\big(\big[P_\tau^{(1)}\big]^{-1}\chi\big) \;\; in \;\; \Omega^{(1)},$$

where

$$\chi = \{\mathcal{T}^{(1)}U^{(1)}\}^+ + \beta\,\{U^{(1)}\}^+ \in [\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(1)})]^6.$$

3. The Existence and Regularity Results

3.1. **Reduction to boundary equations.** Let us return to problem (2.14)–(2.26) and derive the equivalent boundary integral formulation. Keeping in mind (2.27), let

$$G^{(1)} := \begin{cases} Q^{(1)} & \text{on } \; S_N^{(1)}, \\ \widetilde{Q}^{(1)} & \text{on } \; \Gamma_C, \end{cases} \quad G^{(2)} := \begin{cases} Q^{(2)} & \text{on } \; S_N^{(2)}, \\ \widetilde{Q}^{(2)} & \text{on } \; \Gamma_C, \end{cases}$$
$$G^{(1)} \in \big[\,B_{p,p}^{-1/p}(S_N^{(1)} \cup \Gamma_C)\big]^6, \quad G^{(2)} \in \big[\,B_{p,p}^{-1/p}(S_N^{(2)} \cup \Gamma_C)\big]^4, \tag{3.1}$$

and

$$G_0^{(1)} = (G_{01}^{(1)},\dots,G_{06}^{(1)})^\top \in \big[\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(1)})\big]^6, \quad G_0^{(2)} = (G_{01}^{(2)},\dots,G_{04}^{(2)})^\top \in \big[\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(2)})\big]^4 \tag{3.2}$$

be some fixed extensions of the vector functions $G^{(1)}$ and $G^{(2)}$, respectively, onto $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$ preserving the space. It is evident that arbitrary extensions of the same vector functions can then be represented as

$$G^{(1)*} = G_0^{(1)} + \psi + h^{(1)}, \quad G^{(2)*} = G_0^{(2)} + h^{(2)},$$

where

$$\begin{aligned} \psi &:= (\psi_1, \ldots, \psi_6)^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D^{(1)})\,\big]^6, \\ h^{(1)} &:= (h_1^{(1)}, \ldots, h_6^{(1)})^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T)\,\big]^6, \\ h^{(2)} &:= (h_1^{(2)}, \ldots, h_4^{(2)})^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T)\,\big]^4 \end{aligned} \tag{3.3}$$

are arbitrary vector functions.

We look for a solution pair $(U^{(1)}, U^{(2)})$ of the mixed boundary-transmission problem (2.14)–(2.26) in the form of single layer potentials

$$U^{(1)} = (u_1^{(1)}, \ldots, u_6^{(1)})^\top = V_\tau^{(1)}\big([\,P_\tau^{(1)}]^{-1}\,\big[\,G_0^{(1)} + \psi + h^{(1)}\big]\,\big) \ \text{ in } \ \Omega^{(1)}, \tag{3.4}$$

$$U^{(2)} = (u_1^{(2)}, \ldots, u_4^{(2)})^\top = V_\tau^{(2)}\big([\,P_\tau^{(2)}\,]^{-1}\,\big[\,G_0^{(2)} + h^{(2)}\big]\,\big) \ \text{ in } \ \Omega^{(2)}, \tag{3.5}$$

where $P_\tau^{(1)}$ and $P_\tau^{(2)}$ are given by (2.33) and (2.32), and $h^{(1)}$, $h^{(2)}$ and ψ are the unknown vector functions satisfying inclusions (3.3).

Keeping in mind (2.34), we see that the homogeneous differential equations (2.14), (2.15), the boundary conditions (2.16), (2.17) and the crack conditions (2.25), (2.26) are satisfied automatically.

The remaining boundary and transmission conditions (2.21)–(2.24) lead to the system of pseudodifferential equations for the unknown vector functions ψ, $h^{(1)}$ and $h^{(2)}$,

$$r_{_{S_D^{(1)}}}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,\big(G_0^{(1)} + \psi + h^{(1)}\big)\,\big] = f^{(1)} \ \text{ on } \ S_D^{(1)}, \tag{3.6}$$

$$r_{_{\Gamma_T}}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\,G_0^{(1)} + \psi + h^{(1)})\,\big]_j = f_j \ \text{ on } \ \Gamma_T, \ \ j = 4, 5, \tag{3.7}$$

$$\begin{aligned} r_{_{\Gamma_T}}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(G_0^{(1)} + \psi + h^{(1)})\big]_j - r_{_{\Gamma_T}}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(G_0^{(2)} + h^{(2)})\big]_j = f_j \ \text{ on } \ \Gamma_T, \\ j = 1, 2, 3, \end{aligned} \tag{3.8}$$

$$r_{_{\Gamma_T}}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(G_0^{(1)} + \psi + h^{(1)})\big]_6 - r_{_{\Gamma_T}}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(G_0^{(2)} + h^{(2)})\big]_4 = f_6 \ \text{ on } \ \Gamma_T, \tag{3.9}$$

$$r_{_{\Gamma_T}}[G_0^{(1)} + \psi + h^{(1)}]_j + r_{_{\Gamma_T}}[G_0^{(2)} + h^{(2)}]_j = F_j \ \text{ on } \ \Gamma_T, \ \ j = 1, 2, 3, \tag{3.10}$$

$$r_{_{\Gamma_T}}[G_0^{(1)} + \psi + h^{(1)}]_6 + r_{_{\Gamma_T}}[G_0^{(2)} + h^{(2)}]_4 = F_4 \ \text{ on } \ \Gamma_T. \tag{3.11}$$

After some rearrangement we get the system of pseudodifferential equations

$$r_{_{S_D^{(1)}}}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,\big(\psi + h^{(1)}\big)\,\big] = \widetilde{f}^{(1)} \ \text{ on } \ S_D^{(1)}, \tag{3.12}$$

$$r_{_{\Gamma_T}}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\,\psi + h^{(1)})\,\big]_j = \widetilde{f}_j \ \text{ on } \ \Gamma_T, \ \ j = 4, 5, \tag{3.13}$$

$$r_{_{\Gamma_T}}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\psi + h^{(1)})\big]_j - r_{_{\Gamma_T}}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(\,h^{(2)})\big]_j = \widetilde{f}_j \ \text{ on } \ \Gamma_T, \ \ j = 1, 2, 3, \tag{3.14}$$

$$r_{_{\Gamma_T}}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\psi + h^{(1)})\big]_6 - r_{_{\Gamma_T}}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(\,h^{(2)})\big]_4 = \widetilde{f}_6 \ \text{ on } \ \Gamma_T, \tag{3.15}$$

$$r_{_{\Gamma_T}} h_j^{(1)} + r_{_{\Gamma_T}} h_j^{(2)} = \widetilde{F}_j \ \text{ on } \ \Gamma_T, \ \ j = 1, 2, 3, \tag{3.16}$$

$$r_{_{\Gamma_T}} h_6^{(1)} + r_{_{\Gamma_T}} h_4^{(2)} = \widetilde{F}_4 \ \text{ on } \ \Gamma_T, \tag{3.17}$$

where

$$\widetilde{f}_k^{(1)} := f_k^{(1)} - r_{_{S_D^{(1)}}}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,G_0^{(1)}\,\big]_k \in B_{p,p}^{1-\frac{1}{p}}(S_D^{(1)}), \ \ k = \overline{1,6}, \tag{3.18}$$

$$\widetilde{f}_j := f_j - r_{_{\Gamma_T}}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,G_0^{(1)}\big]_j \in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T), \ \ j = 4, 5, \tag{3.19}$$

$$\widetilde{f}_j := f_j + r_{_{\Gamma_T}}\big[\,\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}\,G_0^{(2)}\big]_j - r_{_{\Gamma_T}}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,G_0^{(1)}\big]_j \in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T), \ \ j = 1, 2, 3, \tag{3.20}$$

$$\widetilde{f}_6 := f_6 + r_{\Gamma_T}\,\big[\,\mathcal{H}^{(2)}_\tau\,[P^{(2)}_\tau]^{-1}\,G^{(2)}_0\big]_4 - r_{\Gamma_T}\,\big[\,\mathcal{H}^{(1)}_\tau\,[P^{(1)}_\tau]^{-1}\,G^{(1)}_0\big]_6 \in B^{1-\frac{1}{p}}_{p,p}(\Gamma_T), \tag{3.21}$$

$$\widetilde{F}_j := F_j - r_{\Gamma_T} G^{(1)}_{0j} - r_{\Gamma_T} G^{(2)}_{0j} \in r_{\Gamma_T} \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T), \quad j=1,2,3, \tag{3.22}$$

$$\widetilde{F}_4 := F_4 - r_{\Gamma_T} G^{(1)}_{06} - r_{\Gamma_T} G^{(2)}_{04} \in r_{\Gamma_T} \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T). \tag{3.23}$$

Inclusions (3.22), (3.23) are the *compatibility conditions* for the mixed boundary-transmission problem under consideration. Therefore, in what follows, we assume that $\widetilde{F}_j$ are extended from Γ_T onto the manifold $\partial\Omega^{(2)} \cup \partial\Omega^{(1)} \setminus \Gamma_T$ by zero, i.e., $\widetilde{F}_j \in \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T)$, $j = \overline{1,4}$.

Introduce the Steklov–Poincaré type 6×6 matrix pseudodifferential operators

$$\mathcal{A}^{(1)}_\tau := \mathcal{H}^{(1)}_\tau\,[P^{(1)}_\tau]^{-1}, \quad \mathcal{A}^{(2)}_\tau := \mathcal{H}^{(2)}_\tau\,\big(P^{(2)}_\tau\big)^{-1}.$$

Let

$$\mathcal{B}^{(2)}_\tau := \begin{bmatrix} (A^{(2)}_\tau)_{11} & (A^{(2)}_\tau)_{12} & (A^{(2)}_\tau)_{13} & 0 & 0 & (A^{(2)}_\tau)_{14} \\ (A^{(2)}_\tau)_{21} & (A^{(2)}_\tau)_{22} & (A^{(2)}_\tau)_{23} & 0 & 0 & (A^{(2)}_\tau)_{24} \\ (A^{(2)}_\tau)_{31} & (A^{(2)}_\tau)_{32} & (A^{(2)}_\tau)_{33} & 0 & 0 & (A^{(2)}_\tau)_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ (A^{(2)}_\tau)_{41} & (A^{(2)}_\tau)_{42} & (A^{(2)}_\tau)_{43} & 0 & 0 & (A^{(2)}_\tau)_{44} \end{bmatrix}_{6\times 6}.$$

Taking into account equations (3.16) and (3.17), we can rewrite equations (3.13), (3.14), (3.15) in a matrix form and, finally, the whole system (3.12)-(3.17) can be rewritten as follows:

$$r_{S^{(1)}_D}\,\mathcal{A}^{(1)}_\tau\,(\psi + h^{(1)}) = \widetilde{f}^{(1)} \ \text{ on } \ S^{(1)}_D, \tag{3.24}$$

$$r_{\Gamma_T}\,\mathcal{A}^{(1)}_\tau\,(\psi + h^{(1)}) + r_{\Gamma_T}\,\mathcal{B}^{(2)}_\tau h^{(1)} = \widetilde{g} \ \text{ on } \ \Gamma_T, \tag{3.25}$$

$$r_{\Gamma_T}\,h^{(1)}_j + r_{\Gamma_T}\,h^{(2)}_j = \widetilde{F}_j \ \text{ on } \ \Gamma_T, \ \ j = \overline{1,3}, \tag{3.26}$$

$$r_{\Gamma_T} h^{(1)}_6 + r_{\Gamma_T} h^{(2)}_4 = \widetilde{F}_4 \ \text{ on } \ \Gamma_T, \tag{3.27}$$

where

$$\widetilde{f}^{(1)} := (\widetilde{f}^{(1)}_1, \ldots, \widetilde{f}^{(1)}_6)^\top \in \big[\,B^{1-\frac{1}{p}}_{p,p}(S^{(1)}_D)\,\big]^6, \tag{3.28}$$

$$\widetilde{g} := (\widetilde{g}_1, \ldots, \widetilde{g}_6)^\top \in \big[\,B^{1-\frac{1}{p}}_{p,p}(\Gamma_T)\,\big]^6, \tag{3.29}$$

$$\widetilde{g}_j := \widetilde{f}_j + r_{\Gamma_T}\big[\,\mathcal{H}^{(2)}_\tau[\,P^{(2)}_\tau\,]^{-1}\,\widetilde{F}\big]_j, \quad j = \overline{1,3}, \tag{3.30}$$

$$\widetilde{g}_4 = \widetilde{f}_4, \ \ \widetilde{g}_5 = \widetilde{f}_5, \ \ \widetilde{g}_6 = \widetilde{f}_6 + r_{\Gamma_T}\big[\,\mathcal{H}^{(2)}_\tau[\,P^{(2)}_\tau\,]^{-1}\,\widetilde{F}\big]_4,$$

$$\widetilde{F} := (\widetilde{F}_1, \ldots, \widetilde{F}_4)^\top \in \big[\,\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T)\,\big]^4. \tag{3.31}$$

It is easy to see that the simultaneous equations (3.12)–(3.17) and (3.24)–(3.27), where the right-hand sides are related by equalities (3.18)–(3.23) and (3.28)–(3.31), are equivalent in the following sense: if the triplet $(\psi, h^{(1)}, h^{(2)}) \in [\widetilde{B}^{-\frac{1}{p}}_{p,p}(S^{(1)}_D)]^6 \times [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T)]^6 \times [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T)]^4$ solves the system (3.24)–(3.27), then $(\psi, h^{(1)}, h^{(2)})$ solves the system (3.12)–(3.17), and vice versa.

3.2. **The Existence theorems and regularity of solutions.** Here we show that the system of pseudodifferential equations (3.24)–(3.27) is uniquely solvable in appropriate function spaces. To this end, let us introduce the notation

$$\mathcal{N}_\tau := \begin{bmatrix} r_{S^{(1)}_D}\,\mathcal{A}^{(1)}_\tau & r_{S^{(1)}_D}\,\mathcal{A}^{(1)}_\tau & r_{S^{(1)}_D}\,[\,0\,]_{6\times 4} \\ r_{\Gamma_T}\,\mathcal{A}^{(1)}_\tau & r_{\Gamma_T}[\mathcal{A}^{(1)}_\tau + \mathcal{B}^{(2)}_\tau] & r_{\Gamma_T}\,[\,0\,]_{6\times 4} \\ r_{\Gamma_T}\,[\,0\,]_{4\times 6} & r_{\Gamma_T}\,I_{4\times 6} & r_{\Gamma_T}\,I_4 \end{bmatrix}_{16\times 16},$$

$$I_{4\times 6} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 6}.$$

Further, let

$$\Phi := (\psi,\ h^{(1)},\ h^{(2)})^\top, \quad Y := (\widetilde{f},\ \widetilde{g},\ \widetilde{F})^\top,$$
$$\mathbf{X}_p^s := \big[\,\widetilde{B}^s_{p,p}(S_D^{(1)})\,\big]^6 \times \big[\,\widetilde{B}^s_{p,p}(\Gamma_T)\,\big]^6 \times \big[\,\widetilde{B}^s_{p,p}(\Gamma_T)\,\big]^4,$$
$$\mathbf{Y}_p^s := \big[\,B^{s+1}_{p,p}(S_D^{(1)})\,\big]^6 \times \big[\,B^{s+1}_{p,p}(\Gamma_T)\,\big]^6 \times \big[\,\widetilde{B}^s_{p,p}(\Gamma_T)\,\big]^4,$$
$$\mathbf{X}_{p,q}^s := \big[\,\widetilde{B}^s_{p,q}(S_D^{(1)})\,\big]^6 \times \big[\,\widetilde{B}^s_{p,q}(\Gamma_T)\,\big]^6 \times \big[\,\widetilde{B}^s_{p,q}(\Gamma_T)\,\big]^4,$$
$$\mathbf{Y}_{p,q}^s := \big[\,B^{s+1}_{p,q}(S_D^{(1)})\,\big]^6 \times \big[\,B^{s+1}_{p,q}(\Gamma_T)\,\big]^6 \times \big[\,\widetilde{B}^s_{p,q}(\Gamma_T)\,\big]^4.$$

Note that

$$\mathbf{X}_2^s = \mathbf{X}_{2,2}^s, \quad \mathbf{Y}_2^s = \mathbf{Y}_{2,2}^s, \quad \forall s \in \mathbb{R}.$$

System (3.24)–(3.27) can be rewritten as follows:

$$\mathcal{N}_\tau\,\Phi = Y, \tag{3.32}$$

where $\Phi \in \mathbf{X}_p^s$ is the sought for vector function and $Y \in \mathbf{Y}_p^s$ is a given vector function.

Due to Theorems 5.3 and 5.4, the operator $\mathcal{N}_\tau$ has the following mapping properties:

$$\begin{aligned} \mathcal{N}_\tau &\ :\ \mathbf{X}_p^s \to \mathbf{Y}_p^s, \\ \mathcal{N}_\tau &\ :\ \mathbf{X}_{p,q}^s \to \mathbf{Y}_{p,q}^s, \end{aligned} \tag{3.33}$$

for all $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$.

As it will become clear later, the operator (3.33) is not invertible for all $s \in \mathbb{R}$. The interval $a < s < b$ of invertibility depends on p and on some parameters γ' and γ'' (see (3.40)–(3.43)), which are determined by the eigenvalues of special matrices constructed by means of the principal homogeneous symbol matrices of the operators $\mathcal{A}_\tau^{(1)}$ and $\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}$. Note that the numbers γ' and γ'' define also Hölder's smoothness exponents for the solutions to the original mixed boundary-transmission problem in the neighbourhood of the exceptional curves $\partial S_D^{(1)}$, $\partial\Gamma_C$ and $\partial\Gamma$. We start with the following

Theorem 3.1. *Let the conditions*

$$1 < p < \infty, \quad 1 \leqslant q \leqslant \infty, \quad \frac{1}{p} - 1 + \gamma'' < s + \frac{1}{2} < \frac{1}{p} + \gamma' \tag{3.34}$$

be satisfied with γ' and γ'' given by (3.43)*. Then the operators in* (3.33) *are invertible.*

Proof. We prove the theorem in several steps. First, we show that the operators (3.33) are Fredholm ones with a zero index and afterwards we establish that the corresponding null-spaces are trivial.

Step 1. Let us note that the operators

$$\begin{aligned} r_{_{S_D^{(1)}}}\,\mathcal{A}_\tau^{(1)} &\ :\ \big[\,\widetilde{B}^s_{p,q}(\Gamma_T)\,\big]^6 \to \big[\,B^{s+1}_{p,q}(S_D^{(1)})\,\big]^6, \\ r_{_{\Gamma_T}}\mathcal{A}_\tau^{(1)} &\ :\ \big[\,\widetilde{B}^s_{p,q}(S_D^{(1)})\,\big]^6 \to \big[\,B^{s+1}_{p,q}(\Gamma_T)\,\big]^6 \end{aligned} \tag{3.35}$$

are compact since $S_D^{(1)}$ and Γ_T are disjoint, $\overline{S_D^{(1)}} \cap \overline{\Gamma_T} = \varnothing$. Further, we establish that the operators

$$r_{_{S_D^{(1)}}}\mathcal{A}_\tau^{(1)}\ :\ \big[\,\widetilde{H}_2^{-\frac{1}{2}}(S_D^{(1)})\,\big]^6 \to \big[\,[H_2^{\frac{1}{2}}(S_D^{(1)})\,\big]^6,$$
$$r_{_{\Gamma_T}}\big[\,\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}\,\big]\ :\ \big[\,\widetilde{H}_2^{-\frac{1}{2}}(\Gamma_T)\,\big]^6 \to \big[\,H_2^{\frac{1}{2}}(\Gamma_T)\,\big]^6 \tag{3.36}$$

are strongly elliptic Fredholm pseudodifferential operators of order -1 with a index zero. We note that the principal homogeneous symbol matrices of these operators are strongly elliptic.

