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ON THE ABSOLUTE MATRIX SUMMABILITY FACTORS OF FOURIER SERIES

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Abstract. In this paper, a general theorem on the local property of the $\left|\bar{N}, p_n; \delta\right|_k$ summability of factored Fourier series, which generalizes some known results has been extended to absolute matrix summability factors of Fourier series.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (t_n) of the Riesz means or, simply, the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [4]).

The series $\sum a_n$ is said to be $\left|\bar{N}, p_n; \delta\right|_k$ summable, where $k \geq 1$ and $\delta \geq 0$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \mid t_n - t_{n-1} \mid^k < \infty.$$

In the special case, $p_n = 1$ for all n (resp., $\delta = 0$), the $\left|\bar{N}, p_n; \delta\right|_k$ summability is the same as the $|C, 1; \delta|_k$ $(resp., \left|\bar{N}, p_n\right|_k)$ summability (see [1]).

A sequence (λ_n) is said to be convex if $\Delta^2 \lambda_n \geq 0$ for every positive integer n, where $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$ (see [9]).

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v \quad n = 0, 1, \ldots .$$

The series $\sum a_n$ is said to be $|A, p_n; \delta|_k$ summable, where $k \geq 1$ and $\delta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

If we take $a_{nv} = \frac{p_v}{P_n}$, then the $|A, p_n; \delta|_k$ summability is the same as the $\left|\bar{N}, p_n; \delta\right|_k$ summability. If we take $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$, then the $|A, p_n; \delta|_k$ summability reduces to the $\left|\bar{N}, p_n\right|_k$ summability. Also, if we take $\delta = 0$, then the $|A, p_n; \delta|_k$ summability reduces to the $|A, p_n|_k$ summability (see [6]).

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Given a normal matrix $A=(a_{nv})$, we associate two lower semimatrices $\bar{A}=(\bar{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv}=\sum_{i=v}^{n}a_{ni}, \quad n,v=0,1,\dots \tag{1}$$

and

$$\hat{a}_{00}=\bar{a}_{00}=a_{00}, \quad \hat{a}_{nv}=\bar{a}_{nv}-\bar{a}_{n-1,v}, \quad n=1,2,\dots . \tag{2}$$

It should be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. So, we get

$$A_n(s)=\sum_{v=0}^{n}a_{nv}s_v=\sum_{v=0}^{n}\bar{a}_{nv}a_v$$

and

$$\bar{\Delta}A_n(s)=\sum_{v=0}^{n}\hat{a}_{nv}a_v.$$

2. The Known Results

Let f be a periodic function with period 2π, integrable (L) over $(-\pi,\pi)$. We may assume that the constant term of the Fourier series of f is zero, that is,

$$\int_{-\pi}^{\pi}f(t)dt=0,$$

$$f(t)\sim\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)\equiv\sum_{n=1}^{\infty}C_n(t).$$

In [3], Bor proved the following result dealing with the $\left|\bar{N},p_n;\delta\right|_k$ summability factors of Fourier series.

Theorem 2.1 ([3])**.** *Let $k\geq 1$ and $0\leq\delta<1/k$. If (λ_n) is a convex sequence such that $\sum p_n\lambda_n$ is convergent and*

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}P_v\Delta\lambda_v=O(1) \quad as \quad m\to\infty,$$

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}p_v\lambda_v=O(1) \quad as \quad m\to\infty,$$

$$\sum_{n=v+1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}=O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right),$$

then the $\left|\bar{N},p_n;\delta\right|_k$ summability of the series $\sum C_n(t)\lambda_nP_n$ at a point can be ensured by a local property.

In [7], Sulaiman has obtained a result from which a special case improved the result of [3] in the following form.

Theorem 2.2 ([7]). *Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let (φ_n) be a complex sequence. If $(|\lambda_n|)$ is non-increasing such that $\sum p_n|\lambda_n|$ is convergent and*

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{p_v}{P_v^k}|\lambda_v||\varphi_v|^k = O(1) \quad as \quad m\to\infty, \tag{3}$$

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta}|\Delta\lambda_v||\varphi_v| = O(1) \quad as \quad m\to\infty, \tag{4}$$

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{p_{v+1}^{k-1}}|\lambda_{v+1}||\Delta\varphi_v|^k = O(1) \quad as \quad m\to\infty, \tag{5}$$

$$\sum_{n=v+1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}} = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right),$$

then the $\left|\bar{N},p_n;\delta\right|_k$ summability of the series $\sum C_n(t)\lambda_n P_n$ at a point can be ensured by a local property.

