Transactions of A. Razmadze
Mathematical Institute
Vol. **175** (2021), issue 2, 293–296

ON POISSON TYPE INTEGRALS IN THE POLYBALL

ROMI F. SHAMOYAN

Abstract. We consider some natural extensions of Poisson integral in the unit ball to polyballs and extend some known classical results to the case of product domains (polyballs). In particular, we extend some known results in the unit ball on Poisson integrals related to BMO to the product domain case.

1. Introduction

Let $B_n = |z| < 1$ be the unit ball in $\mathbb{C}^n$, $S^n = \partial B$ be the unit sphere in $\mathbb{C}^n$.

Let $d(z,w) = |1 - \langle z, w\rangle|^{\frac{1}{2}}$, $z, w \in \bar{B}_n$, be the restriction of d on S_n; it is a non-isotropic metric (see [4,6]).

Let also $Q(\xi, r) = \Big\{\eta \in S_n : |1 - \langle \xi, \eta\rangle|^{\frac{1}{2}} < r\Big\}$, $r > 0$, $\xi \in S_n$. We call Q a d-ball putting r sometimes as a subscript for the extension to the ball, $S_n = \{|z| = 1\}$.

We denote various constants appearing in this paper by C, C_1, C_2, c. As usual, we define both a Poisson kernel $P(z,w)$, $z, w \in \bar{B}_n$, $P(z,w) = \frac{(1-|z|^2)^n}{|1-\langle \bar{z}, w\rangle|^{2n}}$, and a Poisson integral of a positive Borel measure μ $P(\mu)$ in a standard way (see, e.g., [6]). For some new and classical results on these objects we refer the reader to [4,6] and [2,5]. By $d\sigma$ we denote the Lebesgue measure on S_n. For the d function we refer to [4,6] (see also below).

In this note we discuss some problems in an open new research area of Poisson type integrals in product domains in $\mathbb{C}^n$. Some new objects in this note will be defined and some interesting new problems will be posed and solved. We provide first known results in a unit ball (see [6]).

It has been shown in [4,6] that for an f function of a certain class (these estimates were used in the study of analytic BMO)

$$\int\limits_{S_n} |f(\eta) - f(a)|^2 P(a,\eta) d\sigma(\eta) \geq \left(\frac{\tilde{c}}{\sigma(Q)} \int\limits_{Q} |f - f(a)|^2 d\sigma\right), \tag{1}$$

we can see that for all $\frac{3}{4} < |a| < 1$,

$$\int\limits_{S_n} |f(\xi) - f(a)|^2 P(a,\xi) d\sigma(\xi) \leq c(\sup)\Big(\frac{1}{\sigma(Q)}\Big) \int\limits_{Q} |f - f_Q|^2 d\sigma < \infty \tag{2}$$

(see [4,6]). Next, it has been shown (see [4,6]) that

$$\Big(\sup_{z \in B_n}\Big) \int\limits_{B_n} P(z,w) d\mu(w) \geq \left(\frac{\mu(Q_r(\xi))}{4^n r^{2n}}\right), \tag{3}$$

for a positive Borel measure μ in B.

And if $c = \Big(\sup\limits_{\xi, r}\Big) \frac{\mu(Q_r(\xi))}{r^{2n}}$, then we have

$$\Big(\sup_{|z| > \frac{3}{4}}\Big) \int\limits_{B_n} (P(z,w)) d\mu(w) \leq (c 16^n) \sum_{k \geq 0}^{\infty} \Big(\frac{1}{2^{nk}}\Big), \tag{4}$$

(see [4,6]) for a positive μ Borel measure.

2020 *Mathematics Subject Classification.* P32,44,46.
Key words and phrases. Poisson integral; Unit ball; d-balls, Carleson type measures; Polyball.

The question is how to extend these and other similar results for a positive μ Borel measure to more general situation if, for example, we consider more general Poisson type kernels of the type $\widetilde{P}(\vec{z},\xi)=\frac{\prod\limits^{n}(1-|z|^2)^{\beta_j}}{\prod\limits_{j=1}^{n}|1-z_j\xi|^{\alpha_j}}$, where

$$\sum_{i=1}^{m}\beta_j=n,\quad \sum_{i=1}^{m}\alpha_j=2n,$$

with $z_j\in B$, $j=1,\dots,n$, $\xi\in S$ (we assume sometimes $|z_j|=|z|$) and to a group of positive Borel (μ_j) measures, where $j=1,\dots,n$.