Using Green's formula and Korn's inequality, for an arbitrary solution vector $U^{(1)} \in [H^1_2(\Omega^{(1)})]^6 = [W^1_2(\Omega^{(1)})]^6$ to the homogeneous equation

$$A^{(1)}(\partial_x, \tau)U^{(1)} = 0 \quad \text{in} \quad \Omega^{(1)},$$

by the standard arguments we derive (see, e.g., [7,8])

$$\operatorname{Re}\big\langle [U^{(1)}]^+, [\mathcal{T}^{(1)}U^{(1)}]^+ \big\rangle_{\partial\Omega^{(1)}} \geqslant c_1 \| U^{(1)} \|^2_{[H^1_2(\Omega^{(1)})]^6} - c_2 \| U^{(1)} \|^2_{[H^0_2(\Omega^{(1)})]^6}, \tag{3.37}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega^{(1)}}$ denotes the duality pairing between the spaces $\big[H^{\frac{1}{2}}(\partial\Omega^{(1)})\big]^6$ and $\big[H^{-\frac{1}{2}}(\partial\Omega^{(1)})\big]^6$.

Substitute here $U^{(1)} = V^{(1)}_\tau([P^{(1)}_\tau]^{-1}\zeta)$ with $\zeta \in [H_2^{-\frac{1}{2}}(\partial\Omega^{(1)})]^6$. Due to the equality

$$\zeta = P^{(1)}_\tau [\mathcal{H}^{(1)}_\tau]^{-1} \{U^{(1)}\}^+$$

and boundedness of the operators involved, we have

$$\|\zeta\|^2_{[H_2^{-\frac{1}{2}}(\partial\Omega^{(1)})]^6} \leqslant c^* \|\{U^{(1)}\}^+\|^2_{[H_2^{\frac{1}{2}}(\partial\Omega^{(1)})]^6}$$

with some positive constant c^*. By the properties of single layer potentials, we have

$$\big\{U^{(1)}\big\}^+ = \mathcal{H}^{(1)}_\tau \big[P^{(1)}_\tau\big]^{-1}\zeta, \quad \big\{\mathcal{T}^{(1)}U^{(1)}\big\}^+ = \big(-\frac{1}{2}I_6 + \mathcal{K}^{(1)}_\tau\big)\big[P^{(1)}_\tau\big]^{-1}\zeta.$$

By the trace theorem, from (3.37), we deduce

$$\begin{aligned} \operatorname{Re}\langle \mathcal{H}^{(1)}_\tau [P^{(1)}_\tau]^{-1}\zeta, \big(-2^{-1} I_6 + \mathcal{K}^{(1)}_\tau + \beta\,\mathcal{H}^{(1)}_\tau\big)\big[P^{(1)}_\tau\big]^{-1}\zeta\rangle_{\partial\Omega^{(1)}} \geqslant c_1' \| \zeta \|^2_{[H_2^{-\frac{1}{2}}(\partial\Omega^{(1)})]^6} \\ + \big\| \beta\,\mathcal{H}^{(1)}[P^{(1)}_\tau]^{-1}\zeta\big\|^2_{[H_2^{\frac{1}{2}}(\partial\Omega^{(1)})]^6} - c_2 \big\| V^{(1)}_\tau([P^{(1)}_\tau]^{-1}\zeta)\big\|^2_{[H^0_2(\Omega^{(1)})]^6}. \end{aligned}$$

Thus we have

$$\begin{aligned} \operatorname{Re}\langle \mathcal{H}^{(1)}_\tau [P^{(1)}_\tau]^{-1}\zeta, \zeta\rangle_{\partial\Omega^{(1)}} \geqslant c_1' \| \zeta \|^2_{[H_2^{-\frac{1}{2}}(\partial\Omega^{(1)})]^6} \\ + \big\| \beta\,\mathcal{H}^{(1)}[P^{(1)}_\tau]^{-1}\zeta\big\|^2_{[H_2^{\frac{1}{2}}(\partial\Omega^{(1)})]^6} - c_2 \big\| V^{(1)}_\tau([P^{(1)}_\tau]^{-1}\zeta)\big\|^2_{[H^0_2(\Omega^{(1)})]^6}. \end{aligned}$$

In particular, in view of Theorem 5.1, for arbitrary $\zeta \in [\widetilde{H}_2^{-\frac{1}{2}}(S^{(1)}_D)]^6$, we have

$$\| U^{(1)} \|^2_{[H^0_2(\Omega^{(1)})]^6} \leqslant c^{**} \| \zeta \|^2_{[\widetilde{H}_2^{-\frac{3}{2}}(S^{(1)}_D)]^6},$$

and, consequently,

$$\operatorname{Re}\big\langle r_{S^{(1)}_D} \mathcal{H}^{(1)}_\tau [P^{(1)}_\tau]^{-1}\zeta, \zeta \big\rangle_{\partial\Omega^{(1)}} \geqslant c_1' \| \zeta \|^2_{[\widetilde{H}_2^{-\frac{1}{2}}(S^{(1)}_D)]^6} - c_2'' \| \zeta \|^2_{[\widetilde{H}_2^{-\frac{3}{2}}(S^{(1)}_D)]^6}. \tag{3.38}$$

From (3.38), it follows that

$$r_{S^{(1)}_D} \mathcal{A}^{(1)}_\tau = r_{S^{(1)}_D} \mathcal{H}^{(1)}_\tau [P^{(1)}_\tau)]^{-1} \ :\ \big[\,\widetilde{H}_2^{-\frac{1}{2}}(S^{(1)}_D)\,\big]^6 \to \big[\,H_2^{\frac{1}{2}}(S^{(1)}_D)\,\big]^6$$

is a strongly elliptic pseudodifferential Fredholm operator with index zero (see [21,23]).

Then the same is true for the operator (3.36), since the principal homogeneous symbol matrix of the operator $\mathcal{B}^{(2)}_\tau$ is nonnegative (see [25]). Therefore, the operator (3.33) is Fredholm with index zero for $s = -1/2$, $p = 2$ and $q = 2$ due to the compactness of operators (3.35).

Step 2. With the help of the uniqueness Theorem 2.1, via representation formulas (3.4) and (3.5) with $G^{(1)}_0 = 0$ and $G^{(2)}_0 = 0$, we can easily show that the operator (3.33) is injective for $s = -1/2$, $p = 2$ and $q = 2$. Since its index is zero, we conclude that it is surjective. Thus the operator (3.33) is invertible for $s = -1/2$, $p = 2$ and $q = 2$.

Step 3. To complete the proof for the general case we proceed as follows. The following block-wise lower triangular operator

$$\mathcal{N}_\tau^{(0)} := \begin{bmatrix} r_{_{S_D^{(1)}}} \mathcal{A}_\tau^{(1)} & r_{_{S_D^{(1)}}} [\,0\,]_{6\times 6} & r_{_{S_D^{(1)}}} [\,0\,]_{6\times 4} \\ r_{\Gamma_T} [\,0\,]_{6\times 6} & r_{\Gamma_T} [\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}] & r_{\Gamma_T} [\,0\,]_{6\times 4} \\ r_{\Gamma_T} [\,0\,]_{4\times 6} & r_{\Gamma_T} I_{4\times 6} & r_{\Gamma_T} I_4 \end{bmatrix}_{16\times 16}$$

is a compact perturbation of the operator $\mathcal{N}_\tau$. Let us analyze the properties of the diagonal entries

$$r_{_{S_D^{(1)}}} \mathcal{A}_\tau^{(1)} \ : \ [\,\widetilde{B}_{p,q}^{s}(S_D^{(1)})]^6 \to \big[\,B_{p,q}^{s+1}(S_D^{(1)})\,\big]^6,$$

$$r_{\Gamma_T} \big[\,\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}\,\big] \ : \ [\,\widetilde{B}_{p,q}^{s}(\Gamma_T)]^6 \to \big[\,B_{p,q}^{s+1}(\Gamma_T)\,\big]^6.$$

Let

$$\mathfrak{S}_1(x,\xi_1,\xi_2) := \mathfrak{S}(\mathcal{A}_\tau^{(1)}; x,\xi_1,\xi_2)$$

be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau^{(1)}$ and let $\lambda_j^{(1)}(x)$ $(j=\overline{1,6})$ be the eigenvalues of the matrix

$$\mathcal{D}_1(x) := \big[\mathfrak{S}_1(x,0,+1)\big]^{-1}\, \mathfrak{S}_1(x,0,-1), \quad x\in\partial S_D^{(1)}.$$

Similarly, let

$$\mathfrak{S}_2(x,\xi_1,\xi_2) = \mathfrak{S}(\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}; x,\xi_1,\xi_2)$$

be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}$ and let $\lambda_j^{(2)}(x)$ $(j=\overline{1,6})$ be the eigenvalues of the corresponding matrix

$$\mathcal{D}_2(x) := \big[\mathfrak{S}_2(x,0,+1)\big]^{-1}\, \mathfrak{S}_2(x,0,-1), \quad x\in\partial\Gamma_T. \tag{3.39}$$

Note that the curve $\partial\Gamma_T$ is the union of the curves, where the interface intersects the exterior boundary $\partial\Gamma$, and the crack edge $\partial\Gamma_C$, $\partial\Gamma_T = \partial\Gamma\cup\partial\Gamma_C$.

Further, we set

$$\gamma_1' := \inf_{x\in\partial S_D^{(1)},\,1\leqslant j\leqslant 6} \frac{1}{2\pi}\,\arg\lambda_j^{(1)}(x), \quad \gamma_1'' := \sup_{x\in\partial S_D^{(1)},\,1\leqslant j\leqslant 6} \frac{1}{2\pi}\,\arg\lambda_j^{(1)}(x), \tag{3.40}$$

$$\gamma_2' := \inf_{x\in\partial \Gamma_T,\,1\leqslant j\leqslant 6} \frac{1}{2\pi}\,\arg\lambda_j^{(2)}(x), \quad \gamma_2'' := \sup_{x\in\partial \Gamma_T,\,1\leqslant j\leqslant 6} \frac{1}{2\pi}\,\arg\lambda_j^{(2)}(x). \tag{3.41}$$

It can be shown that one of the eigenvalues is equal to 1, say $\lambda_6^{(1)} = 1$ (for details see [6, Subsection 4.4], [7, Subsection 5.7]) and [8, Theorem 4.7]. Therefore we have

$$\gamma_1' \leqslant 0, \quad \gamma_1'' \geqslant 0. \tag{3.42}$$

Note that γ_j' and γ_j'' $(j=1,2)$ depend on the material parameters, in general, and belong to the interval $(-\frac{1}{2}\,,\,\frac{1}{2})$. We put

$$\gamma' := \min\{\gamma_1',\,\gamma_2'\}, \quad \gamma'' := \max\{\gamma_1'',\,\gamma_2''\}. \tag{3.43}$$

In view of (3.42), we have

$$-\frac{1}{2} < \gamma' \leqslant 0 \leqslant \gamma'' < \frac{1}{2}\,. \tag{3.44}$$

From Theorem 5.5, we conclude that if the parameters $r_1, r_2\in\mathbb{R}$, $1<p<\infty$, $1\leqslant q\leqslant\infty$, satisfy the conditions

$$\frac{1}{p} - 1 + \gamma_1'' < r_1 + \frac{1}{2} < \frac{1}{p} + \gamma_1', \quad \frac{1}{p} - 1 + \gamma_2'' < r_2 + \frac{1}{2} < \frac{1}{p} + \gamma_2',$$

then the operators

$$r_{_{S_D^{(1)}}} \mathcal{A}_\tau^{(1)} \ : \ \big[\,\widetilde{H}_p^{r_1}(S_D^{(1)})\,\big]^6 \to \big[\,H_p^{r_1+1}(S_D^{(1)})\,\big]^6,$$

$$r_{_{S_D^{(1)}}} \mathcal{A}_\tau^{(1)} \ : \ \big[\,\widetilde{B}_{p,q}^{r_1}(S_D^{(1)})]^6 \to \big[\,[B_{p,q}^{r_1+1}(S_D^{(1)})\,\big]^6,$$

$$r_{\Gamma_T}\big[\,\mathcal{A}_\tau^{(1)}+\mathcal{B}_\tau^{(2)}\,\big]\ :\ \big[\,\widetilde{H}_p^{r_2}(\Gamma_T)\,\big]^6\to\big[\,H_p^{r_2+1}(\Gamma_T)\,\big]^6,$$

$$r_{\Gamma_T}\big[\,\mathcal{A}_\tau^{(1)}+\mathcal{B}_\tau^{(2)}\,\big]\ :\ \big[\,\widetilde{B}_{p,q}^{r_2}(\Gamma_T)\,\big]^6\to\big[\,B_{p,q}^{r_2+1}(\Gamma_T)\,\big]^6$$

are the Fredholm operators with index zero.

Therefore, if conditions (3.34) are satisfied, then the above operators are Fredholm ones with a zero index. Consequently, operators (3.33) are Fredholm with zero index and are invertible due to the results obtained in Step 2 (see [2]) . □

Now we formulate the basic existence and uniqueness results for the mixed boundary-transmission problem under consideration.

Theorem 3.2. *Let inclusions* (2.27) *and compatibility conditions* (3.22)*,* (3.23) *hold and let*

$$\frac{4}{3-2\gamma''}<p<\frac{4}{1-2\gamma'} \tag{3.45}$$

with γ' and γ'' be defined in (3.43)*. Then the mixed boundary-transmission problem* (2.14)–(2.26) *has a unique solution*

$$(U^{(1)},U^{(2)})\in\big[\,W_p^1(\Omega^{(1)})\,\big]^6\times[\,W_p^1(\Omega^{(2)})\,]^4,$$

which can be represented by the formulas

$$U^{(1)}=V_\tau^{(1)}\big([P_\tau^{(1)}]^{-1}\,\big[\,G_0^{(1)}+\psi+h^{(1)}\big]\,\big)\ \ in\ \ \Omega^{(1)}, \tag{3.46}$$

$$U^{(2)}=V_\tau^{(2)}\big(\,\big[\,P_\tau^{(2)}\,\big]^{-1}\,\big[\,G_0^{(2)}+h^{(2)}\big]\,\big)\ \ in\ \ \Omega^{(2)}, \tag{3.47}$$

where the densities ψ, $h^{(1)}$ and $h^{(2)}$ are to be determined from system (3.6)–(3.11) (*or from system* (3.24)–(3.27))*, while $G_0^{(1)}$ and $G_0^{(2)}$ are some fixed extensions of the vector functions $G^{(1)}$ and $G^{(2)}$, respectively, onto $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$, preserving the space* (*see* (3.1) *and* (3.2))*.*

Moreover, the vector functions $G_0^{(1)}+\psi+h^{(1)}$ and $G_0^{(2)}+h^{(2)}$ are defined uniquely by the above systems and are independent of the extension operators.

Proof. From Theorems 5.1, 5.2 and 3.1 with p satisfying (3.45) and $s=-1/p$ it follows immediately that the pair $(U^{(1)},U^{(2)})\in[W_p^1(\Omega^{(1)})]^6\times[W_p^1(\Omega^{(2)})]^4$ given by (3.46), (3.47) represents a solution to the mixed boundary-transmission problem (2.14)–(2.26). Next, we show the uniqueness of solutions.

Due to inequalities (3.44), we have

$$p=2\in\Big(\frac{4}{3-2\gamma''}\,,\,\frac{4}{1-2\gamma'}\Big).$$

Therefore the unique solvability for $p=2$ is a consequence of Theorem 2.1.

To show the uniqueness result for all other values of p from the interval (3.45), we proceed as follows. Let a pair

$$(U^{(1)},U^{(2)})\in\big[\,W_p^1(\Omega^{(1)})\,\big]^6\times[\,W_p^1(\Omega^{(2)})\,]^4$$

with p satisfying (3.45), be a solution to the homogeneous mixed boundary-transmission problem. Then it is evident that

$$\big\{\,U^{(1)}\,\big\}^+\in\big[\,B_{p,p}^{1-\frac1p}(\partial\Omega^{(1)})\,\big]^6,\quad\big\{\,U^{(2)}\,\big\}^+\in\big[\,B_{p,p}^{1-\frac1p}(\partial\Omega^{(2)})\,\big]^4,$$

$$\big\{\,\mathcal{T}^{(1)}U^{(1)}\,\big\}^+\in\big[\,B_{p,p}^{-\frac1p}(\partial\Omega^{(1)})\,\big]^6,\quad\big\{\,\mathcal{T}^{(2)}U^{(2)}\,\big\}^+\in\big[\,B_{p,p}^{-\frac1p}(\partial\Omega^{(2)})\,\big]^4.$$

By Lemmas 2.2 and 2.3, the vectors $U^{(2)}$ and $U^{(1)}$ in $\Omega^{(2)}$ and $\Omega^{(1)}$, respectively, are representable in the form

$$U^{(2)}=V_\tau^{(2)}\big(\,\big[\,P_\tau^{(2)}\big]^{-1}\,h^{(2)}\big)\ \text{ in }\ \Omega^{(2)},\quad h^{(2)}=\big\{\mathcal{T}^{(2)}U^{(2)}\,\big\}^+,$$

$$U^{(1)}=V_\tau^{(1)}\big([P_\tau^{(1)}]^{-1}\,\chi\big)\ \text{ in }\ \Omega^{(1)},\quad\chi=\big\{\mathcal{T}^{(1)}U^{(1)}\big\}^++\beta\,\big\{U^{(1)}\big\}^+.$$

Moreover, due to the homogeneous boundary and transmission conditions, we have

$$h^{(2)}\in\big[\,\widetilde{B}_{p,p}^{-\frac1p}(\Gamma_T)\,\big]^4,\ \ \chi=h^{(1)}+\psi\in[B_{p,p}^{-\frac1p}(S^{(1)})\,\big]^6,\ \ h^{(1)}\in\big[\,\widetilde{B}_{p,p}^{-\frac1p}(\Gamma_T)\,\big]^6,\ \ \psi\in\big[\,\widetilde{B}_{p,p}^{-\frac1p}(S_D^{(1)})\,\big]^6.$$

By the same arguments as above we arrive at the homogeneous system

$$\mathcal{N}_\tau\,\Phi = 0 \ \text{ with } \ \Phi := (\psi,\ h^{(1)},\ h^{(2)})^\top \in \mathbf{X}_p^{-\frac{1}{p}}\,.$$

Due to Theorem 3.1, $\Phi = 0$ and we conclude that $U^{(2)} = 0$ in $\Omega^{(2)}$ and $U^{(1)} = 0$ in $\Omega^{(1)}$.

The last assertion of the theorem is trivial and is an easy consequence of the fact that if the single layer potentials (3.46)and (3.47) vanish identically in $\Omega^{(2)}$ and $\Omega^{(1)}$, then the corresponding densities vanish, as well. □

The following regularity result is true.

Theorem 3.3. *Let the inclusions* (2.27) *and compatibility conditions* (3.22), (3.23) *hold and let* $1 < r < \infty$, $1 \leqslant q \leqslant \infty$,

$$\frac{4}{3-2\gamma''} < p < \frac{4}{1-2\gamma'}\,, \quad \frac{1}{r} - \frac{1}{2} + \gamma'' < s < \frac{1}{r} + \frac{1}{2} + \gamma', \tag{3.48}$$

with γ' *and* γ'' *defined in* (3.43).

