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A NOTE ON THE SUMMATION THEOREM FOR ${}_4F_3[-m,\alpha,\lambda+1,\mu+2;\beta,\lambda,\mu;1]$

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Abstract. This article aims to obtain a summation theorem for ${}_4F_3[-m,\alpha,\lambda+1,\mu+2;\beta,\lambda,\mu;1]$. Further, a general series identity is derived. Applications of the results in terms of interesting Kummer's type transformation formulas are given. Some numerical examples are also discussed.

1. Introduction and Preliminaries

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$ {}_pF_q\left[\begin{matrix}(\alpha_p); \\ \\ (\beta_q);\end{matrix}\quad z\right] := {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p; \\ \\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\quad z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!} \tag{1.1}$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here, p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$\beta_j \neq 0,-1,-2,\ldots; \quad j=1,2,\ldots,q.$$

In contracted notation, the sequence of p numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ is denoted by (α_p) with a similar interpretation for others throughout this paper.

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z|<\infty$, if $p\le q$,
(ii) converges for $|z|<1$, if $p=q+1$,
(iii) diverges for all z, $z\neq 0$, if $p>q+1$.

Chu–Vandermonde theorem [7, p. 69, Q.No. 4]:

$$ {}_2F_1\left[\begin{matrix}-M,\ A \ ; \\ \\ B \ ;\end{matrix}\quad 1\right] = \frac{(B-A)_M}{(B)_M}; \quad M=0,1,2,\ldots, \tag{1.2}$$

such that the ratio of Pochhammer symbols in r.h.s. is well defined and $A,\,B\in\mathbb{C}\setminus\mathbb{Z}_0^-$.

Just as the Gaussian ${}_2F_1$ function was generalized to ${}_pF_q$ by increasing the number of the numerator and denominator parameters, the four Appell functions were unified and generalized by Kampé de Fériet [1,4] who defined a general hypergeometric function of two variables.

We recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation [11, p. 423, Eq. (26)]:

$$F^{p:\ q;\ k}_{\ell:\ m;\ n}\left[\begin{matrix}(a_p)\ :\ (b_q)\ ;\ (c_k)\ ; \\ \\ (\alpha_\ell)\ :\ (\beta_m)\ ;\ (\gamma_n)\ ;\end{matrix}\quad x,\ y\right] = \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!},$$

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where for the convergence,

(i) $p+q<\ell+m+1, \quad p+k<\ell+n+1, \quad |x|<\infty, \quad |y|<\infty,$ or
(ii) $p+q=\ell+m+1, \quad p+k=\ell+n+1$ and

$$\begin{cases} |x|^{1/(p-\ell)}+|y|^{1/(p-\ell)}<1, & \text{if } p>\ell \\ \max\{|x|,|y|\}<1, & \text{if } p\leq \ell. \end{cases}$$

Lemma 1.1 ([10, p.100])**.**

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Omega(m,n)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\Omega(m-n,n),$$

provided that the multiple series involved are absolutely convergent.

It is well known that whenever a generalized hypergeometric function reduces to quotients of the products of the gamma functions, the results are very important from the point of view of applications in numerous areas of physical, mathematical and statistical sciences including (for example) in series systems of symbolic computer algebra manipulation [9].

An important development has been made by various authors in generalizations of the summation and transformation theorems (see [5, 6, 8, 12]). Very recently, several remarkable transformation theorems for the q-series have been proved by W. Chu in [2]. Further, by making use of divided differences, new proofs have been presented in [3] for Dougall's summation theorem for well-poised ${}_7F_6$-series and Whipple's transformation between well-poised ${}_7F_6$-series and balanced ${}_4F_3$-series.

In this work, our main motive is to find the summation theorem for ${}_4F_3[-m,\alpha,\lambda+1,\mu+2;\beta,\lambda,\mu;1]$ and to obtain some applications.

