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A COMPUTATIONAL METHOD FOR SOLVING THE SYSTEM OF HAMILTON–JACOBI–BELLMAN PDES IN NONZERO-SUM FIXED-FINAL-TIME DIFFERENTIAL GAMES

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Abstract. In this study, the shifted Chebyshev-Gauss collocation method (SC-GCM) is used for finding the Nash equilibrium solution of nonzero-sum differential games with fixed-final-time. The search for the Nash equilibrium solutions in a feedback form usually leads to a nonlinear system of Hamilton–Jacobi–Bellman (HJB) PDEs. In the proposed approach, by applying the SC-GCM and pursuing the idea of value functions approximation, the system of HJB PDEs is reduced to a system of algebraic equations. By this method, a Nash equilibrium solution can be approximated as a function of the time and the current state by Chebyshev polynomials. The main advantage of this method is that the boundary conditions of the system of HJB PDEs can be included explicitly in the chosen approximations of value functions, which states that the boundary conditions are satisfied automatically. In view of the convergence of the method, several examples are given to demonstrate the accuracy and efficiency of the proposed method.

1. Introduction

Dynamic game is a practically significant discipline in many different fields such as engineering, ecology, management and economics. Differential game studies the situation that involves several Decision-Makers (or Players) with different objectives, where each Player looks for minimization (or maximization) of his own individual criterion. Nonzero-sum games were introduced in the works of Starr and Ho [44, 45]. For a detail treatment of differential games, we refer the reader to Nash [38], Basar and Olsder [4], Engwerda [13], Friesz [19], Yeung and Petrosyan [49] and Bressan [10].

Research in differential games is focused in the first place on extending control theory to incorporate strategic behavior [49]. Bellman's dynamic programming for solving optimal control problems leads to the Hamilton–Jacobi–Bellman (HJB) equation, which is challenging due to its inherently nonlinear nature. HJB equations have been solved by using different techniques. For example, variational iteration method was applied for nonlinear quadratic optimal control problems in [33]. Saberi and Effati [41] proposed a computational method to generate suboptimal solutions for a class of nonlinear optimal control problems.

The feedback Nash equilibrium strategies in non-zero sum games, where the strategies of players are allowed to depend on time and also on the current state, can be found by solving a highly nonlinear system of Hamilton–Jacobi–Bellman (HJB) PDEs, which are derived from the principle of dynamic programming (see, for example, [4, 10, 13, 19, 33, 41, 49]).

Due to the difficulty in solving nonlinear HJB PDEs, the existence and continuity of the feedback Nash equilibria are mainly considered in linear-quadratic dynamic games. Starr and Ho in [45] derived the sufficient conditions of the existence of a linear feedback equilibrium for a finite-horizon planning, which can be obtained via solving a system of Riccati equations. For more details on nonzero-sum linear-quadratic games see [1, 14–16, 32].

Compared to the linear-quadratic case, not many works are devoted to the nonlinear differential games. Jiménez-Lizárraga et al. [36] studied the state-dependent Riccati equations for a certain

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class of nonlinear polynomial games to obtain open-loop quasi-equilibrium. Kossiorisa et al. [34] provided a solution in a particular case of a nonlinear game representing a pollution and resource management problem. Nikooeinejad et al. employed the pseudospectral method to compute the open-loop Nash and saddle point equilibria for nonlinear nonzero-sum differential games and min-max optimal control problems (M-MOCPs) with uncertainty, respectively [39, 40]. An iterative adaptive dynamic programming method for solving a class of nonlinear zero-sum differential games is used to obtain saddle point of the zero-sum differential games (see [51]). The synchronous PI method in [47] was generalized to solve a multi-player nonzero-sum game for nonlinear continuous-time dynamic systems.

To the extent of our knowledge, the focus of the above paper is on the theoretical analysis rather than the numerical algorithms.

Although, setting up the system of HJB PDEs to obtain feedback Nash equilibrium solutions is not difficult, but in general the difficulty in solving the system of HJB equations remains the biggest problem to the practical application of nonlinear systems.