In this paper, we have found some ways on how to extend (2)–(4) to this more general situation. These results may have various applications in the function theory. Complete proofs will be provided in a separate note. We simply modify already known proofs provided in one domain.

Indeed, a natural idea consists in finding some ways to modify old and known proofs to the product domain case. However, there exist some technical difficulties.

Following the proof for the case $m=1$ (see [4, 6]), for a positive Borel μ measure we find

$$\sup_{z_j\in B_n}\Bigg(\int\limits_{B_n}\cdots\Bigg(\int\limits_{B_n}\frac{\Big[\prod\limits_{j=1}^{m}(1-|z_j|)^{\alpha_j}\Big]^{p_1}d\mu(w_1)}{\prod\limits_{j=1}^{m}|1-\langle z_j,w_j\rangle|^{\beta_jp_1}}\Bigg)^{\frac{p_2}{p_1}}\cdots\mu(w_m)\Bigg)^{\frac{1}{p_m}}\geq c\frac{(\mu(Q_r(\xi)))^{\sum\limits_{i=1}^{m}\frac{1}{p_i}}}{r^{2n}},\tag{$\widetilde{A}$}$$

where $Q_r(\xi)=\{z\in B_n: d(z,\xi)<r\}$, $\xi\in S_n$; $r>0$, $\sum\limits_{j=1}^{m}\alpha_j=n$, $\sum\limits_{j=1}^{m}\beta_j=2n$, $\beta_j>0$; $\alpha_j>0$, $j=1,\dots,m$, $0<p_i<\infty$, $i=1,\dots,m$.

Next, we have the following known estimate (see [6])

$$\Big(\sup_{|z|>\frac{3}{4}}\Big)\int\limits_{B_n}P(z,w)d\mu(w)\leq 16^nc\bigg(\sum_{k=0}^{\infty}\frac{1}{2^{nk}}\bigg),\quad\text{where}\tag{$\tilde{\tilde{A}}$}$$

$$P(z,w)=\frac{(1-|z|^2)^n}{|1-\langle\overline{z},w\rangle|^{2n}},\ c=\Big(\sup_{\xi,r}\Big)\Big(\frac{\mu(Q_r(\xi))}{r^{2n}}\Big),\quad n\in\mathbb{N},\quad z,w\in\mathbb{B}_n.$$

The natural question to give an extension of this $(\tilde{\tilde{A}})$ estimate to

$$\widetilde{M}=\Big(\sup_{|z_j|>\frac{3}{4}}\Big)\bigg(\int\limits_{B_n}\cdots\int\limits_{B_n}(\widetilde{P}(\vec{z},\vec{w}))d\mu_1(w_1)\dots d\mu_m(w_m)\bigg)^{\frac{1}{p_m}},$$

where

$$\widetilde{P}(\vec{z},\vec{w})=\frac{\prod\limits_{j=1}^{m}(1-|z_j|)^{\alpha_j}}{\prod\limits_{j=1}^{m}(1-\langle z_j,w_j\rangle)^{\beta_j}},$$

and

$$\sum_{j=1}^{m}\alpha_j=n,\quad\sum_{j=1}^{m}\beta_j=2n,\quad 0<p_j<\infty,\quad z_j,\ w_j\in B_n,\quad j=1,\dots,m,$$

following carefully one functional known proof (see [4, 6]).

Note that here μ_j are positive Borel measures on B, $i=1,\dots,m$.

We have found the following generalization:

$$\bigg(\sup_{\substack{|z_j|>\tau\\|z_j|=R}}\bigg)\int\limits_{B_n}\frac{\prod\limits_{j=1}^{m}(1-|z_j|)^{\alpha_j}d\mu(w)}{\prod\limits_{j=1}^{m}|(1-z_jw)^{\beta_j}|}\leq(16^n)\mathbb{C}\bigg(\sum_{k=0}^{\infty}\frac{1}{2^{nk}}\bigg)\leq C_n\cdot C;\tag{B}$$

where $\sum\limits_{j=1}^{m}\beta_j=2n$, $\sum\limits_{j=1}^{m}\alpha_j=n$, for some positive constant $\mathbb{C}_n$.

We mention another known estimate and provide below some similar type extensions to the polyballs.