2. Summation Theorem

Theorem 2.1. *If ρ, δ, σ are the non-vanishing zeros of the cubic polynomial Cm^3+Dm^2+Em+G and α, β, λ, μ, $-\rho$, $-\delta$, $-\sigma\in\mathbb{C}\setminus\mathbb{Z}_0^-$; $m\in\mathbb{N}_0$, then the following summation theorem holds true:*

$$ {}_4F_3\left[\begin{matrix}-m,\ \alpha,\ \lambda+1,\ \mu+2 & ; \\ & & 1\\ \beta,\ \lambda,\ \mu & ;\end{matrix}\right]=\frac{(-\rho+1)_m\ (-\delta+1)_m\ (-\sigma+1)_m\ (\beta-\alpha-3)_m}{(-\rho)_m\ (-\delta)_m\ (-\sigma)_m\ (\beta)_m}, \tag{2.1}$$

where the coefficients C, D, E and G are the polynomials in α, β, λ, μ given as follows:

$$C=\alpha^2-\alpha^3-\alpha\lambda+\alpha^2\lambda-\alpha\mu+2\alpha^2\mu+\lambda\mu-2\alpha\lambda\mu-\alpha\mu^2+\lambda\mu^2, \tag{2.2}$$

$$\begin{aligned}D=&-7\alpha^2-\alpha^3+4\alpha^2\beta+6\alpha\lambda-\alpha^2\lambda-\alpha^3\lambda-3\alpha\beta\lambda+\alpha^2\beta\lambda+7\alpha\mu-4\alpha^2\mu-2\alpha^3\mu-4\alpha\beta\mu\\&+2\alpha^2\beta\mu-6\lambda\mu+7\alpha\lambda\mu+4\alpha^2\lambda\mu+3\beta\lambda\mu-4\alpha\beta\lambda\mu+5\alpha\mu^2+2\alpha^2\mu^2-2\alpha\beta\mu^2-6\lambda\mu^2\\&-3\alpha\lambda\mu^2+3\beta\lambda\mu^2,\end{aligned} \tag{2.3}$$

$$\begin{aligned}E=&-4\alpha+6\alpha\beta-2\alpha\beta^2-9\alpha\lambda-6\alpha^2\lambda-\alpha^3\lambda+9\alpha\beta\lambda+3\alpha^2\beta\lambda-2\alpha\beta^2\lambda-12\alpha\mu\\&-7\alpha^2\mu-\alpha^3\mu+13\alpha\beta\mu+4\alpha^2\beta\mu-3\alpha\beta^2\mu+11\lambda\mu-7\alpha^2\lambda\mu-2\alpha^3\lambda\mu-12\beta\lambda\mu\\&+4\alpha\beta\lambda\mu+4\alpha^2\beta\lambda\mu+3\beta^2\lambda\mu-2\alpha\beta^2\lambda\mu-6\alpha\mu^2-5\alpha^2\mu^2-\alpha^3\mu^2+5\alpha\beta\mu^2\\&+2\alpha^2\beta\mu^2-\alpha\beta^2\mu^2+11\lambda\mu^2+12\alpha\lambda\mu^2+3\alpha^2\lambda\mu^2-12\beta\lambda\mu^2-6\alpha\beta\lambda\mu^2+3\beta^2\lambda\mu^2,\end{aligned} \tag{2.4}$$

$$\begin{aligned}G=&-6\lambda\mu-11\alpha\lambda\mu-6\alpha^2\lambda\mu-\alpha^3\lambda\mu+11\beta\lambda\mu+12\alpha\beta\lambda\mu+3\alpha^2\beta\lambda\mu-6\beta^2\lambda\mu-3\alpha\beta^2\lambda\mu\\&+\beta^3\lambda\mu-6\lambda\mu^2-11\alpha\lambda\mu^2-6\alpha^2\lambda\mu^2-\alpha^3\lambda\mu^2+11\beta\lambda\mu^2+12\alpha\beta\lambda\mu^2+3\alpha^2\beta\lambda\mu^2\\&-6\beta^2\lambda\mu^2-3\alpha\beta^2\lambda\mu^2+\beta^3\lambda\mu^2\\=&-C\rho\delta\sigma\\=&\lambda\ \mu\ (\mu+1)\ (\beta-\alpha-1)\ (\beta-\alpha-2)\ (\beta-\alpha-3).\end{aligned} \tag{2.5}$$