The methods from numerical analysis, such as Galerkin's method, can be used to convert the HJB equations from a continuous operator to a discrete problem. The existing references in this area to solve the Hamilton-Jacobi-Isaacs (HJI) equations for zero-sum differential games include Georges [20], Beard [5–7], Alamir [2], and Ferreira [17]. Disadvantage of Galerkin's method is that the evaluation of coefficients depends on the computation of definite integrals.

Our goal of this paper is to introduce a simple computational method that is able to address nonlinear system dynamics. The pseudospectral or collocation methods are the one of best tools for solving ordinary or partial differential equations with a high accuracy [11, 21–31, 37, 50]. A simple way to approximate the value functions of each player is by defining as a linear combinations of polynomial basis functions, and equalizing the residual functions to zero at collocation points to search for the associated coefficients. In this approach, Runge's phenomenon shows that the selection of nodes and the choice of basis function play an important role in the quality of the approximation. The shifted Jacobi polynomials are a well-known class of polynomials exhibiting exponential or sometimes super-exponential convergence, of which particular examples are the first and second kinds of Chebyshev and Legendre polynomials [8, 12, 43]. It is shown that by selecting a limited number of shifted Chebyshev collocation points, the excellent numerical results are obtained. The solution to the system of HJB PDEs (or the value functions for each Player) must be satisfied in the boundary conditions, therefore, the boundary conditions play a much more crucial role in the chosen form for the value functions approximation. In the present paper we intend to extend a simple and efficient numerical method based on value functions approximation and shifted Chebyshev-Gauss collocation method for finding Nash equilibrium solutions of nonzero-sum differential games.

The remainder of this paper is organized as follows. In Section 2, we introduce the nonzero-sum dynamic games and the formulation of the system of HJB PDEs. Some preliminary details about the SC-GCM are given in Section 3. In Section 4, the presented technique is used to approximate the value functions and the Nash equilibrium solutions of nonzero-sum dynamic games. Some numerical examples are given in Section 5 to show the efficiency of the proposed method. Finally, a brief conclusion is drawn in Section 6.

2. Problem Statement

Consider an $n-$person nonzero-sum differential game, where the players' dynamics is governed by the following nonlinear differential equation [4, 49]:

$$\begin{aligned} &\dot{x}(t) = f(t, x(t), u_1(t), u_2(t), \ldots, u_n(t)), \quad t \in [t_0, T], \\ &x(t_0) = x_0, \end{aligned} \tag{1}$$

where $f(t, x(t), u_1(t), u_2(t), \ldots, u_n(t)) = f_0(x(t)) + \sum_{j=1}^{n} g_j(x(t))u_j(t)$. We assume that $f_0(0) = 0$, $f_0(x)$ and $g_j(x)$ are Lipschitz continuous on a compact set $\Omega \in \mathbb{R}^m$ containing the origin, and the system is stabilizable on Ω. Define the finite horizon cost functions associated with Player i as:

$$J_k(u_1, u_2, \ldots, u_n) = \int_{t_0}^{T} L_k(t, x(t), u_1(t), \ldots, u_n(t))dt + \psi_k(x(T)),$$

where $L_k(t, x(t), u_1(t), \ldots, u_n(t)) = x^T Q_k x + \sum_{j=1}^{n} u_j^T R_{kj} u_j$, $x(t) \in \mathbb{R}^m$ is the state vector of the game, $u_k(t) \in U_k \subset \mathbb{R}^{m_k}$ is the control function implemented by the k-th Player and $Q_k \in \mathbb{R}^{m\times m}$, $R_{kj} \in \mathbb{R}^{m_j \times m_j}$ are symmetric positive definite matrices. Also, the functions $f_0(x)$, $g_k(x)$ and $\psi_k(x)$ for $k = 1, 2, \ldots, n$ are the differentiable functions.

It is desirable to find the optimal control vector $\{u_1^*, u_2^*, \ldots, u_n^*\}$ such that for $k = 1, 2, \ldots, n$, controls u_k^* are continuous, u_k^* stabilize (1) on Ω, $\forall x_0 \in \Omega$, $J_k(u_1^*, u_2^*, \ldots, u_n^*)$ are finite, and the cost functions (2) are minimized.

