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ASYMPTOTIC ANALYSIS OF COUPLED OSCILLATORS EQUATIONS IN A NON-UNIFORM PLASMA

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Abstract. We study a set of coupled oscillators equations describing Alfvén's linear coupling and fast magnetosonic waves in a magnetized plasma. Using the methods of asymptotic analysis, we derive analytical expressions for the transformation coefficient, as well as Liouville–Green asymptotic solutions. The obtained results are compared with the mathematically similar Landau–Zener problem in quantum mechanics.

1. Introduction

The aim of the present paper is to study coupled evolution of linear plasma waves in a shear flow. This mechanism is expected to be responsible for generation of compressible perturbations in the solar wind [5].

In a plasma with the uniform background velocity shear $\mathbf{U}_0 = (Ay, 0, 0)$ equations that describe coupled evolution of the Alfvén waves (AW) and fast magnetosonic waves (FMW) are governed by the following coupled oscillators equations [4]:

$$\frac{\mathrm{d}^2 b_y}{\mathrm{d}\tau^2} + \left[1 + K_y^2(\tau)\right] b_y = -K_y(\tau) K_z b_z, \tag{1}$$

$$\frac{\mathrm{d}^2 b_z}{\mathrm{d}\tau^2} + \left[1 + K_z^2\right] b_z = -K_y(\tau) K_z b_y. \tag{2}$$

Here, b_y and b_z are the Fourier amplitudes of the corresponding magnetic field components, K_z is the dimensionless wave number $K_z = k_z/k_x$, k_z and k_x are the components of the wave number vector, $K_y(\tau) = K_y - S\tau$ is the dimensionless wave number, $S = A/k_x V_A$ is a dimensionless shear rate, V_A is the Alfvén speed and $\tau = V_A k_x t$ is a dimensionless time.

The solutions of the characteristic equation of the set of equations (1), (2) are

$$\Omega_F^2(\tau) = 1 + K_z^2 + K_y^2(\tau),\ \Omega_A^2 = 1. \tag{3}$$

They can be easily identified as the frequencies of FMW and AW, respectively.

In the next section we present detailed analysis of equations (1), (2). We study the phenomenon of a mutual transformation of wave modes and derive analytical expression for the transformation coefficient.

2. Asymptotic Analysis

It is well known from the theory of coupled oscillator systems that if inhomogeneity is weak enough (in the considered case the condition implies that the normalized shear rate should be small $S \ll \Omega_A = 1$) and the frequencies of the modes are not close to each other (in the case under consideration this condition of weak coupling implies [4] $\delta \equiv |K_z|/S^{1/3} \ll 1$), then the Liouville–Green approximation [2,6,7] is valid and the asymptotic solutions of equations (1), (2) are given by the following expressions:

$$\Psi_\pm = \frac{D_{F,A\pm}}{\sqrt{\Omega_{F,A}(\tau)}} e^{\pm i \int \Omega_{F,A}(\tau)\mathrm{d}\tau},$$

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where $D_{F,A\pm}$ are the Liouville–Green amplitudes of the corresponding oscillations determined by the initial conditions. It is well known [6] that the signs $\pm$ correspond to the waves propagating along and backward with respect to the x-axis, respectively.

If one considers equations (1), (2) in a complex τ-plane, then the Liouville–Green solution is valid everywhere, except some vicinities of turning points, where $\Omega_F = 0$, and the resonant points, where $\Omega_F = \Omega_A$. If the Liouville–Green approximation is valid, then there is no energy exchange between FMWs and AWs and the energy densities of the modes satisfy the standard relation $E_{F,A\pm} = \Omega_{F,A} D^2_{F,A\pm}$. Analysis of equation (3) shows that if $S \ll 1$, the turning points are not located close to the real τ-axis, i.e., physically speaking, in this case the wave reflection is absent [3]. When solving the equation $\Omega_F = \Omega_A$, one finds that there are two second order resonant points $K_y(\tau_{1,2}) = \pm i K_z$ (the resonant point τ_1 has the order n if $(\Omega_1 - \Omega_2) \sim (\tau_1 - \tau)^{n/2}$ in the neighborhood of τ_1).

As follows from equation (3), the frequencies are closest, i.e., an effective coupling is possible only in some vicinity at the time moment when $K_y(\tau) = 0$. This means that the Liouville–Green approximation is always valid far on the left– and right-hand sides of this point. This circumstance enables to study the wave coupling based on the asymptotic analysis that is usual in the scattering theory. Assume that at the initial moment of time $K_y(0) \ll 1$ and the initial amplitudes of the modes are $D^L_{F,A}$. Denote the amplitudes on the right of the resonant area by $D^R_{F,A}$. If so, the problem reduces to the derivation of the so-called transformation coefficient T_{FA} that connects the initial and final amplitudes $T_{FA} = (D^L_F)^2/(D^R_A)^2$. Physically, T_{FA} represents a part of energy of the initial FMW transformed into the AW energy.