Further, let $U^{(1)} \in [W_p^1(\Omega^{(1)})]^6$ *and* $U^{(2)} \in [W_p^1(\Omega^{(2)})]^4$ *be a unique solution pair to the mixed boundary-transmission problem* (2.14)–(2.26). *Then the following items hold:*

(i) *if*

$$Q_k^{(1)} \in B_{r,r}^{s-1}(S_N^{(1)}), \quad Q_j^{(2)} \in B_{r,r}^{s-1}(S_N^{(2)}), \quad f_k^{(1)} \in B_{r,r}^{s}(S_D^{(1)}), \quad f_k \in B_{r,r}^{s}(\Gamma_T), \quad F_j \in B_{r,r}^{s-1}(\Gamma_T),$$
$$\widetilde{Q}_j^{(2)} \in B_{r,r}^{s-1}(\Gamma_C), \quad \widetilde{Q}_k^{(1)} \in B_{r,r}^{s-1}(\Gamma_C), \ \ k = \overline{1,6}, \ \ j = \overline{1,4},$$

and the compatibility conditions

$$\widetilde{F}_j := F_j - r_{\Gamma_T}\,G_{0j}^{(1)} - r_{\Gamma_T}\,G_{0j}^{(2)} \in r_{\Gamma_T}\,\widetilde{B}_{r,r}^{s-1}(\Gamma_T), \ \ j = \overline{1,3},$$
$$\widetilde{F}_4 := F_4 - r_{\Gamma_T}\,G_{06}^{(1)} - r_{\Gamma_T}\,G_{04}^{(2)} \in r_{\Gamma_T}\,\widetilde{B}_{r,r}^{s-1}(\Gamma_T),$$

are satisfied, then

$$U^{(1)} \in [\,H_r^{s+\frac{1}{r}}(\Omega^{(1)})\,]^6, \ \ U^{(2)} \in [\,H_r^{s+\frac{1}{r}}(\Omega^{(2)})\,]^4;$$

(ii) *if*

$$Q_k^{(1)} \in B_{r,q}^{s-1}(S_N^{(1)}), \quad Q_j^{(2)} \in B_{r,q}^{s-1}(S_N^{(2)}), \quad f_k^{(1)} \in B_{r,q}^{s}(S_D^{(1)}), \quad f_k \in B_{r,q}^{s}(\Gamma_T), \quad F_j \in B_{r,q}^{s-1}(\Gamma_T),$$
$$\widetilde{Q}_j^{(2)} \in B_{r,q}^{s-1}(\Gamma_C), \quad \widetilde{Q}_k^{(1)} \in B_{r,q}^{s-1}(\Gamma_C), \ \ k = \overline{1,6}, \ \ j = \overline{1,4},$$

and the compatibility conditions

$$\widetilde{F}_j := F_j - r_{\Gamma_T}\,G_{0j}^{(1)} - r_{\Gamma_T}\,G_{0j}^{(2)} \in r_{\Gamma_T}\,\widetilde{B}_{r,q}^{s-1}(\Gamma_T), \ \ j = \overline{1,3},$$
$$\widetilde{F}_4 := F_4 - r_{\Gamma_T}\,G_{06}^{(1)} - r_{\Gamma_T}\,G_{04}^{(2)} \in r_{\Gamma_T}\,\widetilde{B}_{r,q}^{s-1}(\Gamma_T),$$

are satisfied, then

$$U^{(1)} \in \big[\,B_{r,q}^{s+\frac{1}{r}}(\Omega^{(1)})\,\big]^6, \quad U^{(2)} \in \big[\,B_{r,q}^{s+\frac{1}{r}}(\Omega^{(2)})\,\big]^4;$$

(iii) *if* $\alpha > 0$ *is not integer and*

$$Q_k^{(1)} \in B_{\infty,\infty}^{\alpha-1}(S_N^{(1)}), \quad Q_j^{(2)} \in B_{\infty,\infty}^{\alpha-1}(S_N^{(2)}), \quad f_k^{(1)} \in C^\alpha(\overline{S_D^{(1)}}), \quad f_k \in C^\alpha(\overline{\Gamma_T}),$$
$$F_j \in B_{\infty,\infty}^{\alpha-1}(\Gamma_T), \quad \widetilde{Q}_j^{(2)} \in B_{\infty,\infty}^{\alpha-1}(\Gamma_C), \quad \widetilde{Q}_k^{(1)} \in B_{\infty,\infty}^{\alpha-1}(\Gamma_C), \ \ k = \overline{1,6}, \ \ j = \overline{1,4},$$

and the compatibility conditions

$$\widetilde{F}_j := F_j - r_{\Gamma_T}\,G_{0j}^{(1)} - r_{\Gamma_T}\,G_{0j}^{(2)} \in r_{\Gamma_T}\,\widetilde{B}_{\infty,\infty}^{\alpha-1}(\Gamma_T), \ \ j = \overline{1,3},$$
$$\widetilde{F}_4 := F_4 - r_{\Gamma_T}\,G_{06}^{(1)} - r_{\Gamma_T}\,G_{04}^{(2)} \in r_{\Gamma_T}\,\widetilde{B}_{\infty,\infty}^{\alpha-1}(\Gamma_T),$$

are satisfied, then

$$U^{(1)} \in \bigcap_{\alpha' < \kappa} \big[\,C^{\alpha'}(\overline{\Omega^{(1)}})\,\big]^6, \quad U^{(2)} \in \bigcap_{\alpha' < \kappa} \big[\,C^{\alpha'}(\overline{\Omega^{(2)}})\,\big]^4,$$

where $\kappa = \min\{\alpha,\ \gamma' + \frac{1}{2}\} > 0$.

Proof. It is word for word repeats the proof of Theorem 5.22 in [7]. □

Regularity results for $u_6^{(1)} = \vartheta^{(1)}$ and $u_4^{(2)} = \vartheta^{(2)}$ are refined in Proposition 3.4 (see also Theorem 4.1).

Proposition 3.4. *Let the conditions of Theorem* 3.3 *(i) and* (3.48) *hold, then*

$$u_6^{(1)} \in C^{\frac{1}{2}-\varepsilon}(\overline{\Omega^{(1)}}), \quad u_4^{(2)} \in C^{\frac{1}{2}-\varepsilon}(\overline{\Omega^{(2)}}), \tag{3.49}$$

where ε *is an arbitrarily small positive number.*

Proof. Due to Theorem 3.3.(i), we deduce

$$U^{(1)} \in \big[H_r^{s+\frac{1}{r}}(\Omega^{(1)}) \big]^6, \quad U^{(2)} \in \big[H_r^{s+\frac{1}{r}}(\Omega^{(2)}) \big]^4,$$

where s and r satisfy (3.48). Note that $u_6^{(1)} = \vartheta^{(1)}$ and $u_4^{(2)} = \vartheta^{(2)}$ solve the following mixed boundary-transmission problem:

$$\begin{cases} \eta_{il}^{(1)}\partial_i\partial_l u_6^{(1)} - \tau^2 h_0^{(1)} u_6^{(1)} = Q^{(1)*} \ \text{ in } \ \Omega^{(1)}, \\ \eta_{il}^{(2)}\partial_i\partial_l u_4^{(2)} - \tau^2 h_0^{(2)} u_4^{(1)} = Q^{(2)*} \ \text{ in } \ \Omega^{(2)}, \\ r_{\Gamma_T}\{u_6^{(1)}\}^+ - r_{\Gamma_T}\{u_4^{(2)}\}^+ = f_6 \ \text{ on } \ \Gamma_T, \\ r_{\Gamma_T}\big\{[\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}]_6\big\}^+ + r_{\Gamma_T}\big\{[\mathcal{T}^{(2)}(\partial_x,\nu,\tau)U^{(2)}]_4\big\}^+ = F_4 \ \text{ on } \ \Gamma_T, \\ r_{S_N^{(1)}\cup\Gamma_C}\big\{[\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}]_6\big\}^+ = G_6^{(1)} \ \text{ on } \ S_N^{(1)}\cup\Gamma_C, \\ r_{S_N^{(2)}\cup\Gamma_C}\big\{[\mathcal{T}^{(2)}(\partial_x,\nu,\tau)U^{(2)}]_4\big\}^+ = G_4^{(2)} \ \text{ on } \ S_N^{(2)}\cup\Gamma_C, \\ r_{S_D^{(1)}}\{u_6^{(1)}\}^+ = f_6^{(1)} \ \text{ on } \ S_D^{(1)}, \end{cases} \tag{3.50}$$

where

$$\big[\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\big]_6 = \eta_{il}^{(1)}\, n_i\, \partial_l\vartheta^{(1)}, \quad \big[\mathcal{T}^{(2)}(\partial_x,\nu,\tau)U^{(2)}\big]_4 = \eta_{il}^{(2)}\, \nu_i\, \partial_l\vartheta^{(2)},$$
$$Q^{(1)*} = \tau\lambda_{kl}^{(1)}\partial_l u_k^{(1)} - \tau p_l^{(1)}\partial_l\varphi^{(1)} - \tau m_l^{(1)}\partial_l\psi^{(1)} + \tau d_0^{(1)}\vartheta^{(1)} \in H_r^{s+\frac{1}{r}-1}(\Omega^{(1)}),$$
$$Q^{(2)*} = \tau\lambda_{kl}^{(1)}\partial_l u_k^{(2)} + \tau d_0^{(2)}\vartheta^{(2)} \in H_r^{s+\frac{1}{r}-1}(\Omega^{(2)}),$$
$$f_6 \in B_{r,r}^{s'}(\Gamma_T), \quad F_4 \in B_{r,r}^{s'-1}(\Gamma_T), \quad f_6^{(1)} \in B_{r,r}^{s'}(S_D^{(1)}), \quad G_6^{(1)} \in B_{r,r}^{s'-1}(S_N^{(1)}\cup\Gamma_C),$$
$$G_4^{(2)} \in B_{r,r}^{s'-1}(S_N^{(2)}\cup\Gamma_C), \quad s < s' < \frac{1}{r} + \frac{1}{2}, \ \ 1 < r < \infty.$$

Since the symbols of the differential operators $-\eta_{il}^{(1)}\partial_i\partial_j$ and $-\eta_{il}^{(2)}\partial_i\partial_j$ are positive, the above problem can be reduced to the strongly elliptic system of pseudodifferential equations. Moreover, the corresponding pseudodifferential operator is positive definite. Therefore (see [25])

$$u_6^{(1)} \in H_r^{s'+\frac{1}{r}}(\Omega^{(1)}), \ \ u_4^{(2)} \in H_r^{s'+\frac{1}{r}}(\Omega^{(2)}), \ \ s < s' < \frac{1}{r} + \frac{1}{2}, \ \ 1 < r < \infty.$$

Due to the embedding theorem (see [33]), for sufficiently small $\delta > 0$, sufficiently large r and $s' > 1/2 + 1/r - \delta$ we have

$$H_r^{s'+\frac{1}{r}}(\Omega^{(1)}) \subset C^{\frac{1}{2}-\frac{1}{r}-\delta}(\overline{\Omega^{(1)}}), \quad H_r^{s'+\frac{1}{r}}(\Omega^{(2)}) \subset C^{\frac{1}{2}-\frac{1}{r}-\delta}(\overline{\Omega^{(2)}}).$$

Therefore (3.49) holds with $\varepsilon = 1/r + \delta$. □

3.3. **Asymptotic behaviour of solutions near the exceptional curves.** Here, we study the asymptotic properties of solutions to the mixed boundary-transmission problem near the interfacial crack edge $\partial\Gamma_C$ and at the curve $\partial\Gamma$, where the interface intersects the exterior boundary. Let us set $\ell := \partial\Gamma_C \cup \partial\Gamma = \partial\Gamma_T$.

Note that the regularity and the asymptotic behaviour of solutions near the collision curve $\partial S_D^{(1)}$ were studied in details in [8].

For the sake of simplicity of description of the method, we assume that the boundary data and the geometrical characteristics of the problem are infinitely smooth. In particular,

$$\begin{gathered}
Q_k^{(1)} \in C^\infty(\overline{S}_N^{(1)}), \;\; Q_j^{(2)} \in C^\infty(\overline{S_N^{(2)}}), \;\; f_k^{(1)} \in C^\infty(\overline{S}_D^{(1)}), \\
f_k \in C^\infty(\overline{\Gamma_T}), \;\; F_j \in C^\infty(\overline{\Gamma_T}), \;\; \widetilde{Q}_k^{(1)} \in C^\infty(\overline{\Gamma_C}), \\
\widetilde{F}_i := F_i - r_{_{\Gamma_T}} G_{0i}^{(1)} - r_{_{\Gamma_T}} G_{0i}^{(2)} \in C_0^\infty(\overline{\Gamma_T}), \quad \widetilde{F}_4 := F_4 - r_{_{\Gamma_T}} G_{06}^{(1)} - r_{_{\Gamma_T}} G_{04}^{(2)} \in C_0^\infty(\overline{\Gamma_T}), \\
\widetilde{Q}_j^{(2)} \in C^\infty(\overline{\Gamma_C}), \;\; i = \overline{1,3}, \;\; j = \overline{1,4}, \;\; k = \overline{1,6},
\end{gathered}$$

where $C_0^\infty(\overline{\Gamma_T})$ denotes a space of infinitely differentiable functions vanishing on $\partial\Gamma_T$ along with all tangential derivatives.

We have already shown that the mixed boundary-transmission problem is uniquely solvable and the pair of solution vectors $(U^{(1)}, U^{(2)})$ are represented by (3.46), (3.47) with the densities defined by the system of pseudodifferential equations (3.6)–(3.11), i.e., (3.24)–(3.27).

Let $\Phi := (\psi, h^{(1)}, h^{(2)})^\top \in \mathbf{X}_p^s$ be a solution of the system (3.24)–(3.27) which is written in matrix form (3.32)

$$\mathcal{N}_\tau \Phi = Y,$$

where

$$Y \in \big[\, C^\infty(\overline{S}_D) \,\big]^6 \times \big[\, C^\infty(\overline{\Gamma_T}) \,\big]^6 \times \big[\, C_0^\infty(\overline{\Gamma_T}) \,\big]^4.$$

To establish asymptotic properties of the solution vectors $U^{(1)}$ and $U^{(2)}$ near the exceptional curve $\ell = \partial\Gamma_T$, we rewrite the representations (3.46), (3.47) in the form

$$\begin{aligned}
&U^{(1)} = V_\tau^{(1)}([P_\tau^{(1)}]^{-1}\psi) + V_\tau^{(1)}([P_\tau^{(1)}]^{-1} h^{(1)}) + R^{(1)} \;\text{ in }\; \Omega^{(1)}, \\
&U^{(2)} = V_\tau^{(2)}([P_\tau^{(2)}]^{-1}\widetilde{h}^{(2)}) + R^{(2)} \;\text{ in }\; \Omega^{(2)},
\end{aligned}$$

where

$$\begin{aligned}
&\psi \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D^{(1)}) \,\big]^6, \;\; h^{(1)} = (h_1^{(1)}, \dots, h_6^{(1)})^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T) \,\big]^6, \\
&\widetilde{h}^{(2)} = -(h_1^{(1)}, h_2^{(1)}, h_3^{(1)}, h_6^{(1)})^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T) \,\big]^4, \quad R^{(1)} := V_\tau^{(1)}([P_\tau^{(1)}]^{-1} G_0^{(1)}) \in \big[\, C^\infty(\overline{\Omega^{(1)}}) \,\big]^6, \\
&R^{(2)} := V_\tau^{(2)}([P_\tau^{(2)}]^{-1} G_0^{(2)}) + V_\tau^{(2)}([P_\tau^{(2)}]^{-1}\widetilde{F}) \in \big[\, C^\infty(\overline{\Omega^{(2)}}) \,\big]^4, \;\; \widetilde{F} = \big(\widetilde{F}_1, \dots, \widetilde{F}_4\big)^\top.
\end{aligned}$$

The vectors $h^{(1)} = (h_1^{(1)}, \dots, h_6^{(1)})^\top$ and $\psi = (\psi_1, \dots, \psi_6)^\top$ solve the following strongly elliptic system of pseudodifferential equations (see (3.24)-(3.27)):

$$\begin{aligned}
&r_{_{S_D^{(1)}}} \mathcal{A}_\tau^{(1)} \psi = \Phi^{(1)} && \text{on} \quad S_D^{(1)}, \\
&r_{_{\Gamma_T}} (\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}) h^{(1)} = \Phi^{(2)} && \text{on} \quad \Gamma_T,
\end{aligned}$$

where

$$\begin{aligned}
\Phi_k^{(1)} &= f_k^{(1)} - r_{_{S_D^{(1)}}} [\mathcal{A}_\tau^{(1)} G_0^{(1)}]_k - r_{_{S_D^{(1)}}} [\mathcal{A}_\tau^{(1)} h^{(1)}]_k, \;\; k = \overline{1,6}, \\
\Phi^{(1)} &= (\Phi_1^{(1)}, \dots, \Phi_6^{(1)})^\top \in \big[\, C^\infty(\overline{S}_D^{(1)}) \big]^6, \\
\Phi_j^{(2)} &= f_j + r_{_{\Gamma_T}} \big[\, \mathcal{H}_\tau^{(2)} (P_\tau^{(2)})^{-1} G_0^{(2)} \,\big]_j - r_{_{\Gamma_T}} [\mathcal{A}_\tau^{(1)} G_0^{(1)}]_j \\
&\quad + r_{_{\Gamma_T}} \big[\, \mathcal{H}_\tau^{(2)} (P_\tau^{(2)})^{-1} \widetilde{F} \,\big]_j - r_{_{\Gamma_T}} [\mathcal{A}_\tau^{(1)} \psi]_j, \;\; j = 1,2,3, \\
\Phi_j^{(2)} &= f_j - r_{_{\Gamma_T}} [\mathcal{A}_\tau^{(1)} G_0^{(1)}]_j - r_{_{\Gamma_T}} [\mathcal{A}_\tau^{(1)} \psi]_j \,, \;\; j = 4,5,
\end{aligned}$$

$$\Phi_6^{(2)} = f_6 + r_{\Gamma_T}\,\big[\,\mathcal{H}_\tau^{(2)}(P_\tau^{(2)})^{-1}\widetilde{F}\,\big]_4 - r_{\Gamma_T}\,[\mathcal{A}_\tau^{(1)}\psi]_6,$$
$$\Phi^{(2)} = (\Phi_1^{(2)},\dots,\Phi_6^{(2)})^\top \in \big[\,C^\infty(\overline{\Gamma_T})\,\big]^6.$$

Applying a partition of unity, natural local coordinate systems and standard rectifying technique based on canonical diffeomorphisms, we can assume that $\ell = \partial\Gamma_T$ is rectified. Then we identify a one-sided neighbourhood on Γ_T of an arbitrary point $\widetilde{x} \in \ell = \partial\Gamma_T$ as a part of the half-plane $x_2 > 0$. Thus we assume that $(x_1, 0) = \widetilde{x} \in \ell = \partial\Gamma_T$ and $(x_1, x_{2,+}) \in \Gamma_T$ for $0 < x_{2,+} < \varepsilon$ with some positive ε.