Proof. Suppose the l.h.s. of equation (2.1) is denoted by Δ, then we have

$$\begin{aligned}
\Delta =& \sum_{r=0}^{m} \frac{(-m)_r\ (\alpha)_r\ (\lambda+1)_r\ (\mu+2)_r}{(\beta)_r\ (\lambda)_r\ (\mu)_r\ r!} \\
=& \sum_{r=0}^{m} \frac{(-m)_r\ (\alpha)_r}{(\beta)_r\ r!} \left[1 + \frac{r(2\lambda+\mu+2)}{\lambda\mu} + \frac{r(r-1)(\lambda+2\mu+4)}{\lambda\mu(\mu+1)} + \frac{r(r-1)(r-2)}{\lambda\mu(\mu+1)}\right] \\
=& {}_2F_1\left[\begin{matrix} -m,\ \alpha & ; & \\ & & 1 \\ \beta & ; & \end{matrix}\right] + \frac{(2\lambda+\mu+2)}{\lambda\mu} \sum_{r=0}^{m-1} \frac{(-m)_{r+1}\ (\alpha)_{r+1}}{(\beta)_{r+1}\ r!} \\
&+ \frac{(\lambda+2\mu+4)}{\lambda\mu(\mu+1)} \sum_{r=0}^{m-2} \frac{(-m)_{r+2}\ (\alpha)_{r+2}}{(\beta)_{r+2}\ r!} + \frac{1}{\lambda\mu(\mu+1)} \sum_{r=0}^{m-3} \frac{(-m)_{r+3}\ (\alpha)_{r+3}}{(\beta)_{r+3}\ r!} \\
=& {}_2F_1\left[\begin{matrix} -m,\ \alpha & ; & \\ & & 1 \\ \beta & ; & \end{matrix}\right] + \frac{(2\lambda+\mu+2)}{\lambda\mu} \frac{(-m)_1\ (\alpha)_1}{(\beta)_1} {}_2F_1\left[\begin{matrix} -(m-1),\ \alpha+1 & ; & \\ & & 1 \\ \beta+1 & ; & \end{matrix}\right] \\
&+ \frac{(\lambda+2\mu+4)}{\lambda\mu(\mu+1)} \frac{(-m)_2\ (\alpha)_2}{(\beta)_2} {}_2F_1\left[\begin{matrix} -(m-2),\ \alpha+2 & ; & \\ & & 1 \\ \beta+2 & ; & \end{matrix}\right] \\
&+ \frac{1}{\lambda\mu(\mu+1)} \frac{(-m)_3\ (\alpha)_3}{(\beta)_3} {}_2F_1\left[\begin{matrix} -(m-3),\ \alpha+3 & ; & \\ & & 1 \\ \beta+3 & ; & \end{matrix}\right]. \qquad (2.6)
\end{aligned}$$

Using the Chu–Vandermonde theorem 1.2 in r.h.s. of equation (2.6), we obtain

$$\begin{aligned}
\Delta =& \frac{(\beta-\alpha)_m}{(\beta)_m} + \frac{(2\lambda+\mu+2)}{\lambda\mu} \frac{(-m)_1\ (\alpha)_1}{(\beta)_1} \frac{(\beta-\alpha)_{m-1}}{(\beta+1)_{m-1}} \\
&+ \frac{(\lambda+2\mu+4)}{\lambda\mu(\mu+1)} \frac{(-m)_2\ (\alpha)_2}{(\beta)_2} \frac{(\beta-\alpha)_{m-2}}{(\beta+2)_{m-2}} + \frac{1}{\lambda\mu(\mu+1)} \frac{(-m)_3\ (\alpha)_3}{(\beta)_3} \frac{(\beta-\alpha)_{m-3}}{(\beta+3)_{m-3}} \\
=& \frac{(\beta-\alpha)_m}{(\beta)_m} + \frac{(2\lambda+\mu+2)(-m)_1\ (\alpha)_1}{\lambda\mu} \frac{(\beta-\alpha)_{m-1}}{(\beta)_m} \\
&+ \frac{(\lambda+2\mu+4)(-m)_2\ (\alpha)_2}{\lambda\mu(\mu+1)} \frac{(\beta-\alpha)_{m-2}}{(\beta)_m} + \frac{(-m)_3\ (\alpha)_3}{\lambda\ \mu\ (\mu+1)} \frac{(\beta-\alpha)_{m-3}}{(\beta)_m} \\
=& \frac{(\beta-\alpha)_m}{(\beta)_m} \left[1 - \frac{(2\lambda+\mu+2)\ m\ \alpha}{\lambda\ \mu\ (\beta-\alpha+m-1)} + \frac{(\lambda+2\mu+4)(-m)_2\ (\alpha)_2}{\lambda\ \mu\ (\mu+1)\ (\beta-\alpha+m-2)_2}\right. \\
&\left. + \frac{(-m)_3\ (\alpha)_3}{\lambda\ \mu\ (\mu+1)\ (\beta-\alpha+m-3)_3}\right] \\
=& \frac{(\beta-\alpha)_m}{(\beta)_m} \left[\frac{\Psi(\alpha,\beta,\lambda,\mu,m)}{\lambda\ \mu\ (\mu+1)\ (\beta-\alpha+m-1)\ (\beta-\alpha+m-2)\ (\beta-\alpha+m-3)}\right], \qquad (2.7)
\end{aligned}$$