If the condition for the effective coupling $\delta \equiv |K_z|/S^{1/3} < 1$ is not satisfied, the transformation coefficient is exponentially small, namely [4],

$$T_{FA} \approx \frac{\pi}{2} \exp\left(-\frac{\delta^3}{3}\right). \tag{4}$$

Analytical expression for the transformation coefficients can be derived also in the opposite limit $\delta \ll 1$. In this case, it can be readily shown that b_y and b_z coincide with the eigenfunctions of FMW and AW, accurate to the terms of order K_z^2. Consequently, the terms on the right-hand sides of equations (1), (2) represent the coupling terms of the same accuracy. Since $K_z \ll S^{1/3}$, the coupling is weak, and if initially there exists only FMW, one can neglect the feedback of AW to FMW. Then, using the well-known expressions for the solution of a linear inhomogeneous second-order differential equation, in the above-considered limit ($\delta \ll 1$), we obtain

$$T_{FA} \approx 2^{2/3}\delta \int\limits_0^\infty x \sin\left(\frac{x^3}{3} - \frac{\delta^2}{2^{2/3}}x\right) dx.$$

Note that

$$\int\limits_0^\infty x \sin\left(\frac{x^3}{3} - \gamma x\right) dx \equiv \pi \frac{\partial}{\partial \gamma} Ai(-\gamma),$$

and using the expansion of the Airy function $Ai(\gamma)$ into power series [1], we finally obtain

$$T_{FA} \approx \frac{2^{2/3}\pi}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}\delta\left(1 - \frac{\Gamma\left(\frac{1}{3}\right)}{2^{7/4}3^{1/3}\Gamma\left(\frac{2}{3}\right)}\delta^4\right). \tag{5}$$

The results of numerical solution of the initial set of equations (1), (2) (solid line), as well as analytical expressions (4) (dash-dotted line) and (5) (dashed line) are presented in Figure 1. It shows that the transformation coefficient reaches its maximal value $(T^2_{FA})_{max} = 1/2$ at δ^{cr} that can be found numerically, or alternatively, by finding the maximum of the analytical expression presented by the equation (5):

$$\delta^{cr} = \left(\frac{2^{7/4}3^{1/3}\Gamma\left(\frac{2}{3}\right)}{5\Gamma\left(\frac{1}{3}\right)}\right)^{1/4}. \tag{6}$$

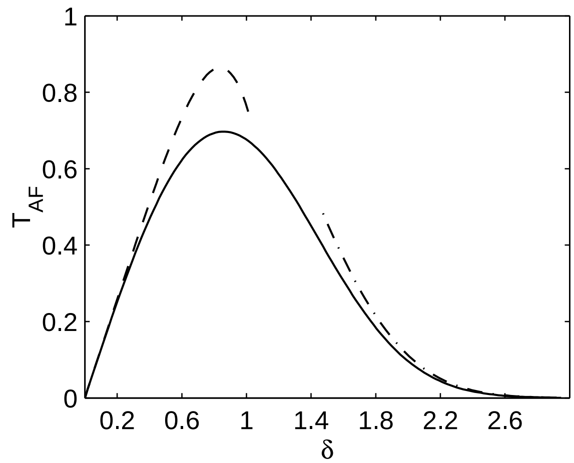

FIGURE 1. The transformation coefficient T_{FA} vs δ. Dash-dotted line and dashed line represent analytical expressions (4) and (5), respectively. Solid line is obtained by numerical solution of equations (1), (2).

Formula (6) is in a perfect accordance with the numerically calculated δ^{cr} (see Figure 1), despite the failure of equation (5) at $\delta \sim 1$. This fact can be explained as follows: the only reason why equation (5) fails is the neglect of the feedback mentioned above. The feedback changes the value of the transformation coefficient, but does not affect the value of δ^{cr}.

3. Discussion and Conclusions

It is well known (see [2, 7] and references therein) that if in the coupled oscillators system with eigenfrequencies $\Omega_{1,2}$, in the neighborhood of the real τ-axis there exist only a pair of complex conjugated first-order resonant points τ_1 and τ_2, the transformation coefficient can be derived from the exact asymptotic formula

$$T_{12} = \exp\left(- \left| \operatorname{Im} \int_{\tau_0}^{\tau_1} (\Omega_1 - \Omega_2) d\tau \right| \right). \tag{7}$$

We shall make two remarks about this equation. Firstly, it shows that in the case of the first-order resonant points only the eigenfrequencies are needed to derive the transformation coefficient. Secondly, equation (7) is valid in the case of strong wave interactions. For instance, if a complex conjugate resonant point of the first order tends to the real τ-axis, then T_{12} tends to unity, i.e., the energy of one wave mode is entirely transformed into another.

None of these properties remain valid in the case of the second order resonant points. Firstly, the transformation coefficient is small in the both limiting cases $\delta \gg 1$ and $\delta \ll 1$, i.e., when the resonant points are both close and far from the real τ-axis. Secondly, only the expressions of the eigenfrequencies are not sufficient for the derivation of the transformation coefficient, the problem needs deeper analysis. Thirdly, the maximum value of the transformation coefficient is $1/2$. This means that even in the optimal regime, only half of the energy of FMW can be transformed into AW, and vice versa. It has to be noted that the Landau–Zener theory [6] provides the same maximum value for the transition probability in the two-level quantum mechanical systems.

The last point we would like to discuss in the present paper is the comparison of our problem with the theory of quantum transitions in the two-level systems. First of all, note that equations (1), (2) correspond to the so-called quantum mechanical diabatic representation. On the other hand, the normal variables that were introduced in [3, 4], correspond to the adiabatic representation. As in the two-level quantum systems, both representations are useful for derivation of a transformation coefficient in different limits. One distinction that makes our problem different and generally more difficult is that in the area of the effective interaction the 'coupling terms' (terms on the right-hand side of equations (1), (2)) cannot be treated as constants. This circumstance does not allow to use another powerful asymptotic method, the so-called momentum representation [6, 7].

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