Denote by m_j the algebraic multiplicities of $\lambda_j^{(2)}(x_1)$, where $\lambda_j^{(2)}$, $j = \overline{1,6}$, are the eigenvalues of the matrix $\mathcal{D}_2(x_1)$ (see (3.39)). Let $\mu_1(x_1),\dots,\mu_l(x_1)$, $1 \leqslant l \leqslant 6$, be the distinct eigenvalues. Evidently, m_j and l depend on x_1, in general, and $m_1 + \cdots + m_l = 6$.

It is well known that the matrix $\mathcal{D}_2(x_1)$ in (3.39) admits the following decomposition (see, e.g., [19]):

$$\mathcal{D}_2(x_1) = \mathcal{D}(x_1)\,\mathcal{J}_{\mathcal{D}_2}(x_1)\,\Big[\mathcal{D}(x_1)\Big]^{-1}, \;\; (x_1, 0) \in \ell = \partial\Gamma_T, \tag{3.51}$$

where $\mathcal{D}$ is the 6×6 nondegenerate matrix with infinitely differentiable entries and $\mathcal{J}_{\mathcal{D}_2}$ is block diagonal

$$\mathcal{J}_{\mathcal{D}_2}(x_1) := \operatorname{diag}\Big\{\mu_1(x_1)B^{(m_1)}(1)\,,\dots,\,\mu_l(x_1)B^{(m_l)}(1)\Big\}.$$

Here, $B^{(r)}(t)$, $r \in \{m_1,\dots,m_l\}$ are upper triangular matrices,

$$B^{(r)}(t) = \big\|b_{jk}^{(r)}(t)\big\|_{r\times r}, \quad b_{jk}^{(r)}(t) = \begin{cases} \dfrac{t^{k-j}}{(k-j)!}\,, & j<k,\\ 1, & j=k,\\ 0, & j>k. \end{cases}$$

Denote

$$B_0(t) := \operatorname{diag}\big\{B^{(m_1)}(t),\,\dots,\,B^{(m_l)}(t)\big\}. \tag{3.52}$$

Applying the results from reference [15], we derive the following asymptotic expansion:

$$\begin{aligned} h^{(1)}(x_1, x_{2,+}) &= \mathcal{D}(x_1)\,x_{2,+}^{-\frac12+\Delta(x_1)}\,B_0\Big(-\frac{1}{2\pi i}\log x_{2,+}\Big)\Big(\mathcal{D}(x_1)\Big)^{-1} b_0(x_1)\\ &+\sum_{k=1}^{M}\mathcal{D}(x_1)\,x_{2,+}^{-\frac12+\Delta(x_1)+k}\,B_k\big(x_1,\log x_{2,+}\big) + h_{M+1}^{(1)}(x_1,x_{2,+}), \end{aligned} \tag{3.53}$$

where $b_0 \in \big[C^\infty(\ell)\big]^6$, $h_{M+1}^{(1)} \in [\,C^\infty(\ell_\varepsilon^+)\,]^6$, $\ell_\varepsilon^+ = \ell\times[0,\varepsilon]$,

$$B_k(x_1,t) = B_0\Big(-\frac{t}{2\pi i}\Big)\sum_{j=1}^{k(2m_0-1)} t^j\,d_{kj}(x_1);$$

$m_0 = \max\{m_1,\dots,m_l\}$, the coefficients $d_{kj} \in [C^\infty(\ell)]^6$, $\Delta := (\Delta_1^{(2)},\dots,\Delta_6^{(2)})^\top$,

$$\Delta_j^{(2)}(x_1) = \frac{1}{2\pi i}\log\lambda_j^{(2)}(x_1) = \frac{1}{2\pi}\,\arg\lambda_j^{(2)}(x_1) + \frac{1}{2\pi i}\,\log|\lambda_j^{(2)}(x_1)|,$$
$$-\pi < \arg\lambda_j^{(2)}(x_1) < \pi, \;\; (x_1,0)\in\ell, \;\; j=\overline{1,6},$$

and

$$x_{2,+}^{-\frac12+\Delta(x_1)+k} := \operatorname{diag}\Big\{x_{2,+}^{-\frac12+\Delta_1^{(2)}(x_1)+k},\,\dots,\,x_{2,+}^{-\frac12+\Delta_6^{(2)}(x_1)+k}\Big\}.$$

Now, having in hand the above asymptotic expansion for the density vector function $h^{(1)}$, we can apply the results of [14] and write the spatial asymptotic expansions of the solution vectors $U^{(1)}$ and $U^{(2)}$:

$$U^{(1)}(x) = \sum_{\mu=\pm1}\sum_{s=1}^{l_0^{(1)}}\Big\{\sum_{j=0}^{n_s^{(1)}-1} x_3^j\,\Big[d_{sj}^{(1)}(x_1,\mu)(z_{s,\mu}^{(1)})^{\frac12+\Delta(x_1)-j}\,B_0(\zeta^{(1)})\Big]\,c_j(x_1)$$

$$+\sum_{\substack{k,l=0\\k+l+j+p\geq 1}}^{M+2}\sum_{j+p=0}^{M+2-l} x_2^l\, x_3^j d^{(1)}_{sljp}(x_1,\mu)\,(z^{(1)}_{s,\mu})^{\frac{1}{2}+\Delta(x_1)+p+k}\, B^{(1)}_{skjp}(x_1,\log z^{(1)}_{s,\mu})\Big\}+U^{(1)}_{M+1}(x)\,, \tag{3.54}$$

$$x_3>0,\ \ \zeta^{(1)}:=-\frac{1}{2\pi i}\,\log z^{(1)}_{s,\mu},$$

$$U^{(2)}(x)=\sum_{\mu=\pm1}\sum_{s=1}^{l_0^{(2)}}\Big\{\sum_{j=0}^{n_s^{(2)}-1} x_3^j\Big[d^{(2)}_{sj}(x_1,\mu)(z^{(2)}_{s,\mu})^{\frac{1}{2}+\Delta(x_1)-j}\,B_0(\zeta^{(2)})\Big]\,c_j(x_1)$$

$$+\sum_{\substack{k,l=0\\k+l+j+p\geq 1}}^{M+2}\sum_{j+p=0}^{M+2-l} x_2^l\, x_3^j d^{(2)}_{sljp}(x_1,\mu)\,(z^{(2)}_{s,\mu})^{\frac{1}{2}+\Delta(x_1)+p+k}\, B^{(2)}_{skjp}(x_1,\log z^{(2)}_{s,\mu})\Big\}+U^{(2)}_{M+1}(x)\,, \tag{3.55}$$

$$x_3>0,\ \ \zeta^{(2)}:=-\frac{1}{2\pi i}\,\log z^{(2)}_{s,\mu}\,.$$

The coefficients $d^{(1)}_{sj}(\cdot\,,\mu)$, $d^{(2)}_{sj}(\cdot\,,\mu)$, $d^{(1)}_{sljp}(\cdot\,,\mu)$ and $d^{(2)}_{sljp}(\cdot\,,\mu)$ are the matrices with entries from the space $C^\infty(\ell)$, $B^{(1)}_{skjp}(x_1,t)$ and $B^{(2)}_{skjp}(x_1,t)$ are polynomials in t with vector coefficients which depend on the variable x_1 and have the order $\nu_{kjp}=k(2m_0-1)+m_0-1+p+j$ with $m_0=\max\{m_1,\dots,m_l\}$,

$$\begin{gathered}
c_j\in[C^\infty(\ell)]^6,\ \ U^{(1)}_{M+1}\in\big[\,C^{M+1}(\overline{\Omega^{(1)}})\,\big]^6,\ \ U^{(2)}_{M+1}\in\big[\,C^{M+1}(\overline{\Omega^{(2)}})\,\big]^4,\\
(z^{(1)}_{s,\mu})^{\kappa+\Delta(x_1)}:=\operatorname{diag}\Big\{(z^{(1)}_{s,\mu})^{\kappa+\Delta^{(2)}_1(x_1)},\,\dots,\,(z^{(1)}_{s,\mu})^{\kappa+\Delta^{(2)}_6(x_1)}\Big\},\\
(z^{(2)}_{s,\mu})^{\kappa+\Delta(x_1)}:=\operatorname{diag}\Big\{(z^{(2)}_{s,\mu})^{\kappa+\Delta^{(2)}_1(x_1)},\,\dots,\,(z^{(2)}_{s,\mu})^{\kappa+\Delta^{(2)}_6(x_1)}\Big\},\\
\kappa\in\mathbb{R},\ \ \mu=\pm1,\ \ (x_1,\,0)\in\ell,\\
z^{(1)}_{s,+1}=-x_2-x_3\tau^{(1)}_{s,+1},\ \ z^{(1)}_{s,-1}=x_2-x_3\tau^{(1)}_{s,-1},\\
z^{(2)}_{s,+1}=-x_2-x_3\tau^{(2)}_{s,+1},\ \ z^{(2)}_{s,-1}=x_2-x_3\tau^{(2)}_{s,-1},\\
-\pi<\arg z_{s,\pm1}<\pi,\quad -\pi<\arg z^{(2)}_{s,\pm1}<\pi,\\
\{\tau^{(1)}_{s,\pm1}\}_{s=1}^{l_0^{(1)}}\in C^\infty(\ell),\quad \{\tau^{(2)}_{s,\pm1}\}_{s=1}^{l_0^{(1)}}\in C^\infty(\ell).
\end{gathered} \tag{3.56}$$

Here, $\{\tau^{(1)}_{s,\pm1}\}_{s=1}^{l_0^{(1)}}$ (respectively, $\{\tau^{(2)}_{s,\pm1}\}_{s=1}^{l_0^{(2)}}$) are the different roots of multiplicity $n_s^{(1)}$, $s=1,\dots,l_0^{(1)}$, (respectively, $n_s^{(2)}$, $s=1,\dots,l_0^{(2)}$) of the polynomial in ζ, $\det A^{(1,0)}([J^\top_{\varkappa^{(1)}}(x_1,0,0)]^{-1}\eta_\pm)$ (respectively, $\det A^{(2,0)}([J^\top_{\varkappa_2}(x_1,0,0)]^{-1}\eta_\pm)$) with $\eta_\pm=(0,\pm1,\zeta)^\top$, satisfying the condition $\operatorname{Re}\tau^{(1)}_{s,\pm1}<0$ (respectively, $\operatorname{Re}\tau^{(2)}_{s,\pm1}<0$). The matrix $J_{\varkappa_1}$ (respectively, $J_{\varkappa_2}$) stands for the Jacobian matrix corresponding to the canonical diffeomorphism $\varkappa_1$ (respectively, $\varkappa_2$) related to the local coordinate system. Under this diffeomorphism, the curve ℓ is locally rectified and we assume that $(x_1,0,0)\in\ell$, $x_2=\operatorname{dist}(x_{_T},\ell)$, $x_3=\operatorname{dist}(x,\Gamma_T)$, where $x_{_T}$ is the projection of the reference point $x\in\Omega^{(1)}$ (respectively, $x\in\Omega^{(2)}$) on the plane corresponding to the image of Γ_T under the diffeomorphism $\varkappa_1$ (respectively, $\varkappa_2$).

Note that the coefficients $d^{(1)}_{sj}(\,\cdot\,,\mu)$ and $d^{(2)}_{sj}(\,\cdot\,,\mu)$ can be calculated explicitly, whereas the coefficients c_j can be expressed by means of the first coefficient b_0 in the asymptotic expansion of (3.53) (see [14]),

$$\begin{aligned}
&d^{(1)}_{sj}(x_1,+1)=\frac{1}{2\pi}\,G_{\varkappa_1}(x_1,0)\,P^{+(1)}_{sj}(x_1)\,\mathcal{D}(x_1),\\
&d^{(1)}_{sj}(x_1,-1)=\frac{1}{2\pi}\,G_{\varkappa_1}(x_1,0)\,P^{-(1)}_{sj}(x_1)\,\mathcal{D}(x_1)\,e^{i\pi(\frac{1}{2}-\Delta(x_1))}, \qquad s=\overline{1,l_0^{(1)}},\ \ j=\overline{0,n_s^{(1)}-1},\\
&d^{(2)}_{sj}(x_1,\,+1)=\frac{1}{2\pi}\,G_{\varkappa_2}(x_1,0)\,P^{+(2)}_{sj}(x_1)\,\widetilde{\mathcal{D}}(x_1),\\
&d^{(2)}_{sj}(x_1,\,-1)=\frac{1}{2\pi}\,G_{\varkappa_2}(x_1,0)\,P^{-(2)}_{sj}(x_1)\,\widetilde{\mathcal{D}}(x_1)\,e^{i\pi(\frac{1}{2}-\Delta(x_1))}, \qquad s=\overline{1,l_0^{(2)}},\ \ j=\overline{0,n_s^{(2)}-1},
\end{aligned}$$

where $\widetilde{\mathcal{D}} = \|\mathcal{D}_{kj}\|_{4\times 6}$, $k = 1,2,3,6$, $j = \overline{1,6}$, is composed of the entries of matrix $\mathcal{D}$ (see (3.51)),

$$P_{sj}^{\pm(1)}(x_1) := V_{-1,\,j}^{(1),s}(x_1,0,0,\pm 1)\Big[\mathfrak{S}\big(-\frac{1}{2}\,I_6 + \mathcal{K}_\tau^{(1)}; x_1,0,0,\pm 1\big)\Big]^{-1},$$

$$P_{sj}^{\pm(2)}(x_1) := V_{-1,\,j}^{(2),s}(x_1,0,0,\pm 1)\Big[\mathfrak{S}\big(-\frac{1}{2}\,I_4 + \mathcal{K}_\tau^{(2)}; x_1,0,0,\pm 1\big)\Big]^{-1},$$

$$V_{-1,\,j}^{(1),s}(x_1,0,0,\pm 1) := -\frac{i^{j+1}}{j!(n_s^{(1)}-1-j)!}\,\frac{d^{n_s^{(1)}-1-j}}{d\zeta^{n_s^{(1)}-1-j}}\,(\zeta-\tau_{s,\pm 1}^{(1)})^{n_s^{(1)}} \times\Big(A^{(1,0)}\big((J_{\varkappa_1}^\top(x_1,0))^{-1}\big)\cdot(0,\pm 1,\zeta)^\top\Big)^{-1}\Big|_{\zeta=\tau_{s,\pm 1}^{(1)}},$$

$$V_{-1,\,j}^{(2),s}(x_1,0,0,\pm 1) := -\frac{i^{j+1}}{j!(n_s^{(2)}-1-j)!}\,\frac{d^{n_s^{(2)}-1-j}}{d\zeta^{n_s^{(2)}-1-j}}\,(\zeta-\tau_{s,\pm 1}^{(2)})^{n_s^{(2)}} \times\Big(A^{(2,\,0)}\big((J_{\varkappa_2}^\top(x_1,0))^{-1}\big)\cdot(0,\pm 1,\zeta)^\top\Big)^{-1}\Big|_{\zeta=\tau_{s,\pm 1}^{(2)}},$$

$G_{\varkappa_1}(x_1,0)$ and $G_{\varkappa_2}(x_1,0)$ are smooth scalar functions explicitly written in terms of diffeomorphisms $\varkappa_1$ and $\varkappa_2$, respectively, and

$$c_j(x_1) = a_j(x_1)\,B_0^-\Big(-\frac{1}{2}+\Delta(x_1)\Big)\,\mathcal{D}^{-1}(x_1)b_0(x_1),$$
$$j = 0,\dots,n_s^{(1)}-1, \quad (j = 0,\dots,n_s^{(2)}-1),$$

where

$$B_0^-\Big(-\frac{1}{2}+\Delta(x_1)\Big) = \operatorname{diag}\Big\{B_-^{m_1}\Big(-\frac{1}{2}+\Delta_1^{(2)}(x_1)\Big),\dots,B_-^{m_l}\Big(-\frac{1}{2}+\Delta_l^{(2)}(x_1)\Big)\Big\},$$

$$B_-^{m_q}(t) = \big\|\widetilde{b}_{kp}^{\,m_q}(t)\big\|_{m_q\times m_q},\ \ q = 1,\dots,l,$$

$$\widetilde{b}_{kp}^{\,m_q}(t) = \begin{cases}\Big(\dfrac{1}{2\pi i}\Big)^{p-k}\dfrac{(-1)^{p-k}}{(p-k)!}\,\dfrac{d^{p-k}}{dt^{p-k}}\,\Gamma(t+1)\,e^{\frac{i\pi(t+1)}{2}}, & \text{for } k\leqslant p,\\ 0, & \text{for } k>p,\end{cases}$$

and $\Gamma(t+1)$ is the Euler integral,

$$a_j(x_1) = \operatorname{diag}\Big\{a^{m_1}(\alpha_1^{(j)}),\dots,a^{m_l}(\alpha_l^{(j)})\Big\},$$

$$\alpha_q^{(j)}(x_1) = -\frac{3}{2}-\Delta_q^{(2)}(x_1)+j,\ \ q=\overline{1,l},\ \ j=\overline{0,n_s^{(1)}-1}\ \ (j=\overline{0,n_s^{(2)}-1}),$$

$$a^{m_q}(\alpha_q^{(j)}) = \big\|a_{kp}^{m_q}(\alpha_q^{(j)})\big\|_{m_q\times m_q},$$

$$a_{kp}^{m_q}(\alpha_q^{(j)}) = \begin{cases}-i\displaystyle\sum_{l=k}^{p}\frac{(-1)^{p-k}(2\pi i)^{l-p}\widetilde{b}_{kl}^{\,m_q}(\mu_q)}{(\alpha_q^{(0)}+1)^{p-l+1}}, \ \ j=0,\ \ k\leqslant p,\\ (-1)^{p-k}\widetilde{b}_{kp}^{\,m_q}(\alpha_q^{(j)}),\ \ j=\overline{1,n_s^{(1)}-1}\ \ (j=\overline{1,n_s^{(2)}-1}),\ \ k\leqslant p,\\ 0,\ \ k>p,\end{cases}$$

$$\mu_q = -\frac{1}{2}-\Delta_q^{(2)}(x_1),\ \ -1<\operatorname{Re}\mu_q<0.$$

Analogous investigation for the basic mixed and interior crack problems for homogeneous piezoelectric bodies has been carried out in reference [8], where the asymptotic properties of solutions have been established near the interior crack's edges and the curves, where the different boundary conditions collide. In [8], it is shown that the stress singularity exponents at the interior crack edges do not depend on the material parameters and are equal to -0.5, while they depend essentially on the material parameters at the collision curves, where different boundary conditions collide.

As it is evident from the above exposed results, the stress singularity exponents at the interfacial crack edges and at the curves, where the interface intersects the exterior boundary, depend essentially on the material parameters, in general. More precise results for particular classes of solids are presented in the next section, where the stress singularity exponents are calculated explicitly.

4. Analysis of Singularities of Solutions

Here, we assume that Γ_T and ℓ are rectified with the help of the diffeomorphisms mentioned in the previous section and for $x'\in\ell=\partial\Gamma_T$ by $\Pi_{x'}$ we denote the plane passing trough the point x' and orthogonal to ℓ. We introduce the polar coordinates (r,α), $r\geqslant 0$, $-\pi\leqslant\alpha\leqslant\pi$, in the plane $\Pi_{x'}$ with the pole at the point x'. Denote by $\Gamma_T^\pm$ the two different faces of the surface Γ_T. It is evident that $(r,\pm\pi)\in\Gamma_T^\pm$.

The intersection of the plane $\Pi_{x'}$ and $\Omega^{(1)}$ is identified with the half-plane $r\geqslant 0$ and $-\pi\leqslant\alpha\leqslant 0$, while the intersection of the plane $\Pi_{x'}$ and $\Omega^{(2)}$ is identified with the half-plane $r\geqslant 0$ and $0\leqslant\alpha\leqslant\pi$.