Note (see [4,6]) that for $Q=Q(\frac{a}{|a|,\ \sqrt{1-|a|^2}})$, $a\in B$, $a\neq 0$,

$$I_a=\int\limits_{S_n}|f(\xi)-f(a)|^2\frac{(1-|a|^2)^n}{|1-\overline{\xi}a|^{2n}}d\sigma(\xi)\geq\frac{1}{4^n(1-|a|^2)^n}$$

and $\int\limits_{Q}|f-f(a)|^2d\sigma\geq\frac{\text{const}}{\sigma(Q_1)}\int\limits_{Q_1}|f-f(a)|^2d\sigma$, where $Q=Q\Big(\frac{a}{|a|},\sqrt{1-|a|^2}\Big)$ as runs over $B_n/0$, whereas the above Q_1 runs over all d balls of radius less than 1 (see [4,6]). We provide some generalizations of such estimates.

Let now $f\in L^{p_1}(S_n\times\cdots\times S_n)$,

$$I_{\vec{a}}=\Bigg(\int\limits_{S_n}\cdots\Bigg(\int\limits_{S_n}|f(\xi_1,\dots,\xi_n)-f(a_1,\dots,a_m)|^{p_1}\times\frac{\prod\limits_{i=1}^{m}(1-|a_i|^2)^{\alpha_i}}{\prod\limits_{i=1}^{m}|1-\overline{\xi}_ia_i|^{\beta_i}}d\sigma(\xi_1)\Bigg)^{\frac{p_2}{p_1}}\cdots d\sigma(\xi_m)\Bigg)^{\frac{1}{p_m}},$$

$$0<p_i<\infty,\quad i=1,\dots,m,\quad \sum_{i=1}^{m}\alpha_i=n,\quad \sum_{i=1}^{m}\beta_j=2n,\quad \alpha_j,\beta_j>0,\quad j=1,\dots,m.$$

Then

$$I_{\vec{a}}\geq\frac{4^{-\frac{n}{p_1}}}{\prod\limits_{i=1}^{m}(1-|a_i|^2)^{\frac{(2\beta_i-\alpha_i)}{p_1}}}\Bigg(\int\limits_{Q}\cdots\Bigg(\int\limits_{Q}|f(\xi_1,\dots,\xi_n)-f(a_1,\dots,a_m)|^{p_1}d\sigma(\xi_1)\Bigg)\cdots d\sigma(\xi_m)\Bigg)^{\frac{1}{p_m}}\qquad\text{(C)}$$

$$Q=Q\Big(\frac{a}{|a|},\sqrt{1-|a|^2}\Big),\quad a\in B_n,\quad a\neq 0,\quad 2\beta_i-\alpha_i>0,\quad i=1,\dots,m.$$

And for same parameters, let

$$\widetilde{I}_{\vec{a}}=\Bigg(\int\limits_{S_n}\cdots\Bigg(\int\limits_{S_n}\prod_{i=1}^{m}|f_i(\xi_i)-f_i(a_i)|^{p_1}\times\frac{\prod\limits_{i=1}^{m}(1-|a_i|)^{\alpha_i}}{\prod\limits_{i=1}^{m}|1-\overline{\xi}_ia_i|^{\beta_i}}d\sigma(\xi_1)\Bigg)^{\frac{p_2}{p_1}}\cdots d\sigma(\xi_m)\Bigg)^{\frac{1}{p_m}}.$$

Then we also have

$$\widetilde{I}_{\vec{a}}\geq\frac{4^{-\frac{n}{p_1}}}{\prod\limits_{i=1}^{m}(1-|a_i|^2)^{\frac{(2\beta_i-\alpha_i)}{p_1}}}\prod_{i=1}^{m}\Bigg(\int\limits_{Q}|f_i(\xi_i)-f_i(a_i)|^{p_i}\Bigg)^{\frac{1}{p_i}},\qquad(\tilde{C})$$

where $f_i\in L^{p_1}(B)$, $i=1,\dots,m$; the proof is based on the estimate

$$(|1-(\overline{a},\xi)|)=1-\Big(\frac{a}{|a|},\xi\Big)+(1-|a|)\Big(\frac{a}{|a|},\xi\Big)$$

and, hence, $|1-a\overline{\xi}|\leq 2(1-|a|^2)$ (see [4,6]), where $\xi\in Q$ and is similar to one domain proof (see [4,6]).

The above estimates may have various applications. In this note we omit the details of proofs of the last estimates refereing to [4,6] for complete elegant proofs of simpler "one domain" cases.