where

$$\begin{aligned}
\Psi(\alpha,\beta,\lambda,\mu,m) =& \lambda\ \mu\ (\mu+1)(\beta-\alpha+m-1)(\beta-\alpha+m-2)(\beta-\alpha+m-3) \\
&- m\alpha(2\lambda+\mu+2)(\mu+1)(\beta-\alpha+m-2)(\beta-\alpha+m-3) \\
&+ (-m)(-m+1)(\lambda+2\mu+4)(\alpha)(\alpha+1)(\beta-\alpha+m-3) \\
&+ (-m)(-m+1)(-m+2)(\alpha)(\alpha+1)(\alpha+2).
\end{aligned}$$

Equation (2.7) can be written as

$$\Delta = \frac{(\beta-\alpha)_m}{(\beta)_m} \left[\frac{Cm^3 + Dm^2 + Em + G}{\lambda\ \mu\ (\mu+1)\ (\beta-\alpha+m-1)\ (\beta-\alpha+m-2)\ (\beta-\alpha+m-3)}\right], \qquad (2.8)$$

Since $\rho,\ \delta,\ \sigma$ are the roots of the cubic equation $Cm^3+Dm^2+Em+G=0$, therefore equation (2.8) can be written as follows:

$$\Delta=\frac{(\beta-\alpha)_m}{(\beta)_m}\left[\frac{C(m-\rho)(m-\delta)(m-\sigma)}{\lambda\ \mu\ (\mu+1)\ (\beta-\alpha+m-1)\ (\beta-\alpha+m-2)\ (\beta-\alpha+m-3)}\right].$$

On simplification, we get the assertion (2.1). □

3. General Double Series Identity

The application of the summation Theorem 2.1 is given by proving the following general double series identity:

Theorem 3.1. *Let $\{\Phi(\ell)\}_{\ell=1}^{\infty}$ be a bounded sequence of arbitrary complex numbers such that $\Phi(0)\neq 0$ and $\alpha,\ \beta,\ \lambda,\ \mu,\ -\rho,\ -\delta,\ -\sigma\in\mathbb{C}\setminus\mathbb{Z}_0^-$. Then*

$$\sum_{m,n=0}^{\infty}\frac{\Phi(m+n)\ (\alpha)_n\ (\lambda+1)_n\ (\mu+2)_n\ (-1)^n}{(\beta)_n\ (\lambda)_n\ (\mu)_n}\frac{z^{m+n}}{m!\ n!}$$
$$=\sum_{m=0}^{\infty}\frac{\Phi(m)\ (-\rho+1)_m\ (-\delta+1)_m\ (-\sigma+1)_m\ (\beta-\alpha-3)_m}{(-\rho)_m\ (-\delta)_m\ (-\sigma)_m\ (\beta)_m}\frac{z^m}{m!},\tag{3.1}$$

where $\rho,\ \delta,\ \sigma$ are the roots of the cubic equation $Cm^3+Dm^2+Em+G=0$ and C,D,E,G are given by equations (2.2)–(2.5) with each absolutely convergent multiple series involved.