The roots given by (3.56) are represented as follows:

$$z^{(1)}_{s,+1}=-r\big[\cos\alpha+\tau^{(1)}_{s,+1}(x')\sin\alpha\big],\quad z^{(1)}_{s,-1}=r\big[\cos\alpha-\tau^{(1)}_{s,-1}(x')\sin\alpha\big],$$
$$s=1,\dots,l^{(1)}_0,\;\; x'\in\ell,$$
$$z^{(2)}_{s,+1}=-r\big[\cos\alpha+\tau^{(2)}_{s,+1}(x')\sin\alpha\big],\quad z^{(2)}_{s,-1}=r\big[\cos\alpha-\tau^{(2)}_{s,-1}(x')\sin\alpha\big],$$
$$s=1,\dots,l^{(2)}_0,\;\; x'\in\ell.$$

From the asymptotic expansions (3.54) and (3.55) we get

$$U^{(1)}(x)=\sum_{\mu=\pm1}\sum_{s=1}^{l_0^{(1)}}\sum_{j=0}^{n_s^{(1)}-1}c^{(1)}_{sj\mu}(x',\,\alpha)\,r^{\gamma+i\delta}B_0(\zeta)\,\widetilde{c}^{(1)}_{sj\mu}(x',\,\alpha)+\cdots,\tag{4.1}$$

$$U^{(2)}(x)=\sum_{\mu=\pm1}\sum_{s=1}^{l_0^{(2)}}\sum_{j=0}^{n_s^{(2)}-1}c^{(2)}_{sj\mu}(x',\,\alpha)\,r^{\gamma+i\delta}B_0(\zeta)\,\widetilde{c}^{(2)}_{sj\mu}(x',\,\alpha)+\cdots,\tag{4.2}$$

where

$$r^{\gamma+i\delta}=\operatorname{diag}\big\{r^{\gamma_1+i\delta_1},\,\dots,\,r^{\gamma_6+i\delta_6}\big\},\;\;\zeta=-\frac{1}{2\pi i}\,\log r,$$
$$\gamma_j=\frac12+\frac{1}{2\pi}\,\arg\lambda_j(x'),\;\;\delta_j=-\frac{1}{2\pi}\,\log|\lambda_j(x')|,\;\; x'\in\ell,\;\; j=\overline{1,6},\tag{4.3}$$

and $\lambda_j=\lambda^{(2)}_j$, $j=\overline{1,6}$, are eigenvalues of the matrix

$$\mathcal{D}_2(x')=\big[\mathfrak{S}_2(x',0,+1)\big]^{-1}\,\mathfrak{S}_2(x',0,-1),\;\; x'\in\ell.\tag{4.4}$$

Note that the subsequent terms in expansion (4.1) and (4.2) have higher regularity, i.e., the real parts of the corresponding exponents are greater than γ_j.

The coefficients $c^{(1)}_{sj\mu}$, $\widetilde{c}^{(1)}_{sj\mu}$, $c^{(2)}_{sj\mu}$ and $\widetilde{c}^{(2)}_{sj\mu}$ in asymptotic expansions (4.1) and (4.2) read as

$$c^{(1)}_{sj\mu}(x',\,\alpha)=\sin^j\alpha\;d^{(1)}_{sj}(x',\,\mu)\big[\psi^{(1)}_{s,\mu}(x',\,\alpha)\big]^{\gamma+i\delta-j},\quad \widetilde{c}^{(1)}_{sj\mu}(x',\,\alpha)=B_0\Big(-\frac{1}{2\pi i}\log\psi^{(1)}_{s,\mu}(x',\,\alpha)\Big)\,c_j(x'),$$
$$j=\overline{0,n_s^{(1)}-1},\;\;\mu=\pm1,\;\; s=\overline{1,l_0^{(1)}},$$
$$c^{(2)}_{sj\mu}(x',\,\alpha)=\sin^j\alpha\;d^{(2)}_{sj}(x',\,\mu)\big[\psi^{(2)}_{s,\mu}(x',\,\alpha)\big]^{\gamma+i\delta-j},\quad \widetilde{c}^{(2)}_{sj\mu}(x',\,\alpha)=B_0\Big(-\frac{1}{2\pi i}\log\psi^{(2)}_{s,\mu}(x',\,\alpha)\Big)\,c_j(x'),$$
$$j=\overline{0,n_s^{(2)}-1},\;\;\mu=\pm1,\;\; s=\overline{1,l_0^{(2)}},$$

where

$$\psi^{(1)}_{s,\mu}(x',\,\alpha)=-\mu\cos\alpha-\tau^{(1)}_{s,\mu}(x')\sin\alpha,\;\; s=\overline{1,l_0^{(1)}},$$
$$\psi^{(2)}_{s,\mu}(x',\,\alpha)=-\mu\cos\alpha-\tau^{(2)}_{s,\mu}(x')\sin\alpha,\;\; s=\overline{1,l_0^{(2)}},$$

$$c^{(1)}_{sj\mu}(x',\,\alpha)=\big\|c^{(1,kp)}_{sj\mu}(x',\,\alpha)\big\|_{6\times 6},\quad c^{(2)}_{sj\mu}(x',\,\alpha)=\big\|c^{(2,kp)}_{sj\mu}(x',\,\alpha)\big\|_{4\times 6}.$$

In what follows, for special classes of elastic materials we will analyze the exponents $\gamma_j+i\delta_j$, which determine the behaviour of $U^{(1)}$ and $U^{(2)}$ near the line ℓ.

As it was mention above, $\lambda_6=1$ (for details see [7, Section 5.7]). Therefore, $\gamma_6=1/2$ and $\delta_6=0$ in accordance with (4.3). This implies that one could not expect better smoothness for solutions than $C^{1/2}$, in general.

More detailed analysis leads to the following refined asymptotic behaviour for the temperature functions (cf. [8]).

Theorem 4.1. *Near the exceptional curve ℓ the functions $\vartheta^{(1)}$ and $\vartheta^{(2)}$ possess the following asymptotic behaviour:*

$$\vartheta^{(1)}=b^{(1)}_0 r^{\frac{1}{2}}+\mathcal{R}^{(1)}, \tag{4.5}$$

$$\vartheta^{(2)}=b^{(2)}_0 r^{\frac{1}{2}}+\mathcal{R}^{(2)}, \tag{4.6}$$

where $b^{(i)}_0\in C^{1+\gamma'-\varepsilon}$, $\mathcal{R}^{(i)}\in C^{\frac{3}{2}+\gamma'-\varepsilon}$, $i=1,2$, in the corresponding one-sided neighbourhoods of ℓ and $1+\gamma'-\varepsilon>\frac{1}{2}$ for sufficiently small $\varepsilon>0$.

Proof. Indeed, $u^{(1)}_6=\vartheta^{(1)}$ and $u^{(2)}_4=\vartheta^{(2)}$ are the solutions of the transmission problem (3.50) with C^∞ data. Since the matrices $[\eta^{(1)}_{ij}]_{3\times 3}$ and $[\eta^{(2)}_{ij}]_{3\times 3}$ are positive definite, this transmission problem can be reduced to a system of pseudodifferential equations, where the principal part is described by the scalar positive-definite invertible pseudodifferential operators

$$\begin{gathered}\mathcal{H}^{(1)}_{scalar}\big(-2^{-1}I+\mathcal{K}^{(1)}_{scalar}\big)^{-1}+\mathcal{H}^{(2)}_{scalar}\big(-2^{-1}I+\mathcal{K}^{(2)}_{scalar}\big)^{-1}\ :\ \widetilde{H}^{s-1}_p(\Gamma_T)\to H^s_p(\Gamma_T)\\ \mathcal{H}^{(1)}_{scalar}\big(-2^{-1}I+\mathcal{K}^{(1)}_{scalar}\big)^{-1}+\mathcal{H}^{(2)}_{scalar}\big(-2^{-1}I+\mathcal{K}^{(2)}_{scalar}\big)^{-1}\ :\ \widetilde{B}^{s-1}_{p,p}(\Gamma_T)\to B^s_{p,p}(\Gamma_T),\\ \frac{1}{p}-\frac{1}{2}<s<\frac{1}{p}+\frac{1}{2},\ \ 1<p<\infty,\end{gathered}$$

where $\mathcal{K}^{(i)}_{scalar}$, $i=1,2$, are compact. These pseudodifferential operators have principal homogeneous symbol $-2\mathfrak{S}\big(\mathcal{H}^{(1)}_{scalar}+\mathcal{H}^{(2)}_{scalar};x,\xi\big)$, which is positive and even in ξ. Hence we can establish refined explicit asymptotic relations of type (4.5), (4.6) for the temperature functions $u^{(1)}_6=\vartheta^{(1)}$ and $u^{(2)}_4=\vartheta^{(2)}$ in the corresponding one-sided neighbourhoods of ℓ (see [14, 15, 17, 18]). ∎

From (4.5) and (4.6), it follows that

(i) The leading exponents for $u^{(1)}_6=\vartheta^{(1)}$ and $u^{(2)}_4=\vartheta^{(2)}$ in the neighborhood of line ℓ are equal to $\frac{1}{2}$;
(ii) Logarithmic factors are absent in the first terms of the asymptotic expansions of $\vartheta^{(1)}$ and $\vartheta^{(2)}$;
(iii) The temperature functions $\vartheta^{(1)}$ and $\vartheta^{(2)}$ do not oscillate in the neighbourhood of the collision curve ℓ and for the heat flux vector we have no oscillating singularities;
(iv) The temperature functions $\vartheta^{(1)}$ and $\vartheta^{(2)}$ belong to $C^{\frac{1}{2}}(\overline{\Omega^{(1)}})$ and $C^{\frac{1}{2}}(\overline{\Omega^{(2)}})$, respectively, (cf. [8], Theorem 6.4).

Non-zero parameters δ_j in (4.3) lead to the so-called oscillating singularities for the first order derivatives of $U^{(1)}$ and $U^{(2)}$, in general. In turn, this yields oscillating stress singularities, which sometimes lead to mechanical contradictions, for example, to an overlapping of materials. So, from the practical point of view, it is important to single out the classes of solids for which the oscillating singularities do not occur.

Let us consider the above investigated mixed boundary-transmission problem for particular elastic components. We assume that the medium occupying the domain $\Omega^{(1)}$ belongs to the **422** (Tetragonal) or **622** (Hexagonal) class of crystals. The corresponding system of differential equations reads as (see, e.g., [16])

$$
\begin{aligned}
&(c_{11}\,\partial_1^2+c_{66}\,\partial_2^2+c_{44}\,\partial_3^2)u_1^{(1)}+(\,c_{12}+c_{66}\,)\,\partial_1\partial_2u_2^{(1)}+(\,c_{13}+c_{44}\,)\,\partial_1\partial_3u_3^{(1)}\\
&\qquad\qquad -e_{14}\,\partial_2\partial_3\varphi^{(1)}-q_{15}\,\partial_2\partial_3\psi^{(1)}-\widetilde{\gamma}_1\,\partial_1\vartheta^{(1)}-\varrho^{(1)}\,\tau^2\,u_1^{(1)}=F_1,\\
&(\,c_{12}+c_{66}\,)\,\partial_2\partial_1u_1^{(1)}+(\,c_{66}\,\partial_1^2+c_{11}\,\partial_2^2+c_{44}\,\partial_3^2\,)\,u_2^{(1)}+(\,c_{13}+c_{44}\,)\,\partial_2\partial_3u_3^{(1)}\\
&\qquad\qquad +e_{14}\partial_1\partial_3\varphi^{(1)}+q_{15}\partial_1\partial_3\psi^{(1)}-\widetilde{\gamma}_1\partial_2\vartheta^{(1)}-\varrho^{(1)}\,\tau^2\,u_2^{(1)}=F_2,\\
&(\,c_{13}+c_{44}\,)\,\partial_3\partial_1u_1^{(1)}+(\,c_{13}+c_{44}\,)\,\partial_3\partial_2u_2^{(1)}+(\,c_{44}\,\partial_1^2+c_{44}\,\partial_2^2+c_{33}\,\partial_3^2\,)\,u_3^{(1)}\\
&\qquad\qquad -\widetilde{\gamma}_3\,\partial_3\vartheta^{(1)}-\varrho^{(1)}\,\tau^2\,u_3^{(1)}=F_3,\\
&e_{14}\partial_2\partial_3u_1^{(1)}-e_{14}\partial_1\partial_3u_2^{(1)}+(\,\varkappa_{11}\,\partial_1^2+\varkappa_{11}\,\partial_2^2+\varkappa_{33}\,\partial_3^2\,)\,\varphi^{(1)}-(1+\nu_0\tau)p_3\,\partial_3\vartheta^{(1)}=F_4,\\
&q_{15}\partial_2\partial_3u_1^{(1)}-q_{15}\partial_1\partial_3u_2^{(1)}+(\,\mu_{11}\,\partial_1^2+\mu_{11}\,\partial_2^2+\mu_{33}\,\partial_3^2\,)\,\psi^{(1)}-(1+\nu_0\tau)m_3\,\partial_3\vartheta^{(1)}=F_5,\\
&-\tau\,T_0\,(\,\widetilde{\gamma}_1\,\partial_1u_1^{(1)}+\widetilde{\gamma}_1\,\partial_2u_2^{(1)}+\widetilde{\gamma}_3\,\partial_3u_3^{(1)}\,)+\tau\,T_0\,p_3\,\partial_3\varphi^{(1)}+\tau\,T_0\,m_3\,\partial_3\psi^{(1)}\\
&\qquad\qquad +(\,\eta_{11}\,\partial_1^2+\eta_{11}\,\partial_2^2+\eta_{33}\,\partial_3^2\,)\,\vartheta^{(1)}-(\tau\,d_0+\tau^2h^{(1)})\,\vartheta^{(1)}=F_6,
\end{aligned}
$$

where c_{11}, c_{12}, c_{13}, c_{33}, c_{44}, c_{66} are the elastic constants, e_{14} is the piezoelectric constant, q_{15} is the piezomagnetic constant, $\varkappa_{11}$ and $\varkappa_{33}$ are the dielectric constants, μ_{11} and μ_{33} are the magnetic permeability constants, $\widetilde{\gamma}_1=(1+\nu_0\tau)\lambda_{11}=(1+\nu_0\tau)\lambda_{21}$ and $\widetilde{\gamma}_3=(1+\nu_0\tau)\lambda_{31}$ are the thermal strain constants, η_{11} and η_{33} are the thermal conductivity constants, p_3 is the pyroelectric constant and m_3 is the pyromagnetic constant. In the case of Hexagonal crystals (**622** class), we have $c_{66}=(c_{11}-c_{12})/2$.

Note that some important polymers and bio-materials are modelled by the above partial differential equations, for example, the *collagen-hydroxyapatite* is one example of such a material. This material is widely used in biology and medicine (see [31]). Another important example is TeO_2 [16].

In this model, the generalized stress operator is defined as

$$
T(\partial_x,n,\tau)=\big\|\,T_{jk}(\partial_x,n,\tau)\,\big\|_{6\times 6}
$$

with

$$
\begin{aligned}
&T_{11}(\partial_x,n,\tau)=c_{11}n_1\partial_1+c_{66}n_2\partial_2+c_{44}n_3\partial_3, && T_{12}(\partial_x,n,\tau)=c_{12}n_1\partial_2+c_{66}n_2\partial_1,\\
&T_{13}(\partial_x,n,\tau)=c_{13}n_1\partial_3+c_{44}n_3\partial_1, && T_{14}(\partial_x,n,\tau)=-e_{14}n_3\partial_2,\\
&T_{15}(\partial_x,n,\tau)=-q_{15}n_3\partial_2, && T_{16}(\partial_x,n,\tau)=-\widetilde{\gamma}_1\,n_1,\\
&T_{21}(\partial_x,n,\tau)=c_{66}n_1\partial_2+c_{12}n_2\partial_1, && T_{22}(\partial_x,n,\tau)=c_{66}n_1\partial_1+c_{11}n_2\partial_2+c_{44}n_3\partial_3,\\
&T_{23}(\partial_x,n,\tau)=c_{13}n_2\partial_3+c_{44}n_3\partial_2, && T_{24}(\partial_x,n,\tau)=e_{14}n_3\partial_1,\\
&T_{25}(\partial_x,n,\tau)=q_{15}n_3\partial_1, && T_{26}(\partial_x,n,\tau)=-\widetilde{\gamma}_1\,n_2,\\
&T_{31}(\partial_x,n,\tau)=c_{44}n_1\partial_3+c_{13}n_3\partial_1, && T_{32}(\partial_x,n,\tau)=c_{44}n_2\,\partial_3+c_{13}n_3\partial_2,\\
&T_{33}(\partial_x,n)=c_{44}n_1\partial_1+c_{44}n_2\partial_2+c_{33}n_3\partial_3, && T_{34}(\partial_x,n,\tau)=0,\\
&T_{35}(\partial_x,n,\tau)=0, && T_{36}(\partial_x,n,\tau)=-\widetilde{\gamma}_3\,n_3,\\
&T_{41}(\partial_x,n,\tau)=e_{14}n_2\partial_3, && T_{42}(\partial_x,n,\tau)=-e_{14}n_1\partial_3,\\
&T_{43}(\partial_x,n,\tau)=0, && T_{44}(\partial_x,n,\tau)=\varkappa_{11}(n_1\partial_1+n_2\partial_2)+\varkappa_{33}n_3\partial_3,\\
&T_{45}(\partial_x,n,\tau)=0, && T_{46}(\partial_x,n,\tau)=-p_3n_3,\\
&T_{51}(\partial_x,n,\tau)=q_{15}n_2\partial_3, && T_{52}(\partial_x,n,\tau)=-q_{15}n_1\partial_3,
\end{aligned}
$$

$$
\begin{aligned}
&T_{53}(\partial_x,n,\tau)=0, && T_{54}(\partial_x,n,\tau)=0,\\
&T_{55}(\partial_x,n,\tau)=\mu_{11}(n_1\partial_1+n_2\partial_2)+\mu_{33}n_3\partial_3, && T_{56}(\partial_x,n,\tau)=-m_3n_3,\\
&T_{6j}(\partial_x,n,\tau)=0,\ \text{for}\ j=\overline{1,5}, && T_{66}(\partial_x,n,\tau)=\eta_{11}(n_1\partial_1+n_2\partial_2)+\eta_{33}n_3\partial_3.
\end{aligned}
$$

The material constants satisfy the following system of inequalities

$$\begin{aligned} &c_{11} > |c_{12}|, \quad c_{44} > 0, \quad c_{66} > 0, \quad c_{33}(c_{11}+c_{12}) > 2c_{13}^2, \\ &\varkappa_{11} > 0, \quad \varkappa_{33} > 0, \quad \eta_{11} > 0, \quad \eta_{33} > 0, \quad \mu_{11} > 0, \quad \mu_{33} > 0, \end{aligned} \tag{4.7}$$

which are equivalent to the positive definiteness of the internal energy form (see (2.7), (2.8)).

From (2.9), (2.12), (2.13), and (4.7) it follows also that

$$\varkappa_{33} > p_3^2 T_0 d_0^{-1}, \quad \mu_{33} > m_3^2 T_0 d_0^{-1}, \quad c_{11}c_{33} > c_{13}^2. \tag{4.8}$$

Under these conditions the mixed boundary-transmission problem in question is uniquely solvable.

