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ON NONMEASURABLE UNIFORM SUBSETS OF THE EUCLIDEAN PLANE

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Abstract. It is shown that the cardinality continuum is not measurable in the Ulam sense if and only if for every nonzero σ-finite diffused measure μ on $\mathbf{R}^2$ there is a μ-nonmeasurable uniform subset of $\mathbf{R}^2$. Several related results are also considered.

The main goal of this communication is to discuss briefly uniform subsets of the Euclidean plane $\mathbf{R}^2$ in the context of their nonmeasurability in some generalized sense.

Let l be a straight line in the plane $\mathbf{R}^2$ considered as a certain direction in $\mathbf{R}^2$.

A set $Z \subset \mathbf{R}^2$ is called uniform in direction l if any line of $\mathbf{R}^2$, parallel to l, meets Z at most at one point.

A set $Z \subset \mathbf{R}^2$ is called a graph in direction l if any line of $\mathbf{R}^2$, parallel to l, meets Z exactly at one point.

Accordingly, we say that a set $Z \subset \mathbf{R}^2$ is uniform in $\mathbf{R}^2$ (is a graph in $\mathbf{R}^2$) if there exists a line l in $\mathbf{R}^2$ such that Z is uniform (is a graph) in direction l.

There were established interesting properties of uniform subsets of the plane, which are closely related to the Continuum Hypothesis (**CH**) and to certain propositions in the plane geometry (see, e.g., [1–3, 8, 9]).

Some other properties of uniform sets in $\mathbf{R}^2$ are connected (more or less) with the notion of measurability. To illustrate the above-said, let us give several examples.

1. Every uniform set is G-negligible, where G denotes the group of all translations of $\mathbf{R}^2$ (see [5, 6]).
2. There exist uniform sets which are not G-absolutely negligible (see again [5, 6]).
3. For any straight line l in $\mathbf{R}^2$, there exists a G-invariant measure μ_l on $\mathbf{R}^2$ which extends the standard Lebesgue measure λ_2 on $\mathbf{R}^2$ and is such that all uniform sets in direction l belong to $\mathrm{dom}(\mu_l)$ (it is clear that if Z is uniform in direction l, then $\mu_l(Z) = 0$).
4. There exists a graph in direction l, which is a Hamel basis of $\mathbf{R}^2$. Since every Hamel basis of $\mathbf{R}^2$ is G-absolutely negligible (see [4]), one can conclude that there exist G-absolutely negligible graphs in $\mathbf{R}^2$.
5. No finite family of uniform subsets of $\mathbf{R}^2$ can be a covering of $\mathbf{R}^2$ (see [8]).

Observe that the last fact easily follows from Banach's classical result stating that there exists a finitely additive translation invariant measure on $\mathbf{R}^2$, which extends λ_2 and is defined for all bounded subsets of $\mathbf{R}^2$. Notice also that the analogous fact remains valid for uniform hyper-surfaces in the multi-dimensional Euclidean spaces.

In the sequel, we need a simple auxiliary proposition.

Let l be any fixed straight line in $\mathbf{R}^2$ and let $Z \subset \mathbf{R}^2$ be uniform in direction l. The following two assertions are valid:

(a) every subset of Z is uniform in the same direction l;

(b) $Z = Z_1 \cap Z_2$, where Z_1 and Z_2 are two graphs in the same direction l.

Recall that a measure μ defined on some σ-algebra of subsets of a ground set E is diffused (or continuous) if all singletons in E belong to $\mathrm{dom}(\mu)$ and μ vanishes on all of them.

Also, recall that a cardinal number $\mathbf{a}$ is measurable in Ulam's sense if there exists a probability diffused measure whose domain is the power set of $\mathbf{a}$.

Theorem 1. *Let $\{l_j : j \in J\}$ be a countably infinite family of pairwise non-parallel directions in $\mathbf{R}^2$. The following two assertions are equivalent:*

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(1) *the cardinality continuum* $\mathbf{c}$ *is not measurable in Ulam's sense;*

(2) *for any nonzero* σ*-finite diffused measure* μ *on* $\mathbf{R}^2$*, there exist a direction* l_j *and a graph in this direction, which is nonmeasurable with respect to* μ.

The proof of Theorem 1 is essentially based on the profound result of Davies [3].

Remark 1. Let $\{l_k : k \in K\}$ be a fixed finite family of pairwise non-parallel directions in $\mathbf{R}^2$ and suppose that for any nonzero σ-finite diffused measure μ on $\mathbf{R}^2$ there exist a direction l_k and a uniform set in this direction, which is nonmeasurable with respect to μ. Then, using the result from [1], it can be shown that $\mathbf{c} = \omega_n$ for some natural number n. So, in this case, $\mathbf{c}$ is substantially restricted in its size and automatically turns out to be nonmeasurable in Ulam's sense.

Theorem 2. *Assume Martin's Axiom* (**MA**) *and let* $\{l_j : j \in J\}$ *be a countably infinite family of pairwise non-parallel directions in* $\mathbf{R}^2$.

Then there exists a countable family $\{Z_t : t \in T\}$ *of sets in the plane* $\mathbf{R}^2$ *such that:*

(1) *every set* Z_t *is a graph in some direction* $l_{j(t)}$*, where* $j(t) \in J$*;*

(2) *for any nonzero* σ*-finite diffused measure* μ *on* $\mathbf{R}^2$*, at least one set from the family* $\{Z_t : t \in T\}$ *is nonmeasurable with respect to* μ.

The proof of Theorem 2 is again based on the result of Davies [3] and on the fact that under **MA** there exists a countable family $\{B_i : i \in I\}$ of subsets of $\mathbf{R}^2$, which is absolutely nonmeasurable with respect to the family of all nonzero σ-finite diffused measures on $\mathbf{R}^2$. Actually, the role of $\{B_i : i \in I\}$ can be played by a countable topological base of some generalized Luzin subset of $\mathbf{R}^2$.

Remark 2. Under the assumption that $\mathbf{c}$ is not measurable in Ulam's sense, the problem of generalized nonmeasurability can be considered for other classes of point sets in $\mathbf{R}^2$, e.g., for the class of all Vitali subsets of $\mathbf{R}^2$, for the class of all Bernstein subsets of $\mathbf{R}^2$, or for the class of all Hamel bases of $\mathbf{R}^2$ (cf. [6, 7]).

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