We have also the following known estimate (see [4,6]):

$$\int\limits_{S_n}|f(\xi)-f(a)|^P\frac{(1-|a|^2)}{|1-a\xi|^{2n}}d\sigma\leq c\,C_*^{t,p},\qquad\text{(A)}$$

$r_0 < |a| < 1$, for some constant C, where

$$C_*^{t,p} = (\sup)\Big(\frac{1}{\sigma(Q)^t}\Big)\int\limits_Q |f - f_Q|^P d\sigma < \infty, \quad f \in H^P(B), \quad (f_Q) = \frac{1}{\sigma(Q)}\Bigg(\int\limits_Q f d\sigma\Bigg).$$

We refer to [4,6] for $t = 1$, $p = 2$ case of (A).

We wish to extend (A) again using an extension of classical Poisson kernel $P(z, \xi)$ and carefully studying the classical known proof in [1,2]. We have found the following result, the main result of this note. Here we formulate this result.

Theorem 1. *Let $f \in L^P(S)$, $p \geq 1$. Then we have*

$$\Bigg(\sup_{\substack{z_0<|a_j|<1\\ j=1,\dots,m}}\Bigg)\int\limits_{S^n} |f(\xi) - f_{(Q_0)}|^P \times \prod_{j=1}^m \frac{(1-|a_j|)^{\alpha_j}}{|1-\xi a_j|^{\beta_j}} d\sigma(\xi) \leq \widetilde{C} C_*^{1,p}, \tag{1}$$

for some positive constant C, where α_j, $\beta_j > 0$, $j = 1, \dots, m$, $Q_0 = Q\left(\frac{a_1}{|\widetilde{a}|}, \sqrt{1-|\widetilde{a}|}\right)$, $a_1 \in B$ $|\widetilde{a}| > r_0$, $\sum\limits_{j=1}^m \beta_j = \sum\limits_{j=1}^m \alpha_j + n$.

Also,

$$\Big(\sup_j\Big)\int\limits_{S_n} |f(\xi) - f(a_j)|^P \cdot \widetilde{P}(\vec{a}, \xi) d\sigma(\xi) \leq C_\sigma\left(C_*^{1,p}\right), \tag{2}$$

for all $|a_j| \in [r_0, 1)$, $|a_j| = |\tilde{a}|$, $a_j = \tilde{a}|\varphi_j$, $\widetilde{P}(\vec{a}, \xi) = \tilde{P} = \left(\prod\limits_{j=1}^m \frac{(1-|a_j|)^{\alpha_j}}{|1-\xi a_j|^{\beta_j}}\right)$, where

$$\sum_{j\geq 1}^m \beta_j = \sum_{j\geq 1}^m \alpha_j + n, \ \ f \in H^P, \ \ 1 \leq p < \infty, \ \ j = 1, \dots, m.$$

Putting $f_Q^S = \frac{1}{\sigma(Q)^S}\Big(\int\limits_Q f d\sigma\Big)$, $S > 0$, we can similarly provide another version of our theorem with other restrictions on α_j, β_j, $j = 1, \dots, \bar{m}$, extending classical results.

It will be interesting to consider another object

$$f_Q^S = \frac{1}{\sigma(\tilde{Q})^S}\Bigg(\int\limits_{\tilde{Q}} f d\sigma\Bigg), \quad \tilde{Q} = Q \times Q, \quad S > 0,$$

on the product domains and to prove similar to our theorem result for those domains, as well.

References

1. I. Chyzhykov, O. Zolota, Growth of the Poisson-Stieltjes integral in a polydisc. *Zh. Mat. Fiz. Anal. Geom.* **7** (2011), no. 2, 141–157, 196, 198.
2. G. Kresin, V. Maz'ya, Sharp estimates for the gradient of the generalized Poisson integral for a half-space. *Georgian Math. J.* **25** (2018), no. 2, 283–290.
3. V. Maz'ya, G. Kresin, Generalized Poisson integral and sharp estimates for harmonic and biharmonic functions in the half-space. *Math. Model. Nat. Phenom.* **13** (2018), no. 4, 37pp.
4. W. Rudin, *Function Theory in the Unit Ball of C^n*. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2008.
5. R. Shamoyan, A note on Poisson type integrals in pseudoconvex and convex domains of finite type and some related results. *Acta Univ. Apulensis Math. Inform.* **50** (2017), 19–27.
6. K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball.* Graduate Texts in Mathematics, 226. Springer-Verlag, New York, 2005.

(Received 04.06.2020)

Department of Mathematics, Bryansk State Technical University, Bryansk 241050, Russia
E-mail address: rsham@mail.ru