Furthermore, we assume that $e_{14} \neq 0$, $e_{15} \neq 0$, $\dfrac{\mu_{11}}{\varkappa_{11}} = \dfrac{\mu_{33}}{\varkappa_{33}} = \alpha$, the surface Γ_C is parallel to the plane of isotropy (i.e., to the plane $x_3 = 0$) in some neighbourhood of $\partial\Gamma_C$, and the domain $\Omega^{(2)}$ is occupied by an isotropic material modeled by the generalized thermoelasticity equations (see (2.1), (2.2))

$$\begin{gathered} \mu\Delta u^{(2)} + (\lambda+\mu)\operatorname{grad}\operatorname{div} u^{(2)} - (1+\nu_0\tau)\lambda^{(2)}\operatorname{grad}\vartheta^{(2)} - \varrho^{(2)}\tau^2 u^{(2)} = 0, \\ \eta^{(2)}\Delta\vartheta^{(2)} - (\tau d_0^{(2)} + \tau^2 h_0^{(2)})\vartheta^{(2)} - \tau\lambda^{(2)} u^{(2)} = 0, \\ \mu > 0, \quad 3\lambda + 2\mu > 0, \quad \eta^{(2)} > 0, \quad h_0^{(2)} > 0, \quad d_0^{(2)} - \nu_0 h_0^{(2)} > 0, \end{gathered}$$

In the case of this particular mixed boundary-transmission problem we find the exponents involved in the asymptotic expansions of solutions explicitly in terms of the material constants. To this end, we find the eigenvalues of the matrix (4.4) explicitly and calculate the exponents $\gamma + i\delta$ involved in the asymptotic expansions (4.1) and (4.2).

Taking into account the relations

$$\mathfrak{S}\Big(-2^{-1}I_6 \pm \mathcal{K}_\tau^{(1)}; x', 0, 1\Big) = \mathfrak{S}\Big(-2^{-1}I_6 + \mathcal{K}_\tau^{(1)}; x', 0, \pm 1\Big), \quad \mathfrak{S}(\mathcal{H}_\tau^{(1)}; x', 0, -1) = \mathfrak{S}(\mathcal{H}_\tau^{(1)}; x', 0, 1),$$

for these symbol matrices we introduce the short notation

$$\sigma\Big(-2^{-1}I_6 \pm \mathcal{K}_\tau^{(1)}\Big) := \mathfrak{S}\Big(-2^{-1}I_6 \pm \mathcal{K}_\tau^{(1)}; x', 0, 1\Big) = \mathfrak{S}\Big(-2^{-1}I_6 + \mathcal{K}_\tau^{(1)}; x', 0, \pm 1\Big)$$

and

$$\sigma(\mathcal{H}_\tau^{(1)}) := \mathfrak{S}(\mathcal{H}_\tau^{(1)}; x', 0, \pm 1).$$

These symbols can be calculated explicitly (see [8], Appendix B):

$$\sigma\Big(-\frac{1}{2}I_6 \pm \mathcal{K}_\tau^{(1)}\Big) = \begin{bmatrix} -\frac12 & 0 & 0 & \pm A_{14} & \pm A_{15} & 0 \\ 0 & -\frac12 & \pm A_{23} & 0 & 0 & 0 \\ 0 & \pm A_{32} & -\frac12 & 0 & 0 & 0 \\ \pm A_{41} & 0 & 0 & -\frac12 & 0 & 0 \\ \pm A_{51} & 0 & 0 & 0 & -\frac12 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac12 \end{bmatrix},$$

where

$$A_{14} = -i\,\frac{e_{14}\,c_{66}\,(b_2-b_1)}{2\,b_1\,b_2\sqrt{B}} - i\,\frac{e_{14}q_{15}^2}{\alpha\varkappa_{11}\widetilde{e}_{14}^{\,2}}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1 b_2 + \varkappa_{11})}{\sqrt{B}}\right],$$

$$A_{15} = -i\,\frac{q_{15}\,c_{66}\,(b_2-b_1)}{2\alpha\,b_1\,b_2\sqrt{B}} - i\,\frac{q_{15}e_{14}^2}{\alpha\varkappa_{11}\widetilde{e}_{14}^{\,2}}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}} - \frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1 b_2 + \varkappa_{11})}{\sqrt{B}}\right],$$

$$A_{41} = -i\,\frac{e_{14}\,\varkappa_{33}\,(b_2-b_1)}{2\sqrt{B}}, \quad A_{51} = -i\,\frac{q_{15}\,\varkappa_{33}\,(b_2-b_1)}{2\sqrt{B}},$$

$$b_1 = \sqrt{\frac{A-\sqrt{B}}{2\,c_{44}\,\varkappa_{33}}}, \qquad b_2 = \sqrt{\frac{A+\sqrt{B}}{2\,c_{44}\,\varkappa_{33}}}, \quad \widetilde{e}_{14} = \Big(e_{14}^2 + \alpha^{-1}q_{15}^2\Big)^{1/2}, \quad \alpha = \frac{\mu_{11}}{\varkappa_{11}} = \frac{\mu_{33}}{\varkappa_{33}} > 0,$$

$$A=\widetilde{e}_{14}^2+c_{44}\,\varkappa_{11}+c_{66}\,\varkappa_{33}>0,\qquad B=A^2-4\,c_{44}\,c_{66}\,\varkappa_{11}\,\varkappa_{33}>0,\quad A>\sqrt{B}.$$

Note that $b_1b_2=\sqrt{\dfrac{c_{66}\varkappa_{11}}{c_{44}\varkappa_{33}}}$.

It can be proved that $A_{14}A_{41}<0$, $A_{15}A_{51}<0$ (see [8], Appendix B).

Let us calculate the entries A_{23} and A_{32}. Introduce the notation

$$C:=c_{11}\,c_{33}-c_{13}^2-2\,c_{13}\,c_{44},\quad D:=C^2-4\,c_{44}^2\,c_{33}c_{11}. \tag{4.9}$$

Consider two cases.

Case 1. Let $D>0$. Then

$$A_{23}=i\,\frac{c_{44}\,(d_2-d_1)\,(c_{11}-c_{13}\,d_1d_2)}{2\,d_1\,d_2\sqrt{D}}\,,\quad A_{32}=-i\,\frac{c_{44}\,(d_2-d_1)\,(c_{33}\,d_1\,d_2-c_{13})}{2\,d_1\,d_2\sqrt{D}}\,, \tag{4.10}$$

where

$$d_1=\sqrt{\frac{C-\sqrt{D}}{2\,c_{44}\,c_{33}}}\,,\qquad d_2=\sqrt{\frac{C+\sqrt{D}}{2\,c_{44}\,c_{33}}}.$$

Inequalities (4.7) imply $C>\sqrt{D}$ and

$$d_1d_2=\frac{\sqrt{c_{11}}}{\sqrt{c_{33}}},\quad (d_2-d_1)^2=\frac{C-2c_{44}\sqrt{c_{33}}\sqrt{c_{11}}}{c_{44}c_{33}}>0. \tag{4.11}$$

Then, from (4.10), we obtain $A_{23}\,A_{32}>0$.

Case 2. Let $D<0$. In this case,

$$A_{23}=i\,\frac{a\,c_{44}(\sqrt{c_{11}\,c_{33}}-c_{13})}{\sqrt{-D}},\quad A_{32}=-i\,\frac{a\,c_{44}(\sqrt{c_{11}\,c_{33}}-c_{13})}{\sqrt{-D}}\,\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}}\,, \tag{4.12}$$

where

$$a=\frac{1}{2}\sqrt{\frac{-C+2c_{44}\sqrt{c_{11}c_{33}}}{c_{44}c_{33}}}>0 \tag{4.13}$$

and we get again

$$A_{23}A_{32}=\frac{c_{44}^2\,a^2(\sqrt{c_{11}\,c_{33}}-c_{13})^2}{-D}\,\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}}>0.$$

The symbol matrix $\sigma(\mathcal{H}_\tau^{(1)})$ has the following block-wise structure:

$$\sigma(\mathcal{H}_\tau^{(1)})=\begin{bmatrix}\mathbf{C}_{11} & 0 & 0 & 0 & 0 & 0\\ 0 & \mathbf{C}_{22} & 0 & 0 & 0 & 0\\ 0 & 0 & \mathbf{C}_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbf{C}_{44} & \mathbf{C}_{45} & 0\\ 0 & 0 & 0 & \mathbf{C}_{45} & \mathbf{C}_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & \mathbf{C}_{66}\end{bmatrix}_{6\times 6},$$

where

$$\mathbf{C}_{11}=-\frac{b_2-b_1}{2\sqrt{B}}\left(\varkappa_{33}+\frac{\varkappa_{11}}{b_1b_2}\right),$$

$$\mathbf{C}_{22}=\begin{cases}-\dfrac{d_2-d_1}{2\sqrt{D}}\left(c_{33}+c_{44}\sqrt{\dfrac{c_{33}}{c_{11}}}\right) & \text{if } D>0,\\[2mm] -\dfrac{a}{\sqrt{D}}\left(c_{33}+c_{44}\sqrt{\dfrac{c_{33}}{c_{11}}}\right), & \text{if } D<0,\end{cases}$$

$$\mathbf{C}_{33}=\begin{cases}-\dfrac{d_2-d_1}{2\sqrt{D}}\,(c_{44}+\sqrt{c_{11}c_{33}}\,), & \text{if } D>0,\\[2mm] -\dfrac{a}{\sqrt{D}}\,(c_{44}+\sqrt{c_{11}c_{33}}\,), & \text{if } D<0,\end{cases}.$$

$$\mathbf{C}_{44}=-\left\{\frac{b_2-b_1}{2\sqrt{B}}\left(c_{44}+\frac{c_{66}}{b_1b_2}\right)+\frac{q_{15}^2}{2\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}}-\frac{c_{44}(b_2-b_1)(\varkappa_{33}b_1b_2+\varkappa_{11})}{\sqrt{B}}\right]\right\},$$

$$\mathbf{C}_{55}=-\left\{\frac{b_2-b_1}{2\sqrt{B}}\left(c_{44}+\frac{c_{66}}{b_1b_2}\right)+\frac{e_{14}^2}{2\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}}-\frac{c_{44}(b_2-b_1)(\varkappa_{33}b_1b_2+\varkappa_{11})}{\sqrt{B}}\right]\right\},$$

$$\mathbf{C}_{45}=\mathbf{C}_{54}=\frac{e_{14}q_{15}}{2\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}}-\frac{c_{44}(b_2-b_1)(\varkappa_{33}b_1b_2+\varkappa_{11})}{\sqrt{B}}\right],\quad \mathbf{C}_{66}=-\frac{1}{2\sqrt{\eta_{11}\eta_{33}}}.$$

Remark that $C_{jj}<0,\ j=\overline{1,6}$ (see [8], Appendix B).

The symbol matrix $\sigma^{\pm}(\mathcal{B}_\tau^{(2)}):=\mathfrak{S}(\mathcal{B}_\tau^{(2)};x',0,\pm1)$ reads as

$$\sigma^{\pm}(\mathcal{B}_\tau^{(2)})=\begin{bmatrix}\frac{1}{\mu}&0&0&0&0&0\\0&a&\pm ib&0&0&0\\0&\mp ib&a&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\end{bmatrix}_{6\times6},\quad a:=\frac{2(\lambda+2\mu)\mu}{\lambda+\mu},\quad b:=\frac{\mu^2}{\lambda+3\mu}.$$

Then the symbol matrix of the Poincaré–Steklov type operator has the form

$$\sigma^{\pm}(\mathcal{A}_\tau^{(1)}):=\sigma(\mathcal{H}_\tau^{(1)})\sigma\Big(-\frac{1}{2}I_6\pm\mathcal{K}_\tau^{(1)}\Big)^{-1}=\big[\mathcal{A}_{jk}^{\pm}\big]_{6\times6},$$

where

$$\begin{aligned}
&\mathcal{A}_{11}^{\pm}=\mathcal{A}_{11}=\frac{2\mathbf{C}_{11}}{Q_1},\quad \mathcal{A}_{12}^{\pm}=\mathcal{A}_{13}^{\pm}=\mathcal{A}_{16}^{\pm}=0,\quad \mathcal{A}_{14}^{\pm}=\mp\frac{4A_{14}\mathbf{C}_{11}}{Q_1},\quad \mathcal{A}_{15}^{\pm}=\pm\frac{4A_{15}\mathbf{C}_{11}}{Q_1},\\
&\mathcal{A}_{21}^{\pm}=0,\quad \mathcal{A}_{22}^{\pm}=\mathcal{A}_{22}=\frac{2\mathbf{C}_{22}}{Q_2},\quad \mathcal{A}_{23}^{\pm}=\mp\frac{4A_{23}\mathbf{C}_{22}}{Q_2},\quad \mathcal{A}_{24}^{\pm}=\mathcal{A}_{25}^{\pm}=\mathcal{A}_{26}^{\pm}=0,\\
&\mathcal{A}_{31}^{\pm}=0,\quad \mathcal{A}_{32}^{\pm}=\mp\frac{4A_{32}\mathbf{C}_{33}}{Q_2},\quad \mathcal{A}_{33}^{\pm}=\mathcal{A}_{33}=\frac{2\mathbf{C}_{33}}{Q_2},\quad \mathcal{A}_{34}^{\pm}=\mathcal{A}_{35}^{\pm}=\mathcal{A}_{36}^{\pm}=0,\\
&\mathcal{A}_{41}^{\pm}=\mp\left(\frac{4A_{41}\mathbf{C}_{44}}{Q_1}+\frac{4A_{51}\mathbf{C}_{45}}{Q_1}\right),\quad \mathcal{A}_{42}^{\pm}=\mathcal{A}_{43}^{\pm}=\mathcal{A}_{46}^{\pm}=0,\\
&\mathcal{A}_{44}^{\pm}=\mathcal{A}_{44}=\frac{(2-8A_{15}A_{51})\mathbf{C}_{44}}{Q_1}+\frac{8A_{14}A_{51}\mathbf{C}_{45}}{Q_1},\\
&\mathcal{A}_{45}^{\pm}=\mathcal{A}_{45}=-\frac{8A_{15}A_{41}\mathbf{C}_{44}}{Q_1}-\frac{(2-8A_{14}A_{41})\mathbf{C}_{45}}{Q_1},\\
&\mathcal{A}_{51}^{\pm}=\pm\left(\frac{4A_{41}\mathbf{C}_{45}}{Q_1}+\frac{4A_{51}\mathbf{C}_{55}}{Q_1}\right),\quad \mathcal{A}_{52}^{\pm}=\mathcal{A}_{53}^{\pm}=\mathcal{A}_{56}^{\pm}=0,\\
&\mathcal{A}_{54}^{\pm}=\mathcal{A}_{54}=-\frac{(2-8A_{15}A_{51})\mathbf{C}_{45}}{Q_1}-\frac{8A_{14}A_{51}\mathbf{C}_{55}}{Q_1},\\
&\mathcal{A}_{55}^{\pm}=\mathcal{A}_{55}=\frac{8A_{15}A_{41}\mathbf{C}_{45}}{Q_1}+\frac{(2-8A_{14}A_{41})\mathbf{C}_{55}}{Q_1},\\
&\mathcal{A}_{61}^{\pm}=\mathcal{A}_{62}^{\pm}=\mathcal{A}_{63}^{\pm}=\mathcal{A}_{64}^{\pm}=\mathcal{A}_{65}^{\pm}=0,\quad \mathcal{A}_{66}^{\pm}=\mathcal{A}_{66}=-2\mathbf{C}_{66}.
\end{aligned}$$

Introduce the notation

$$Q_1:=-1+4A_{14}A_{41}+4A_{15}A_{51}<0,\quad Q_2:=-1+4A_{23}A_{32}.$$

Lemma 4.2. *The following inequality* $Q_2=-1+4A_{23}A_{32}<0$ *holds.*

Proof. Consider two cases.

Case 1: $D>0$. Then inequality $4A_{23}A_{32}<1$ can be equivalently reduced to the inequality

$$c_{44}^2(d_2-d_1)^2(c_{11}-c_{13}d_1d_2)(c_{33}d_1d_2-c_{13})<d_1^2d_2^2D.$$

By replacing here $d_1 d_2$ by its expression from (4.11), we get

$$c_{44}^2(d_2-d_1)^2\big(\sqrt{c_{11}c_{33}}-c_{13}\big)^2<\frac{\sqrt{c_{11}}}{\sqrt{c_{33}}}D.$$

Now, replace $(d_2-d_1)^2$ and D by their expressions from (4.11) and (4.9), respectively, to obtain

$$c_{44}^2\frac{(C-2c_{44}\sqrt{c_{11}c_{33}})}{c_{44}c_{33}}(\sqrt{c_{11}c_{33}}-c_{13})^2<\frac{\sqrt{c_{11}}}{\sqrt{c_{33}}}(C^2-4c_{44}^2c_{33}c_{11}).$$

From the above inequality we deduce

$$c_{44}(C-2c_{44}\sqrt{c_{11}c_{33}})(\sqrt{c_{11}c_{33}}-c_{13})^2<\sqrt{c_{11}c_{33}}(C+2c_{44}\sqrt{c_{11}c_{33}})(C-2c_{44}\sqrt{c_{11}c_{33}}).$$

Substituting here the expression of C from (4.9) to obtain

$$c_{44}(\sqrt{c_{11}c_{33}}-c_{13})^2<\sqrt{c_{11}c_{33}}(c_{11}c_{33}-c_{13}^2-2c_{13}c_{44}+2c_{44}\sqrt{c_{11}c_{33}}),$$

i.e.,

$$c_{44}(\sqrt{c_{11}c_{33}}-c_{13})^2<\sqrt{c_{11}c_{33}}\big[(\sqrt{c_{11}c_{33}}+c_{13})(\sqrt{c_{11}c_{33}}-c_{13})+2c_{44}(\sqrt{c_{11}c_{33}}-c_{13})\big],$$

we arrive at the inequality

$$c_{44}(\sqrt{c_{11}c_{33}}-c_{13})<\sqrt{c_{11}c_{33}}(\sqrt{c_{11}c_{33}}+c_{13}+2c_{44}). \tag{4.14}$$

But (4.14) holds, since

$$c_{44}(\sqrt{c_{11}c_{33}}-c_{13})<2c_{44}\sqrt{c_{11}c_{33}}<\sqrt{c_{11}c_{33}}(\sqrt{c_{11}c_{33}}+c_{13}+2c_{44}),$$

due to the inequality $\sqrt{c_{11}c_{33}}>|c_{13}|$ (see (4.8)).