The control vector $\{u_1^*, u_2^*, \ldots, u_n^*\}$ corresponds to the Nash equilibrium solution of the game.

To find Nash equilibrium solutions, we need to consider a family of problems having a unique Nash equilibrium solution. Here we describe an important class of problems where this assumption is satisfied.

Lemma 2.1 ([10]). *Assume that the dynamics and the running costs take the decoupled form*

$$f(t, x, u_1, \ldots, u_n) = f_0(x) + \sum_{k=1}^{n} g_k(x)u_k, \tag{2}$$

$$L_k(t, x, u_1, \ldots, u_n) = \sum_{j=1}^{n} L_{kj}(t, x, u_j), \quad k = 1, \ldots, n.$$

Also, assume that

(i) *The domains $U_k(k = 1, \ldots, n)$ are closed and the convex subsets of R^{m_k} are, possibly, unbounded.*
(ii) *The functions $g_k(x)$ depend continuously on t, x.*
(iii) *The functions $u_k \mapsto L_{kk}(t, x, u_k)$ are strictly convex.*
(iv) *For each $k = 1, \ldots, n$, either U_k is compact, or L_{kk} has superlinear growth*

$$\lim_{|\omega|\to\infty} \frac{L_{kk}(t, x, \omega)}{u_k} = +\infty, \quad k = 1, \ldots, n.$$

Then for every $(t, x) \in [0, T] \times R^m$ and any vector $p_k \in R^m(k = 1, \ldots, n)$, there exists a unique set $(u_1^*(t), \ldots, u_n^*(t)) \in U_1 \times \cdots \times U_n$ such that

$$u_k^* = \arg\min_{\omega\in U_k} \{L_{kk}(t, x, \omega) + p_k.g_k(x)\omega\}.$$

We consider here the case, where both players can observe the current state of the system. The value functions $V_k(t, x)$, $k = 1, 2, \ldots, n$ associated with the admissible control policies $u_k \in U_k$ are defined as follows:

$$V_k(t, x) = \min_{u_k\in U_k} \left\{ \int_{t}^{T} L_k(t, x, u_1, \ldots, u_n)dt + \psi_k(x(T))\right\}. \tag{3}$$

Assume that the value functions (3) are continuously differentiable. By Bellman's optimality and the dynamic programming principle, the optimal cost functions defined in (3) are satisfied the following system of Hamilton–Jacobi–Bellman (HJB) PDEs:

$$\begin{aligned} 0 =& \frac{\partial}{\partial t} V_k(t, x) \\ &+ \min_{u_k\in U_k} \left\{ L_k(t, x, u_1, \ldots, u_n) + \Big(\frac{\partial}{\partial x} V_k(t, x)\Big)^T \Big(f_0(x) + \sum_{j=1}^{n} g_j(x)u_j \Big) \right\}, \\ k =& 1, 2, \ldots, n, \end{aligned} \tag{4}$$

with the boundary conditions $V_k(T,x)=\psi_k(x),\ k=1,2,\ldots,n$. Define the Hamiltonian functions as

$$H_k\left(t,x,u_1,\ldots,u_n,\frac{\partial}{\partial x}V_k\right)=\left(\frac{\partial}{\partial x}V_k\right)^T\left(f_0(x)+\sum_{j=1}^{n}g_j(x)u_j\right)$$
$$+L_k(t,x,u_1,\ldots,u_n),\quad k=1,2,\ldots,n.$$

Then the associated state feedback control policies can be obtained by

$$\frac{\partial H_k}{\partial u_k}=0\Rightarrow u_k^*(t,x)=-\frac{1}{2}R_{kk}^{-1}g_k^T(x)\frac{\partial}{\partial x}V_k(t,x),\quad k=1,2,\ldots,n. \tag{5}$$