3. Main Result

The aim of this paper is to generalize Sulaiman's result in [7] for the $|A,p_n;\delta|_k$ summability method.

Theorem 3.1. *Let (φ_n) be a complex sequence. Let $k \geq 1$ and $0 \leq \delta < 1/k$. Suppose that $A = (a_{nv})$ is a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \; n = 0,1,\ldots, \tag{6}$$

$$a_{n-1,v} \geq a_{nv}, \text{ for } n \geq v+1, \tag{7}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right).$$

If $(|\lambda_n|)$ is non-increasing such that $\sum p_n|\lambda_n|$ is convergent and satisfy conditions (3)–(5) of Theorem 2.2 and the conditions

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{p_v}{P_v}\right) \quad as \quad m\to\infty, \tag{8}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}| = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\right) \quad as \quad m\to\infty \tag{9}$$

are satisfied, then the $|A,p_n;\delta|_k$ summability of the series

$$\sum_{n=1}^{\infty}C_n(t)\lambda_n\varphi_n,$$

at any point is a local property of f.

Lemma 3.1 ([8]). *From conditions* (1), (2) *and* (6), (7), *we have*

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})| \leq a_{nn},$$

$$|\hat{a}_{n,v+1}| \leq a_{nn}.$$

Lemma 3.2 ([7]). *If $(|\lambda_n|)$ is non-increasing such that $\sum p_n|\lambda_n| < \infty$, then $P_n|\lambda_n| = O(1)$, as $n\to\infty$.*

Lemma 3.3. *Let (φ_n) be a complex sequence. If (s_n) is bounded, and all the conditions of Theorem* 3.1 *are satisfied, then the series*

$$\sum_{n=1}^{\infty}a_n\lambda_n\varphi_n$$

is $|A,p_n;\delta|_k$ *summable, where* $k\geq 1$ *and* $0\leq\delta<1/k$*, and* $(|\lambda_n|)$ *is the same as in Theorem* 3.1.

Proof. Let (I_n) denotes the A-transform of the series $\sum_{n=1}^{\infty} a_n\lambda_n\varphi_n$, then

$$\bar{\Delta} I_n=\sum_{v=1}^{n}\hat{a}_{nv}a_v\lambda_v\varphi_v.$$

Applying Abel's transformation to this sum, we have

$$\begin{aligned}\bar{\Delta} I_n &=\sum_{v=1}^{n-1}\Delta_v(\hat{a}_{nv}\lambda_v\varphi_v)\sum_{r=1}^{v}a_r+a_{nn}\lambda_n\varphi_n\sum_{v=1}^{n}a_v\\ &=\sum_{v=1}^{n-1}\Delta_v(\hat{a}_{nv}\lambda_v\varphi_v)s_v+a_{nn}\lambda_n\varphi_n s_n\\ &=\sum_{v=1}^{n-1}\Delta(\hat{a}_{nv})\lambda_v\varphi_v s_v+\sum_{v=1}^{n-1}\Delta\lambda_v\varphi_v\hat{a}_{n,v+1}s_v+\sum_{v=1}^{n-1}\Delta\varphi_v\lambda_{v+1}\hat{a}_{n,v+1}s_v+a_{nn}\lambda_n\varphi_n s_n\\ &=I_{n,1}+I_{n,2}+I_{n,3}+I_{n,4}.\end{aligned}$$

To complete the proof of Lemma 3.3, it suffices to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid I_{n,r}\mid^k<\infty,\quad \text{for}\quad r=1,2,3,4.$$