Proof. Suppose l.h.s. of equation (3.1) is denoted by Ξ. Then in view of Lemma 1.1, we have

$$\Xi=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{\Phi(m)\ (\alpha)_n\ (\lambda+1)_n\ (\mu+2)_n\ (-1)^n}{(\beta)_n\ (\lambda)_n\ (\mu)_n}\frac{z^m}{(m-n)!\ n!}$$
$$=\sum_{m=0}^{\infty}\Phi(m)\frac{z^m}{m!}\sum_{n=0}^{m}\frac{(-m)_n\ (\alpha)_n\ (\lambda+1)_n\ (\mu+2)_n}{(\beta)_n\ (\lambda)_n\ (\mu)_n\ n!}$$
$$=\sum_{m=0}^{\infty}\Phi(m)\frac{z^m}{m!}\ {}_4F_3\left[\begin{matrix}-m,\ \alpha,\ \lambda+1,\ \mu+2 & ; & \\ & & 1\\ \beta,\ \lambda,\ \mu & ; & \end{matrix}\right].$$

Using Theorem 2.1 in r.h.s. of the above equation, relation (3.1) follows. □

4. Applications

If $\rho,\ \delta,\ \sigma$ are the roots of the cubic equation $Cm^3+Dm^2+Em+G=0$ and $\alpha,\ \beta,\ \lambda,\ \mu,\ -\rho,\ -\delta,\ -\sigma\in\mathbb{C}\setminus\mathbb{Z}_0^-$; $m\in\mathbb{N}_0$, then we have the following applications:

I. Taking $\Phi(p)=\frac{\prod_{i=1}^{A}(a_i)_p}{\prod_{i=1}^{B}(b_i)_p}$ in equation (3.1), we get the following reduction formula:

$$F_{B:0;3}^{A:0;3}\left[\begin{matrix}(a_A) & : & - & ; & \alpha,\ \lambda+1,\ \mu+2 & ; & \\ & & & & & & z,\ -z\\ (b_B) & : & - & ; & \beta,\ \lambda,\ \mu & ; & \end{matrix}\right]$$
$$={}_{A+4}F_{B+4}\left[\begin{matrix}a_1,\dots,a_A,\ -\rho+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3 & ; & \\ & & z\\ b_1,\dots,b_B,\ -\rho,\ -\delta,\ -\sigma,\ \beta & ; & \end{matrix}\right],$$

subject to the convergence conditions:

$$\begin{cases}|z|<\frac{1}{2}, & \text{if } A=B+1\\ |z|<\infty, & \text{if } A\leq B.\end{cases}$$

II. Taking $A = 1$, $a_1 = d$, $B = 0$ in equation (3.1) and putting $z = \frac{-y}{1-y}$, we get the following Pfaff-Kummer-type transformation formula:

$$
{}_4F_3\left[\begin{matrix} d,\ \alpha,\ \lambda+1,\ \mu+2 & ; \\ & & y \\ \beta,\ \lambda,\ \mu & ; \end{matrix}\right]
$$

$$
=(1-y)^{-d}\ {}_5F_4\left[\begin{matrix} d,\ -\rho+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3 & ; \\ & & \frac{-y}{1-y} \\ -\rho,\ -\delta,\ -\sigma,\ \beta & ; \end{matrix}\right], \tag{4.1}
$$

where $|y| < 1$, $Re(y) < \frac{1}{2}$ and $\alpha,\ \beta,\ \lambda,\ \mu,\ -\rho,\ -\delta,\ -\sigma \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

III. Taking $A = B = 0$ in equation (3.1) and putting $z = -y$, we get the following Kummer's type first transformation formula:

$$
{}_3F_3\left[\begin{matrix} \alpha,\ \lambda+1,\ \mu+2 & ; \\ & & y \\ \beta,\ \lambda,\ \mu & ; \end{matrix}\right]
$$

$$
=\exp(y)\ {}_4F_4\left[\begin{matrix} -\rho+1,\ -\delta+1,\ -\sigma+1,\ \beta-\alpha-3 & ; \\ & & -y \\ -\rho,\ -\delta,\ -\sigma,\ \beta & ; \end{matrix}\right], \tag{4.2}
$$

where $|y| < \infty$ and $\alpha,\ \beta,\ \lambda,\ \mu,\ -\rho,\ -\delta,\ -\sigma \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

In this section, we consider some numerical examples.