So, we finally obtain

$$Q_2=-1+A_{23}A_{32}<0.$$

Case 2: $D<0$. In this case, due to (4.12), we have

$$4A_{23}A_{32}=\frac{4a^2c_{44}^2(\sqrt{c_{11}c_{33}}-c_{13})^2}{-D}\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}}<1.$$

Therefore

$$4a^2c_{44}^2(\sqrt{c_{11}c_{33}}-c_{13})^2\sqrt{c_{33}}<-D\sqrt{c_{11}}.$$

Inserting here a and D from (4.13) and (4.9), respectively, we rewrite the above inequality as

$$\left(\frac{-C+2c_{44}\sqrt{c_{11}c_{33}}}{\sqrt{c_{33}}}\right)c_{44}(\sqrt{c_{11}c_{33}}-c_{13})^2<\big(-C^2+4c_{44}^2c_{33}c_{11}\big)\sqrt{c_{11}}.$$

Replacing here C with it's expression from (4.9), we get

$$c_{44}(\sqrt{c_{11}c_{33}}-c_{13})^2<\big(2c_{44}\sqrt{c_{11}c_{33}}+c_{11}c_{33}-c_{13}^2-2c_{13}c_{44}\big)\sqrt{c_{11}c_{33}},$$

implying

$$c_{44}(\sqrt{c_{11}c_{33}}-c_{13})^2<\Big[2c_{44}(\sqrt{c_{11}c_{33}}-c_{13})+(\sqrt{c_{11}c_{33}}+c_{13})(\sqrt{c_{11}c_{33}}-c_{13})\Big]\sqrt{c_{11}c_{33}}.$$

Dividing the inequality by $\sqrt{c_{11}c_{33}}-c_{13}$, we obtain

$$c_{44}(\sqrt{c_{11}c_{33}}-c_{13})<\sqrt{c_{11}c_{33}}\,(2c_{44}+\sqrt{c_{11}c_{33}}+c_{13}). \tag{4.15}$$

Thus, the inequality $Q_2<0$ is equivalently reduced to the relation (4.15), which coincides with (4.14) and which is true as is shown above. This completes the proof. □

Introduce the notation

$$\sigma_2^{\pm}=\sigma_2^{\pm}(\mathcal{A}_\tau^{(1)}+\mathcal{B}_\tau^{(2)}):=\mathfrak{S}\big(\mathcal{A}_\tau^{(1)}+\mathcal{B}_\tau^{(2)};x',0,\pm1\big),\ x'\in\ell.$$

The characteristic polynomial of the matrix $(\sigma_2^+)^{-1}\sigma_2^-$ can be represented as follows:

$$\det\big(\sigma_2^- -\lambda\sigma_2^+\big)=\det\Big[\sigma_2^-\big(\mathcal{A}_\tau^{(1)}+\mathcal{B}_\tau^{(2)}\big)-\lambda\sigma_2^+\big(\mathcal{A}_\tau^{(1)}+\mathcal{B}_\tau^{(2)}\big)\Big]$$

$$= \det\left\{\Big[\sigma(\mathcal{H}_\tau^{(1)})\sigma\big(-\frac{1}{2}I_6-\mathcal{K}_\tau^{(2)}\big)^{-1}+\sigma^-(\mathcal{B}_\tau^{(2)})\Big]-\lambda\Big[\sigma(\mathcal{H}_\tau^{(1)})\sigma\big(-\frac{1}{2}I_6+\mathcal{K}_\tau^{(1)}\big)^{-1}+\sigma^+(\mathcal{B}_\tau^{(2)})\Big]\right\}$$

$$=\det\begin{bmatrix}(1-\lambda)\widetilde{\mathcal{A}}_{11} & 0 & 0 & -(1+\lambda)\widetilde{\mathcal{A}}_{14}^+ & -(1+\lambda)\widetilde{\mathcal{A}}_{15}^+ & 0\\ 0 & (1-\lambda)\widetilde{\mathcal{A}}_{22} & -(1+\lambda)\widetilde{\mathcal{A}}_{23}^+ & 0 & 0 & 0\\ 0 & -(1+\lambda)\widetilde{\mathcal{A}}_{32}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{33} & 0 & 0 & 0\\ -(1+\lambda)\widetilde{\mathcal{A}}_{41}^+ & 0 & 0 & (1-\lambda)\widetilde{\mathcal{A}}_{44} & (1-\lambda)\widetilde{\mathcal{A}}_{45}^+ & 0\\ -(1+\lambda)\widetilde{\mathcal{A}}_{51}^+ & 0 & 0 & (1-\lambda)\widetilde{\mathcal{A}}_{54}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & (1-\lambda)\widetilde{\mathcal{A}}_{66}\end{bmatrix}_{6\times 6}, \tag{4.16}$$

where

$$\begin{array}{llll}\widetilde{\mathcal{A}}_{11}=\mathcal{A}_{11}+\frac{1}{\mu}, & \widetilde{\mathcal{A}}_{14}^+=\mathcal{A}_{14}^+, & \widetilde{\mathcal{A}}_{15}^+=\mathcal{A}_{15}^+ & \widetilde{\mathcal{A}}_{22}=\mathcal{A}_{22}+a,\\ \widetilde{\mathcal{A}}_{23}^+=\mathcal{A}_{23}^++ib, & \widetilde{\mathcal{A}}_{32}^+=\mathcal{A}_{32}^+-ib, & \widetilde{\mathcal{A}}_{33}=\mathcal{A}_{33}+a, & \widetilde{\mathcal{A}}_{41}^+=\mathcal{A}_{41}^+,\\ \widetilde{\mathcal{A}}_{44}=\mathcal{A}_{44}, & \widetilde{\mathcal{A}}_{45}^+=\mathcal{A}_{45}, & \widetilde{\mathcal{A}}_{51}^+=\mathcal{A}_{51}^+, & \widetilde{\mathcal{A}}_{54}^+=\mathcal{A}_{54},\\ & \widetilde{\mathcal{A}}_{55}=\mathcal{A}_{55}, & \widetilde{\mathcal{A}}_{66}=\mathcal{A}_{66}+1. & \end{array}$$

From (4.16), one can easily deduce

$$\det\left(\sigma_2^- -\lambda\sigma_2^+\right)=\det\begin{bmatrix}(1-\lambda)\widetilde{\mathcal{A}}_{22} & -(1+\lambda)\widetilde{\mathcal{A}}_{23}^+,\\ -(1+\lambda)\widetilde{\mathcal{A}}_{32}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{33}\end{bmatrix}$$

$$\times\det\begin{bmatrix}(1-\lambda)\widetilde{\mathcal{A}}_{11} & -(1+\lambda)\mathcal{A}_{14}^+ & -(1+\lambda)\mathcal{A}_{15}^+\\ -(1+\lambda)\mathcal{A}_{41}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{44} & (1-\lambda)\mathcal{A}_{45}\\ -(1+\lambda)\mathcal{A}_{51}^+ & (1-\lambda)\mathcal{A}_{54} & (1-\lambda)\widetilde{\mathcal{A}}_{55}\end{bmatrix}(1-\lambda)\widetilde{\mathcal{A}}_{66}=0.$$

Therefore, one of the eigenvalues, say λ_6, is equal to 1 and other eigenvalues are defined by the following equations:

$$\det\begin{bmatrix}(1-\lambda)\widetilde{\mathcal{A}}_{22} & -(1+\lambda)\widetilde{\mathcal{A}}_{23}^+\\ -(1+\lambda)\widetilde{\mathcal{A}}_{32}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{33}\end{bmatrix}=(1-\lambda)^2\widetilde{\mathcal{A}}_{22}\widetilde{\mathcal{A}}_{33}-(1+\lambda)^2\widetilde{\mathcal{A}}_{23}^+\widetilde{\mathcal{A}}_{32}^+=0, \tag{4.17}$$

$$\det\begin{bmatrix}(1-\lambda)\widetilde{\mathcal{A}}_{11} & -(1+\lambda)\mathcal{A}_{14}^+ & -(1+\lambda)\mathcal{A}_{15}^+\\ -(1+\lambda)\mathcal{A}_{41}^+ & (1-\lambda)\widetilde{\mathcal{A}}_{44} & (1-\lambda)\mathcal{A}_{45}\\ -(1+\lambda)\mathcal{A}_{51}^+ & (1-\lambda)\mathcal{A}_{54} & (1-\lambda)\widetilde{\mathcal{A}}_{55}\end{bmatrix}=0. \tag{4.18}$$

Equation (4.17) can be rewritten as

$$\left(\frac{1-\lambda}{1+\lambda}\right)^2=\frac{\widetilde{\mathcal{A}}_{23}^+\widetilde{\mathcal{A}}_{32}^+}{\widetilde{\mathcal{A}}_{22}\widetilde{\mathcal{A}}_{33}}. \tag{4.19}$$

Lemma 4.3. *The expression* $q:=\dfrac{\widetilde{\mathcal{A}}_{23}^+\widetilde{\mathcal{A}}_{32}^+}{\widetilde{\mathcal{A}}_{22}\widetilde{\mathcal{A}}_{33}}$ *is positive.*

Proof. We have

$$\widetilde{\mathcal{A}}_{23}^+=\mathcal{A}_{23}^++ib,\quad \widetilde{\mathcal{A}}_{32}^+=\mathcal{A}_{32}^+-ib,\quad \widetilde{\mathcal{A}}_{22}=\mathcal{A}_{22}+a,\quad \widetilde{\mathcal{A}}_{33}=\mathcal{A}_{33}+a,$$

where

$$\mathcal{A}_{23}^+=-\frac{4A_{23}\mathbf{C}_{22}}{Q_2},\quad \mathcal{A}_{32}^+=-\frac{4A_{32}\mathbf{C}_{33}}{Q_2},\quad \mathcal{A}_{22}=\frac{2\mathbf{C}_{22}}{Q_2},\quad \mathcal{A}_{33}=\frac{2\mathbf{C}_{33}}{Q_2}.$$

Since
$$Q_2=-1+4A_{23}A_{32}<0,\quad \mathbf{C}_{22}<0,\quad \mathbf{C}_{33}<0,\quad a>0,$$
we have
$$\widetilde{\mathcal{A}}_{22}>0,\quad \widetilde{\mathcal{A}}_{33}>0.$$
Further, we show that $\widetilde{\mathcal{A}}^+_{23}\widetilde{\mathcal{A}}^+_{32}>0$. Using the relations
$$\mathbf{C}_{22}=\mathbf{C}_{33}\frac{\sqrt{c_{33}}}{\sqrt{c_{11}}},\quad A_{23}=-A_{32}\frac{\sqrt{c_{11}}}{\sqrt{c_{33}}},$$
we deduce $A_{23}\mathbf{C}_{22}=-\mathbf{C}_{33}A_{32}$ and, consequently, $\mathcal{A}^+_{32}=-\mathcal{A}^+_{23}$. Since $\mathcal{A}^+_{23}$ is pure imaginary, we get
$$\widetilde{\mathcal{A}}^+_{23}\widetilde{\mathcal{A}}^+_{32}=-\big(\mathcal{A}^+_{23}+ib\big)^2>0,$$
which implies $q>0$. □

Now, consider equation (4.18),
$$\det\begin{bmatrix}(1-\lambda)\widetilde{\mathcal{A}}_{11} & -(1+\lambda)\mathcal{A}^+_{14} & -(1+\lambda)\mathcal{A}^+_{15}\\ -(1+\lambda)\mathcal{A}^+_{41} & (1-\lambda)\widetilde{\mathcal{A}}_{44} & (1-\lambda)\mathcal{A}_{45}\\ -(1+\lambda)\mathcal{A}^+_{51} & (1-\lambda)\mathcal{A}_{54} & (1-\lambda)\widetilde{\mathcal{A}}_{55}\end{bmatrix}=(1-\lambda)^3\widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{44}\widetilde{\mathcal{A}}_{55}$$
$$-(1-\lambda)^3\widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{54}\widetilde{\mathcal{A}}_{45}-(1+\lambda)^2(1-\lambda)\widetilde{\mathcal{A}}^+_{14}\widetilde{\mathcal{A}}^+_{41}\widetilde{\mathcal{A}}_{55}+(1+\lambda)^2(1-\lambda)\widetilde{\mathcal{A}}^+_{14}\widetilde{\mathcal{A}}^+_{51}\widetilde{\mathcal{A}}_{45}$$
$$+(1+\lambda)^2(1-\lambda)\widetilde{\mathcal{A}}^+_{15}\widetilde{\mathcal{A}}^+_{41}\widetilde{\mathcal{A}}_{54}-(1+\lambda)^2(1-\lambda)\widetilde{\mathcal{A}}^+_{15}\widetilde{\mathcal{A}}^+_{51}\widetilde{\mathcal{A}}_{44}=0,$$
which can be rewritten as
$$(1-\lambda)\big[(1-\lambda)^2A+(1+\lambda)^2B\big]=0.$$
Consequently, we get $\lambda_5=1$ and two other eigenvalues are defined by the equation
$$\Big(\frac{1-\lambda}{1+\lambda}\Big)^2=-\frac{B}{A}=:-p, \tag{4.20}$$
where
$$A=\widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{44}\widetilde{\mathcal{A}}_{55}-\widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{54}\widetilde{\mathcal{A}}_{45},$$
$$B=-\widetilde{\mathcal{A}}^+_{14}\widetilde{\mathcal{A}}^+_{41}\widetilde{\mathcal{A}}_{55}+\widetilde{\mathcal{A}}^+_{14}\widetilde{\mathcal{A}}^+_{51}\widetilde{\mathcal{A}}_{45}+\widetilde{\mathcal{A}}^+_{15}\widetilde{\mathcal{A}}^+_{41}\widetilde{\mathcal{A}}_{54}-\widetilde{\mathcal{A}}^+_{15}\widetilde{\mathcal{A}}^+_{51}\widetilde{\mathcal{A}}_{44}.$$

Lemma 4.4. *The inequality* $p=\frac{B}{A}>0$ *holds.*

Proof. We have
$$\widetilde{\mathcal{A}}_{44}\widetilde{\mathcal{A}}_{55}-\widetilde{\mathcal{A}}_{54}\widetilde{\mathcal{A}}_{45}=\bigg[\frac{(2-8A_{15}A_{51})\mathbf{C}_{44}}{Q_1}+\frac{8A_{14}A_{51}\mathbf{C}_{45}}{Q_1}\bigg]\bigg[\frac{8A_{15}A_{41}\mathbf{C}_{45}}{Q_1}+\frac{(2-8A_{14}A_{41})\mathbf{C}_{55}}{Q_1}\bigg]$$
$$-\bigg[\frac{(2-8A_{15}A_{51})\mathbf{C}_{45}}{Q_1}+\frac{8A_{14}A_{51}\mathbf{C}_{55}}{Q_1}\bigg]\bigg[\frac{8A_{15}A_{41}\mathbf{C}_{44}}{Q_1}+\frac{(2-8A_{14}A_{41})\mathbf{C}_{45}}{Q_1}\bigg]$$
$$=\frac{(2-8A_{15}A_{51})\mathbf{C}_{44}}{Q_1}\cdot\frac{(2-8A_{14}A_{41})\mathbf{C}_{55}}{Q_1}+\frac{64A_{14}A_{51}A_{15}A_{41}\mathbf{C}^2_{45}}{Q_1^2}$$
$$-\frac{(2-8A_{15}A_{51})\mathbf{C}_{45}}{Q_1}\cdot\frac{(2-8A_{14}A_{41})\mathbf{C}_{45}}{Q_1}-\frac{64A_{14}A_{51}A_{15}A_{41}\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}$$
$$=M(\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}^2_{45})+N(\mathbf{C}^2_{45}-\mathbf{C}_{44}\mathbf{C}_{55})=(\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}^2_{45})(M-N),$$
where
$$M:=\frac{(2-8A_{15}A_{51})(2-8A_{14}A_{41})}{Q_1^2},\quad N:=\frac{64A_{14}A_{51}A_{15}A_{41}}{Q_1^2}.$$
Note that $M-N>0$, since $A_{14}A_{41}<0$ and $A_{15}A_{51}<0$. Indeed, we have
$$M-N=\frac{(2-8A_{15}A_{51})(2-8A_{14}A_{41})}{Q_1^2}-\frac{64A_{14}A_{51}A_{15}A_{41}}{Q_1^2}=\frac{4}{Q_1^2}\big[1-4A_{14}A_{41}-4A_{15}A_{51}\big]>0.$$

Now we show that $\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2>0$. Rewrite $\mathbf{C}_{44},\ \mathbf{C}_{55}$ and $\mathbf{C}_{45}$ in the form

$$\mathbf{C}_{44}=-(m+q_{15}^2 n),\quad \mathbf{C}_{55}=-(m+e_{14}^2 n),\quad \mathbf{C}_{45}=e_{14}q_{15}n,$$

where

$$m=\frac{(b_2-b_1)}{2\sqrt{B}}\left(c_{44}+\frac{c_{66}}{b_1b_2}\right)>0,\quad n=\frac{1}{2\alpha\varkappa_{11}\widetilde{e}_{14}^2}\left[\sqrt{\frac{\varkappa_{11}}{\varkappa_{33}}}-\frac{c_{44}(b_2-b_1)(\varkappa_{33}\,b_1b_2+\varkappa_{11})}{\sqrt{B}}\right]>0$$

(see [8], Appendix B) and

$$\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2=m^2+(e_{14}^2+q_{15}^2)mn>0.$$

Consequently,

$$\widetilde{\mathcal{A}}_{44}\widetilde{\mathcal{A}}_{55}-\widetilde{\mathcal{A}}_{54}\widetilde{\mathcal{A}}_{45}>0$$

and, since $\widetilde{\mathcal{A}}_{11}>0$, we have

$$A=\widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{44}\widetilde{\mathcal{A}}_{55}-\widetilde{\mathcal{A}}_{11}\widetilde{\mathcal{A}}_{54}\widetilde{\mathcal{A}}_{45}>0.$$

Now, we show that

$$B=-\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{55}+\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{45}+\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{54}-\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{44}>0.$$

First, we prove the inequality $-\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{55}+\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{45}>0$. Indeed,

$$\begin{aligned}
&-\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{55}+\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{45}=\widetilde{\mathcal{A}}_{14}^+(-\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{55}+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{45})\\
&=-\frac{4A_{14}\mathbf{C}_{11}}{Q_1}\Big[\Big(\frac{4A_{41}\mathbf{C}_{44}}{Q_1}+\frac{4A_{51}\mathbf{C}_{45}}{Q_1}\Big)\Big(\frac{8A_{15}A_{41}\mathbf{C}_{45}}{Q_1}+\frac{(2-8A_{14}A_{41})\mathbf{C}_{55}}{Q_1}\Big)\\
&-\Big(\frac{4A_{41}\mathbf{C}_{45}}{Q_1}+\frac{4A_{51}\mathbf{C}_{55}}{Q_1}\Big)\Big(\frac{8A_{15}A_{41}\mathbf{C}_{44}}{Q_1}+\frac{(2-8A_{14}A_{41})\mathbf{C}_{45}}{Q_1}\Big)\Big]\\
&=-\frac{4A_{14}\mathbf{C}_{11}}{Q_1}\Big[\frac{4A_{41}(2-8A_{14}A_{41})\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}+\frac{32A_{51}A_{15}A_{41}\mathbf{C}_{45}^2}{Q_1^2}\\
&-\frac{4A_{41}(2-8A_{14}A_{41})\mathbf{C}_{45}^2}{Q_1^2}-\frac{32A_{51}A_{15}A_{41}\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}\Big]\\
&=-\frac{32A_{14}A_{41}\mathbf{C}_{11}}{Q_1}\Big[\frac{(1-4A_{14}A_{41})}{Q_1^2}(\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2)+\frac{4A_{51}A_{15}}{Q_1^2}(\mathbf{C}_{45}^2-\mathbf{C}_{44}\mathbf{C}_{55})\Big]\\
&=-\frac{32A_{14}A_{41}\mathbf{C}_{11}}{Q_1}\Big[\frac{1-4A_{14}A_{41}-4A_{51}A_{15}}{Q_1^2}\Big](\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2)=\frac{32A_{14}A_{41}C_{11}}{Q_1^2}(C_{44}C_{55}-C_{45}^2).
\end{aligned}$$

Therefore, taking into account the inequalities $A_{14}A_{41}<0,\quad \mathbf{C}_{11}<0,\quad \mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2>0$, we conclude that

$$-\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{55}+\widetilde{\mathcal{A}}_{14}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{45}>0.$$

Further, we prove that

$$\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{54}-\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{44}>0.$$