Substitution of (5) into (4) yields the following n-coupled HJB equations:

$$\begin{aligned}0=&\frac{\partial}{\partial t}V_k(t,x)+x^TQ_kx+\left(\frac{\partial}{\partial x}V_k(t,x)\right)^Tf_0(x)\\&-\frac{1}{2}\left(\frac{\partial}{\partial x}V_k(t,x)\right)^T\sum_{j=1}^{n}g_j(x)R_{jj}^{-1}g_j^T(x)\frac{\partial}{\partial x}V_j(t,x)\\&+\frac{1}{4}\sum_{j=1}^{n}\left(\frac{\partial}{\partial x}V_j(t,x)\right)^Tg_j(x)R_{jj}^{-1}R_{jj}R_{jj}^{-1}g_j^T(x)\frac{\partial}{\partial x}V_j(t,x),\\&k=1,2,\ldots,n,\end{aligned} \tag{6}$$

with the boundary conditions

$$V_k(T,x)=\psi_k(x),\quad k=1,2,\ldots,n. \tag{7}$$

The system of Hamilton–Jacobi–Bellman PDE equations (6) with boundary conditions (7) cannot generally be solved due to its nonlinear nature. We intend to solve system (6) and (7) by the shifted Chebyshev-Gauss collocation method (SC-GCM).

3. Some Properties of Chebyshev Polynomials

In this section, we introduce some basic properties of the Chebyshev polynomials that we use in the CSCM as the function approximation structures.

The Chebyshev polynomials $T_n(z), n=0,1,2,\ldots$ are the eigenfunctions of the singular Sturm-Liouville problem

$$(1-z^2)T_n''(z)-zT_n'(z)+n^2T_n(z)=0.$$

They are orthogonal with respect to the L_w^2 inner product on the interval $[-1,1]$ with the weight function $w(z)=\frac{1}{\sqrt{1-z^2}}$. The Chebyshev polynomials satisfy the recurrence formula as follows:

$$T_{n+1}(z)=2zT_n(z)-T_{n-1}(z),\quad n=1,2,\ldots,$$

where $T_0(z)=1$ and $T_1(z)=z$. For practical use of the Chebyshev polynomials on the interval $[a,b]$, it is necessary to shift the defining domain by the following variable substitution:

$$z=\frac{2}{b-a}t-\frac{b+a}{b-a}.$$

Let the shifted Chebyshev polynomials $T_n(\frac{2}{b-a}t-\frac{b+a}{b-a})$ be denoted by $T_n^*(t)$. Then these polynomials can be obtained by using the following recurrence formula:

$$T_{n+1}^*(t)=\left(4\left(\frac{t}{b-a}\right)-2\left(\frac{b+a}{b-a}\right)\right)T_n^*(t)-T_{n-1}^*(t),\quad n=1,2,\ldots,$$

where $T_0^*(t)=1$ and $T_1^*(t)=\frac{2}{b-a}t-\frac{b+a}{b-a}$.

Now, let the shifted Chebyshev polynomials $T_n(\frac{2}{b-a}t-\frac{b+a}{b-a})$ and $T_n(\frac{2}{d-c}x-\frac{d+c}{d-c})$ be denoted by $T_n^*(t)$ and $T_n^*(x)$, respectively.

Similarly, an arbitrary function of two variables $f(t,x) \in L_w^2([a,b]\times[c,d])$, can be approximated by the shifted Chebyshev polynomials as:

$$f(t,x) \simeq \tilde{f}(t,x) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} f_{ij} T_i^*(t) T_j^*(x),$$

with

$$f_{ij} = \frac{4}{\pi^2 c_i c_j} \int_{-1}^{1}\int_{-1}^{1} \frac{f(\frac{b-a}{2}t + \frac{b+a}{2}, \frac{d-c}{2}x + \frac{d+c}{2}) T_i(t) T_j(x)}{\sqrt{1-t^2}\sqrt{1-x^2}} dt dx,$$
$$i = 0,1,\ldots,N_1, \quad j = 0,1,\ldots,N_2.$$

The fundamental results of the proposed method are based on the remarkable Weierstrass Theorem and approximability of orthogonal polynomials [9, 43].