5. Numerical Examples

Taking $\alpha = 2$, $\beta = \frac{3}{2}$, $\lambda = \frac{5}{4}$, $\mu = \frac{7}{3}$ in equations (2.2)–(2.5), the numerical values of C, D, E and G are obtained as follows:

$$
C = -\frac{1}{3}, \quad D = \frac{14}{3}, \quad E = -\frac{1579}{24}, \quad G = -\frac{6125}{48}.
$$

The cubic polynomial equation $Cm^3 + Dm^2 + Em + G = 0$ becomes

$$
16m^3 - 224m^2 + 3158m + 6125 = 0. \tag{5.1}
$$

The roots ρ, δ and σ of equation (5.1) are obtained approximately as:

$$
\rho = -1.70749, \quad \delta = 7.85375 + i12.7481, \quad \sigma = 7.85375 - i12.7481.
$$

Now, substituting the values of α, β, λ, μ, ρ, δ and σ in equations (4.1) and (4.2), we get the following Kummer-type transformation formulas:

$$
{}_4F_3\left[\begin{matrix} d,\ 2,\ \frac{9}{4},\ \frac{13}{3} & ; \\ & & y \\ \frac{3}{2},\ \frac{5}{4},\ \frac{7}{3} & ; \end{matrix}\right]
$$

$$
=(1-y)^{-d}\ {}_5F_4\left[\begin{matrix} d,\ 2.70749,\ -6.85375-i12.7481,\ -6.85375+i12.7481,\ -\frac{7}{2} & ; \\ & & \frac{-y}{1-y} \\ 1.70749,\ -7.85375-i12.7481,\ -7.85375+i12.7481,\ \frac{3}{2} & ; \end{matrix}\right],
$$

where $|y| < 1$, $Re(y) < \frac{1}{2}$ and

$$
{}_3F_3\left[\begin{matrix} 2,\ \frac{9}{4},\ \frac{13}{3} & ; \\ & & y \\ \frac{3}{2},\ \frac{5}{4},\ \frac{7}{3} & ; \end{matrix}\right]
$$

$$
=\exp(y)\ {}_4F_4\left[\begin{matrix} 2.70749,\ -6.85375-i12.7481,\ -6.85375+i12.7481,\ -\frac{7}{2} & ; \\ & & -y \\ 1.70749,\ -7.85375-i12.7481,\ -7.85375+i12.7481,\ \frac{3}{2} & ; \end{matrix}\right],
$$

where $|y| < \infty$.

Several other examples can be obtained in a similar manner by considering different values of α, β, λ and μ. The extensions of the summation theorems to hypergeometric functions containing arbitrary number of pairs of numerator and denominator parameters will be taken as future aspect.

Appendix

The roots ρ, δ, σ of the cubic equation $Cm^3+Dm^2+Em+G=0$ are calculated by using *Wolfram Mathematica 9.0* Software. The values of ρ, δ and σ are given as follows:

$$\begin{aligned}\rho = &-\frac{D}{3C}\\ &-\frac{2^{1/3}\left(-D^2+3CE\right)}{3C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}\\ &+\frac{\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{3\times 2^{1/3}C}\end{aligned}$$

$$\begin{aligned}\delta = &-\frac{D}{3C}\\ &+\frac{\left(1+i\sqrt{3}\right)\left(-D^2+3CE\right)}{3\times 2^{2/3}C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}\\ &-\frac{\left(1-i\sqrt{3}\right)\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{6\times 2^{1/3}C}\end{aligned}$$

$$\begin{aligned}\sigma = &-\frac{D}{3C}\\ &+\frac{\left(1-i\sqrt{3}\right)\left(-D^2+3CE\right)}{3\times 2^{2/3}C\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}\\ &-\frac{\left(1+i\sqrt{3}\right)\left(-2D^3+9CDE-27C^2G+\sqrt{4\left(-D^2+3CE\right)^3+\left(-2D^3+9CDE-27C^2G\right)^2}\right)^{1/3}}{6\times 2^{1/3}C}.\end{aligned}$$

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