First, applying Hölder's inequality, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid I_{n,1}\mid^k &\leq\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v||\varphi_v||s_v|\right\}^k\\ &\leq\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta(\hat{a}_{nv})|\,|s_v|^k|\lambda_v|^k|\varphi_v|^k\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}\\ &=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|\varphi_v|^k\right\}\\ &=O(1)\sum_{v=1}^{m}|\lambda_v|^k|\varphi_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|\\ &=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{p_v}{P_v}|\lambda_v|^k|\varphi_v|^k\\ &=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}(P_v|\lambda_v|)^{k-1}|\lambda_v||\varphi_v|^k\frac{p_v}{P_v^k}\\ &=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\lambda_v||\varphi_v|^k\frac{p_v}{P_v^k}\\ &=O(1)\quad\text{as}\quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Lemma 3.3 and by using condition (3) of Theorem 2.2, condition (8) of Theorem 3.1 and also taking into account Lemma 3.1 and Lemma 3.2. Now, using Hölder's inequality, we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid I_{n,2}\mid^k\leq\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v||\varphi_v||s_v|\right\}^k$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}\left(\frac{p_v}{P_v}\right)^{\delta k-\delta}|\hat{a}_{n,v+1}||\Delta\lambda_v||\varphi_v||s_v|^k\times\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^{\delta}|\hat{a}_{n,v+1}||\Delta\lambda_v||\varphi_v|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}\left(\frac{p_v}{P_v}\right)^{\delta k-\delta}|\hat{a}_{n,v+1}||\Delta\lambda_v||\varphi_v|\times\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^{\delta}|\Delta\lambda_v||\varphi_v|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left\{\sum_{v=1}^{n-1}\left(\frac{p_v}{P_v}\right)^{\delta k-\delta}|\hat{a}_{n,v+1}||\Delta\lambda_v||\varphi_v|\right\}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{p_v}{P_v}\right)^{\delta k-\delta}|\Delta\lambda_v||\varphi_v|\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{p_v}{P_v}\right)^{\delta k-\delta}\left(\frac{P_v}{p_v}\right)^{\delta k}|\Delta\lambda_v||\varphi_v|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta}|\Delta\lambda_v||\varphi_v|\\
&= O(1) \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Lemma 3.3 and by taking condition (4) of Theorem 2.2 and also condition (9) of Theorem 3.1. Further, we have

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}| I_{n,3} |^k &\le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\varphi_v||\lambda_{v+1}||s_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta\varphi_v|^k|\hat{a}_{n,v+1}|\frac{|\lambda_{v+1}|}{p_{v+1}^{k-1}}\times\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|p_{v+1}\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\Delta\varphi_v|^k|\hat{a}_{n,v+1}|\frac{|\lambda_{v+1}|}{p_{v+1}^{k-1}}\times\left\{\sum_{v=1}^{n-1}|\lambda_{v+1}|p_{v+1}\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\sum_{v=1}^{n-1}|\Delta\varphi_v|^k|\hat{a}_{n,v+1}|\frac{|\lambda_{v+1}|}{p_{v+1}^{k-1}}\\
&= O(1)\sum_{v=1}^{m}|\Delta\varphi_v|^k\frac{|\lambda_{v+1}|}{p_{v+1}^{k-1}}\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}|\Delta\varphi_v|^k\frac{|\lambda_{v+1}|}{p_{v+1}^{k-1}}\\
&= O(1) \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Lemma 3.3 and using condition (5) of Theorem 2.2, condition (9) of Theorem 3.1 and also taking Lemma 3.1 and Lemma 3.2. Finally, by virtue of the hypotheses of Lemma 3.3 and using condition (3) of Theorem 2.2 and taking Lemma 3.2, we have

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}| I_{n,4} |^k &= \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|a_{nn}\lambda_n\varphi_n s_n|^k \le \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^k|\lambda_n|^k|\varphi_n|^k|s_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_n|^k|\varphi_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{p_n}{P_n^k}|\lambda_n||\varphi_n|^k(P_n|\lambda_n|)^{k-1}
\end{aligned}
$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{p_n}{P_n^k}|\lambda_n||\varphi_n|^k = O(1) \quad \text{as} \quad m\to\infty,$$

which completes the proof of Lemma 3.3. □

Proof of Theorem 3.1. Since the convergence of Fourier series at a point is a local property of its generating function f, our theorem follows immediately from Lemma 3.3.

4. Conclusions

If we take $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we have a result of Theorem 2.2. Also, if we take $\delta = 0$ in Theorem 3.1, we have a new result dealing with the $|A, p_n|_k$ summability of Fourier series.

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