Theorem 3.1 ([42])**.** *If the function $f(t,x)$ has the second order continuous derivatives, then*

$$|f_{i,0}| \le \frac{2\gamma_{2,0}}{(i-1)^2}, \quad |f_{i,1}| \le \frac{8\gamma_{2,0}}{\pi(i-1)^2}, \quad i > 1,$$
$$|f_{0,j}| \le \frac{2\gamma_{0,2}}{(j-1)^2}, \quad |f_{1,j}| \le \frac{8\gamma_{0,2}}{\pi(j-1)^2}, \quad j > 1,$$

where $f(t,x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{ij} T_i(t) T_j(x)$, $\tilde{f}(t,x) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} f_{ij} T_i(t) T_j(x)$, $\gamma_{2,0} \ge \max\{|\frac{\partial^2 f}{\partial t^2}(t,x)| : t,x \in [-1,1]\}$, and $\gamma_{0,2} \ge \max\{|\frac{\partial^2 f}{\partial x^2}(t,x)| : t,x \in [-1,1]\}$.

Theorem 3.2 ([42])**.** *If the function $f(t,x)$ has the second order continuous partial derivatives, then $\lim_{N_1,N_2\to\infty} \tilde{f}(t,x) = f(t,x)$ uniformly in $[-1,1]$ and*

$$|f(t,x) - \tilde{f}(t,x)| \le \sqrt{6}\left(\frac{20\gamma_{0,2}^2}{(N_2-1)^2} + \frac{20\gamma_{2,0}^2}{(N_1-1)^2} + \frac{\pi^2\gamma_{1,1}^2}{6N_2} + \frac{\pi^2\gamma_{1,1}^2}{6N_1}\right)^{\frac{1}{2}}.$$

For obtaining the first partial derivatives $\frac{\partial}{\partial t}\tilde{f}(t,x)$ and $\frac{\partial}{\partial x}\tilde{f}(t,x)$, we can rewrite $\tilde{f}(t,x)$ as:

$$\tilde{f}(t,x) = \sum_{i=0}^{N_1} A_i(x) T_i^*(t), \quad \text{with} \quad A_i(x) = \sum_{j=0}^{N_2} f_{ij} T_j^*(x),$$

or

$$\tilde{f}(t,x) = \sum_{j=0}^{N_2} B_j(t) T_j^*(x), \quad \text{with} \quad B_j(t) = \sum_{i=0}^{N_1} f_{ij} T_i^*(t).$$

Then the first partial derivatives of $\tilde{f}(t,x)$ can be obtained as:

$$\frac{\partial}{\partial t}\tilde{f}(t,x) = \frac{2}{b-a}\sum_{i=0}^{N_1} A_i^{(1)}(x) T_i^*(t), \tag{8}$$

$$\frac{\partial}{\partial x}\tilde{f}(t,x) = \frac{2}{d-c}\sum_{j=0}^{N_2} B_j^{(1)}(t) T_i^*(x), \tag{9}$$

where the coefficients $A_i^{(1)}(x)$, $i = 0,1,\ldots,N_1$ and $B_j^{(1)}(t)$, $j = 0,1,\ldots,N_2$ are:

$$A_i^{(1)}(x) = \frac{2}{c_i} \sum_{\substack{p=i+1 \\ (p+i)\ odd}}^{N_1} p A_p(x), \ i = 0,\ldots,N_1 - 1, \ A_{N_1}^{(1)}(x) = 0, \tag{10}$$

$$B_j^{(1)}(t) = \frac{2}{c_j} \sum_{\substack{q=j+1 \\ (p+j)\ odd}}^{N_2} q B_q(t), \ j = 0,\ldots,N_2 - 1, \ B_{N_2}^{(1)}(t) = 0. \tag{11}$$

4. Nonlinear Fixed-Final-Time n-Coupled HJB Solution by the Collocation Method

The n-coupled HJB equations (6) and (7) are difficult to solve for the cost functions $V_k(t,x)$. In this section, the direct collocation method is used to solve approximately the value functions in (6) over Ω by approximating the cost functions $V_k(t,x)$ and their partial derivatives as Chebyshev polynomials. We assume that $V_k(t,x)$, $k=1,2,\dots,n$ are smooth. Therefore, one can use approximate cost functions $V_k(t,x)$ for $t\in[0,T]$ and a compact set $\Omega\subset\mathbb{R}^m$ as follows:

$$\begin{aligned} V_k(t,x) \simeq&\widetilde{V}_k(t,x)\\ =&(t-t_{N_1})\left(\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} v_{ij}^k T_i^*(t)T_j^*(x)\right)+\psi_k(x), \quad k=1,2,\dots,n. \end{aligned} \tag{12}$$