Conducting algebraic transformations as in the previous case, we get

$$\begin{aligned}
&\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{54}-\widetilde{\mathcal{A}}_{15}^+\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{44}=\widetilde{\mathcal{A}}_{15}^+(-\widetilde{\mathcal{A}}_{51}^+\widetilde{\mathcal{A}}_{44}+\widetilde{\mathcal{A}}_{41}^+\widetilde{\mathcal{A}}_{54})\\
&=\frac{4A_{15}\mathbf{C}_{11}}{Q_1}\Big[-\Big(\frac{4A_{41}\mathbf{C}_{45}}{Q_1}+\frac{4A_{51}\mathbf{C}_{55}}{Q_1}\Big)\Big(\frac{(2-8A_{15}A_{51})\mathbf{C}_{44}}{Q_1}+\frac{8A_{14}A_{51}\mathbf{C}_{45}}{Q_1}\Big)\\
&+\Big(\frac{4A_{41}\mathbf{C}_{44}}{Q_1}+\frac{4A_{51}\mathbf{C}_{45}}{Q_1}\Big)\Big(\frac{(2-8A_{15}A_{51})\mathbf{C}_{45}}{Q_1}+\frac{8A_{14}A_{51}\mathbf{C}_{55}}{Q_1}\Big)\Big]\\
&=\frac{4A_{15}\mathbf{C}_{11}}{Q_1}\Big[-\frac{32A_{41}A_{14}A_{51}\mathbf{C}_{45}^2}{Q_1^2}-\frac{4A_{51}(2-8A_{15}A_{51})\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}\\
&+\frac{32A_{41}A_{14}A_{51}\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}+\frac{4A_{51}(2-8A_{15}A_{51})\mathbf{C}_{45}^2}{Q_1^2}\Big]\\
&=\frac{4A_{15}A_{51}\mathbf{C}_{11}}{Q_1}\Big[-\frac{32A_{14}A_{41}\mathbf{C}_{45}^2}{Q_1^2}-\frac{4(2-8A_{15}A_{51})\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}
\end{aligned}$$

$$+\frac{32A_{41}A_{14}\mathbf{C}_{44}\mathbf{C}_{55}}{Q_1^2}+\frac{4(2-8A_{15}A_{51})\mathbf{C}_{45}^2}{Q_1^2}\Big]$$
$$=\frac{4A_{15}A_{51}\mathbf{C}_{11}}{Q_1}\Big[\frac{32A_{14}A_{41}}{Q_1^2}(-\mathbf{C}_{45}^2+\mathbf{C}_{44}\mathbf{C}_{55})-\frac{4(2-8A_{15}A_{51})}{Q_1^2}(-\mathbf{C}_{45}^2+\mathbf{C}_{44}\mathbf{C}_{55})\Big]$$
$$=\frac{32A_{15}A_{51}\mathbf{C}_{11}}{Q_1}\Big[\frac{4A_{14}A_{41}}{Q_1^2}-\frac{(1-4A_{15}A_{51})}{Q_1^2}\Big](-\mathbf{C}_{45}^2+\mathbf{C}_{44}\mathbf{C}_{55})=\frac{32A_{15}A_{51}\mathbf{C}_{11}}{Q_1^2}(-\mathbf{C}_{45}^2+\mathbf{C}_{44}\mathbf{C}_{55}).$$

Taking into account the inequalities $A_{51}A_{15}<0$, $\mathbf{C}_{11}<0$ and $\mathbf{C}_{44}\mathbf{C}_{55}-\mathbf{C}_{45}^2>0$, we obtain

$$\widetilde{\mathcal{A}}_{15}^{+}\widetilde{\mathcal{A}}_{41}^{+}\widetilde{\mathcal{A}}_{54}-\widetilde{\mathcal{A}}_{15}^{+}\widetilde{\mathcal{A}}_{51}^{+}\widetilde{\mathcal{A}}_{44}>0.$$

Thus, $B>0$ and, consequently, $p=\dfrac{A}{B}>0$. □

Due to (4.19) and (4.20), we have the following expressions for the eigenvalues of the matrix $(\sigma_2^+)^{-1}\sigma_2^-$ (i.e., the roots of polynomial (4.16) with respect to λ),

$$\lambda_1=\frac{1-i\sqrt{p}}{1+i\sqrt{p}},\quad \lambda_2=\lambda_1^{-1}=\overline{\lambda_1},\quad \lambda_3=\frac{1-\sqrt{q}}{1+\sqrt{q}},\quad \lambda_4=\lambda_3^{-1},\quad \lambda_5=\lambda_6=1.$$

Note that $|\lambda_1|=|\lambda_2|=1$. Moreover, since λ_3 and λ_4 are real, they are positive (see Appendix, Subsection 5.2).

Applying the above results, we can explicitly write the exponents of the first terms of the asymptotic expansions of the solutions (see (4.3)):

$$\begin{aligned}
\gamma_1&=\frac12+\frac{1}{2\pi}\arg\lambda_1=\frac12+\frac{1}{2\pi}\arg\frac{1-i\sqrt{p}}{1+i\sqrt{p}}\\
&=\frac12+\frac{1}{2\pi}\Big(\arg(1-i\sqrt{p})-\arg(1+i\sqrt{p})\Big)=\frac12-\frac1\pi\arctan\sqrt{p},\\
\gamma_1&=\frac12-\frac1\pi\arctan\sqrt{p}, && \delta_1=0,\\
\gamma_2&=\frac12+\frac1\pi\arctan\sqrt{p}, && \delta_2=0,\\
\gamma_3&=\gamma_4=\frac12, && \delta_3=-\delta_4=\widetilde{\delta}=-\frac{1}{2\pi}\log\frac{1-\sqrt{q}}{1+\sqrt{q}},\\
\gamma_5&=\gamma_6=\frac12, && \delta_5=\delta_6=0.
\end{aligned}$$

It is evident that $0<\gamma_1<\frac12$ and $\frac12<\gamma_2<1$.

Note that in this case $B_0(t)$ has the following form (see (3.52)):

$$B_0(t)=\begin{bmatrix} I_4 & [0]_{4\times2}\\ [0]_{2\times4} & B^{(2)}(t)\end{bmatrix},\qquad \text{where}\quad B^{(2)}(t)=\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}.$$

Now, we can draw the following conclusions:

(1) In view of Theorem 4.1, the solutions of the problem possess the following asymptotic behaviour near the edge curve $\ell=\partial\Gamma_T$:

$$\begin{aligned}
(u^{(1)},\varphi^{(1)},\psi^{(1)})^\top &= c_0^{(1)}r^{\gamma_1}+c_1^{(1)}r^{\frac12}\ln r+c_2^{(1)}r^{\frac12+i\widetilde{\delta}}+c_3^{(1)}r^{\frac12-i\widetilde{\delta}}+c_4^{(1)}r^{\frac12}+c_5^{(1)}r^{\gamma_2}+\cdots,\\
\vartheta^{(1)} &= b_0^{(1)}r^{\frac12}+b_1^{(1)}r^{\gamma_2}+\cdots,\\
u^{(2)} &= c_0^{(2)}r^{\gamma_1}+c_1^{(2)}r^{\frac12}\ln r+c_2^{(2)}r^{\frac12+i\widetilde{\delta}}+c_3^{(2)}r^{\frac12-i\widetilde{\delta}}+c_4^{(2)}r^{\frac12}+c_5^{(2)}r^{\gamma_2}+\cdots,\\
\vartheta^{(2)} &= b_0^{(2)}r^{\frac12}+b_1^{(2)}r^{\gamma_2}+\cdots,
\end{aligned}$$

where coefficients $c_j^{(1)}$, $j=0,...5$, are the 5-dimensional vectors, $c_j^{(2)}$, $j=0,...5$, are the 3-dimensional vectors and $b_j^{(k)}$, $j=0,1$, $k=1,2$, are scalars.

As we can see, the exponent γ_1 characterizing the behaviour of $u^{(1)}$, $\varphi^{(1)}$, $\psi^{(1)}$ and $u^{(2)}$ near the line ℓ depends on the elastic, piezoelectric, piezomagnetic, dielectric and permeability constants, and does not depend on the thermal constants. Moreover, γ_1 takes values from the interval $(0,\frac{1}{2})$.

For the general anisotropic case, these exponents also depend on the geometry of the line ℓ, in general.

(2) In general, we have the following smoothness of mechanical and electromagnetic fields:

$$(u^{(1)},\varphi^{(1)},\psi^{(1)})\in\big[C^{\gamma_1}(\overline{\Omega}_1)\big]^5,\quad u^{(2)}\in\big[C^{\gamma_1}(\overline{\Omega}_2)\big]^3,\ \ 0<\gamma_1<\frac{1}{2}.$$

(3) Since $\gamma_1<\frac{1}{2}$, we have no oscillating stress singularities for physical fields in the neighbourhood of the curve ℓ.

Note that in the classical elasticity theory (for both isotropic and anisotropic solids) for mixed boundary value and mixed transmission problems the dominant exponents are $\frac{1}{2}$, $\frac{1}{2}\pm i\,\widetilde{\delta}$ with $\widetilde{\delta}\neq 0$ and, consequently, there occur oscillating stress singularities at the line ℓ (for details see [12,13]).

5. Appendix

5.1. **Properties of Potentials and Boundary Operators.** Here we collect some theorems describing the mapping properties of potentials and the corresponding boundary integral (pseudodifferential) operators. The proof of these theorems can be found in references [7,8,20].

Theorem 5.1. *Let $1<p<\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$. Then the single layer potentials can be extended to the following continuous operators:*

$$V_\tau^{(2)}\ :\big[B^s_{p,q}(S)\big]^4\to\big[B^{s+1+\frac{1}{p}}_{p,q}(\Omega^{(2)})\big]^4,\qquad V_\tau^{(1)}\ :\big[B^s_{p,p}(S)\big]^6\to\big[H^{s+1+\frac{1}{p}}_{p}(\Omega^{(1)})\big]^6,$$
$$V_\tau^{(2)}\ :\big[H^s_{p}(S)\big]^4\to\big[H^{s+1+\frac{1}{p}}_{p}(\Omega^{(2)})\big]^4,\qquad V_\tau^{(1)}\ :\big[H^s_{p}(S)\big]^6\to\big[H^{s+1+\frac{1}{p}}_{p}(\Omega^{(1)})\big]^6.$$

Theorem 5.2. *Let $1<p<\infty$, $1\leqslant q\leqslant\infty$, $h^{(2)}\in\big[B^{-\frac{1}{p}}_{p,q}(\partial\Omega^{(2)})\big]^4$, $h^{(1)}\in\big[B^{-\frac{1}{p}}_{p,q}(\partial\Omega^{(1)})\big]^6$. Then*

$$\big\{V_\tau^{(2)}(h^{(2)})\big\}^+=\big\{V_\tau^{(2)}(h^{(2)})\big\}^-=\mathcal{H}_\tau^{(2)}\,(h^{(2)})\ \ on\ \ \partial\Omega^{(2)},$$
$$\big\{\mathcal{T}^{(2)}(\partial,\nu,\tau)V_\tau^{(2)}(h^{(2)})\big\}^\pm=\big[\,\mp\,2^{-1}I_4+\mathcal{K}_\tau^{(2)}\,\big]\,(h^{(2)})\ \ on\ \ \partial\Omega^{(2)},$$
$$\big\{V_\tau^{(1)}(h^{(1)})\big\}^+=\big\{V_\tau^{(1)}(h^{(1)})\big\}^-=\mathcal{H}_\tau^{(1)}\,(h^{(1)})\ \ on\ \ \partial\Omega^{(1)},$$
$$\big\{\mathcal{T}^{(1)}(\partial,n,\tau)V_\tau^{(1)}(h^{(1)})\big\}^\pm=\big[\,\mp\,2^{-1}I_6+\mathcal{K}_\tau^{(1)}\,\big]\,(h^{(1)})\ \ on\ \ \partial\Omega^{(1)},$$

where I_k stands for the $k\times k$ unit matrix.

The operators $\mathcal{H}_\tau^{(1)}$, $\mathcal{H}_\tau^{(2)}$, $\mathcal{K}_\tau^{(1)}$ and $\mathcal{K}_\tau^{(2)}$ possess the mapping and the Fredholm properties [7].

Theorem 5.3. *Let $1<p<\infty$, $1\leqslant q\leqslant\infty$, $s\in\mathbb{R}$. The operators*

$$\mathcal{H}_\tau^{(2)}\ :\ \big[H^s_p(\partial\Omega^{(2)})\big]^4\to\big[H^{s+1}_p(\partial\Omega^{(2)})\big]^4,\qquad \mathcal{H}_\tau^{(1)}\ :\ \big[H^s_p(\partial\Omega^{(1)})\big]^6\to\big[H^{s+1}_p(\partial\Omega^{(1)})\big]^6,$$
$$\mathcal{H}_\tau^{(2)}\ :\ \big[B^s_{p,q}(\partial\Omega^{(2)})\big]^4\to\big[B^{s+1}_{p,q}(\partial\Omega^{(2)})\big]^4,\qquad \mathcal{H}_\tau^{(1)}\ :\ \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6\to\big[B^{s+1}_{p,q}(\partial\Omega^{(1)})\big]^6,$$
$$\mathcal{K}_\tau^{(2)}\ :\ \big[H^s_p(\partial\Omega^{(2)})\big]^4\to\big[H^{s}_p(\partial\Omega^{(2)})\big]^4,\qquad \mathcal{K}_\tau^{(1)}\ :\ \big[H^s_p(\partial\Omega^{(1)})\big]^6\to\big[H^{s}_p(\partial\Omega^{(1)})\big]^6,$$
$$\mathcal{K}_\tau^{(2)}\ :\ \big[B^s_{p,q}(\partial\Omega^{(2)})\big]^4\to\big[B^{s}_{p,q}(\partial\Omega^{(2)})\big]^4,\qquad \mathcal{K}_\tau^{(1)}\ :\ \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6\to\big[B^{s}_{p,q}(\partial\Omega^{(1)})\big]^6,$$

are continuous.

Theorem 5.4. *Let $1<p<\infty$, $1\leqslant q\leqslant\infty$, $s\in\mathbb{R}$ and $\tau=\sigma+i\,\omega$. The operators*

$$\mathcal{H}_\tau^{(2)}:\big[H^s_p(\partial\Omega^{(2)})\big]^4\to\big[H^{s+1}_p(\partial\Omega^{(2)})\big]^4,\qquad \mathcal{H}_\tau^{(1)}\ :\big[H^s_p(\partial\Omega^{(1)})\big]^6\to\big[H^{s+1}_p(\partial\Omega^{(1)})\big]^6,$$

$$\mathcal{H}_\tau^{(2)} : \big[B^s_{p,q}(\partial\Omega^{(2)})\big]^4 \to \big[B^{s+1}_{p,q}(\partial\Omega^{(2)})\big]^4, \qquad \mathcal{H}_\tau^{(1)} : \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6 \to \big[B^{s+1}_{p,q}(\partial\Omega^{(1)})\big]^6,$$

are invertible if $\sigma > 0$ or $\tau = 0$.

The operators

$$\begin{aligned}
\pm 2^{-1} I_4 + \mathcal{K}_\tau^{(2)} &: \big[H^s_p(\partial\Omega^{(2)})\big]^4 \to \big[H^s_p(\partial\Omega^{(2)})\big]^4,\\
\pm 2^{-1} I_4 + \mathcal{K}_\tau^{(2)} &: \big[B^s_{p,q}(\partial\Omega^{(2)})\big]^4 \to \big[B^s_{p,q}(\partial\Omega^{(2)})\big]^4,\\
2^{-1} I_6 + \mathcal{K}_\tau^{(1)} &: \big[H^s_p(\partial\Omega^{(1)})\big]^6 \to \big[H^s_p(\partial\Omega^{(1)})\big]^6,\\
2^{-1} I_6 + \mathcal{K}_\tau^{(1)} &: \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6 \to \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6,
\end{aligned}$$

are invertible if $\sigma > 0$.

The operators

$$\begin{aligned}
-2^{-1} I_6 + \mathcal{K}_\tau^{(1)} &: \big[H^s_p(\partial\Omega^{(1)})\big]^6 \to \big[H^s_p(\partial\Omega^{(1)})\big]^6,\\
-2^{-1} I_6 + \mathcal{K}_\tau^{(1)} &: \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6 \to \big[B^s_{p,q}(\partial\Omega^{(1)})\big]^6
\end{aligned}$$

are Fredholm ones with the index, equal to zero for any $\tau \in \mathbb{C}$.

5.2. **Fredholm properties of pseudodifferential operators on manifolds with boundary.** Let $\mathcal{M}$ be a compact, n-dimensional, smooth, nonselfintersecting manifold with the smooth boundary $\partial\mathcal{M} \neq \varnothing$ and let $\mathbf{A}(x,D)$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(\mathbf{A};x,\xi)$ the principal homogeneous symbol matrix of the operator $\mathbf{A}(x,D)$ in some local coordinate system ($x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \dots, \lambda_N(x)$ be the eigenvalues of the matrix

$$\big[\,\mathfrak{S}(\mathbf{A};x,0,\dots,0,+1)\,\big]^{-1}\big[\,\mathfrak{S}(\mathbf{A};x,0,\dots,0,-1)\,\big], \ \ x \in \partial\mathcal{M},$$

and introduce the notation

$$\delta_j(x) = \operatorname{Re}\big[\,(2\pi i)^{-1}\ln\lambda_j(x)\,\big], \ \ j = 1,\dots,N.$$

Here $\ln\zeta$ denotes the branch of the logarithmic function, analytic in the complex plane cut along $(-\infty,0]$. Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system and the strong inequality $-1/2 < \delta_j(x) < 1/2$ holds for all $x \in \overline{\mathcal{M}}$, $j = \overline{1,N}$, due to the strong ellipticity of $\mathbf{A}$. In a particular case, when $\mathfrak{S}(\mathbf{A};x,\xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\delta_1(x) = \cdots = \delta_N(x) = 0$, since the eigenvalues $\lambda_1(x), \dots, \lambda_N(x)$ are positive for all $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudo-differential operators on manifolds with boundary are characterized by the following theorem (see [2, 4, 18, 30]).

Theorem 5.5. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and let $\mathbf{A}(x,D)$ be a pseudodifferential operator of order $\nu \in \mathbb{R}$ with the strongly elliptic symbol $\mathfrak{S}(\mathbf{A};x,\xi)$, that is, there is a positive constant c_0 such that*

$$\operatorname{Re}\mathfrak{S}(\mathbf{A};x,\xi)\,\eta\cdot\eta \geqslant c_0\,|\eta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\eta \in \mathbb{C}^N$.

Then the operators

$$\begin{aligned}
\mathbf{A} &: \big[\widetilde{H}^s_p(\mathcal{M})\big]^N \to \big[H^{s-\nu}_p(\mathcal{M})\big]^N\\
\mathbf{A} &: \big[\widetilde{B}^s_{p,q}(\mathcal{M})\big]^N \to \big[B^{s-\nu}_{p,q}(\mathcal{M})\big]^N
\end{aligned} \tag{5.1}$$

are Fredholm and have the trivial index $\operatorname{Ind}\mathbf{A} = 0$ *if*

$$\frac{1}{p} - 1 + \sup_{\substack{x\in\partial\mathcal{M},\\ 1\leqslant j\leqslant N}} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{\substack{x\in\partial\mathcal{M},\\ 1\leqslant j\leqslant N}} \delta_j(x). \tag{5.2}$$

Moreover, the null-spaces and indices of the operators (5.1) *coincide for all values of the parameter $q \in [1,+\infty]$ provided p and s satisfy inequality* (5.2).

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