Before describing the method, it should be pointed out that this method is introduced for $x\in\mathbb{R}$, however, it can be extended easily to $x\in\mathbb{R}^m$. Our aim is to approximate the solution of system (6) and (7) for the time horizon $[t_0,T]$ and the state domain $\Omega=[x_{\min},x_{\max}]$. So, we define:

$$\begin{aligned} t_r =&\frac{T-t_0}{2}\left(\cos\left(\frac{(N_1-r)\pi}{N_1}\right)\right)+\frac{T+t_0}{2}, \quad r=0,1,\dots,N_1, \\ x_s =&\frac{x_{\max}-x_{\min}}{2}\left(\cos\left(\frac{(N_2-s)\pi}{N_2}\right)\right)+\frac{x_{\max}+x_{\min}}{2}, \quad s=0,1,\dots,N_2, \end{aligned} \tag{13}$$

which are named as shifted Chebyshev–Gauss–Lobatto nodes. In fact, these points are zeros of the $(t-t_0)(T-t)\dot{T}^*_{N_1}(t)$ and $(x-x_{\min})(x_{\max}-x)\dot{T}^*_{N_2}(x)$, respectively.

By the grid points defined in (13), and substituting t_{N_1} into (12), we have:

$$\widetilde{V}_k(t_{N_1},x)=\widetilde{V}_k(T,x)=\psi_k(x), \quad k=1,2,\dots,n,$$

which guarantee the boundary conditions for the cost functions $V_k(t,x)$ and for $k=1,2,\dots,n$ are satisfied automatically.

In addition, from equations (8) and (9), we can get the partial derivatives $\frac{\partial}{\partial t}\widetilde{V}_k(t,x)$ and $\frac{\partial}{\partial x}\widetilde{V}_k(t,x)$ as follows:

$$\begin{aligned} \frac{\partial}{\partial t}\widetilde{V}_k(t,x) =&\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} v_{ij}^k T_i^*(t)T_j^*(x)+\frac{2(t-t_{N_1})}{T-t_0}\sum_{i=0}^{N_1}A_i^{(1)}(x)T_i^*(t),\\ \frac{\partial}{\partial x}\widetilde{V}_k(t,x) =&\frac{2(t-t_{N_1})}{x_{\max}-x_{\min}}\sum_{j=0}^{N_2}B_j^{(1)}(t)T_j^*(x)+\frac{\partial\psi_k(x)}{\partial x},\\ k =&1,2,\dots,n, \end{aligned}$$

where the coefficients $A_i^{(1)}(x)$, $i=0,1,\dots,N_1$ and $B_j^{(1)}(t)$, $j=0,1,\dots,N_2$ can be obtained from equations (10) and (11).

Approximating $V_k(t,x)$, $\frac{\partial}{\partial t}V_k(t,x)$ and $\frac{\partial}{\partial x}V_k(t,x)$ in the n-coupled HJB equations (6) by $\widetilde{V}_k(t,x)$, $\frac{\partial}{\partial t}\widetilde{V}_k(t,x)$ and $\frac{\partial}{\partial x}\widetilde{V}_k(t,x)$, respectively, we have

$$\begin{aligned} \mathrm{Res}_{V_k}(t,x) =&\frac{\partial}{\partial t}\widetilde{V}_k(t,x)+x^TQ_ix+\left(\frac{\partial}{\partial x}\widetilde{V}_k(t,x)\right)^T f_0(x)\\ &-\frac{1}{2}\left(\frac{\partial}{\partial x}\widetilde{V}_k(t,x)\right)^T\sum_{j=1}^{n}g_j(x)R_{jj}^{-1}g_j^T(x)\frac{\partial}{\partial x}\widetilde{V}_j(t,x)\\ &+\frac{1}{4}\sum_{j=1}^{n}\left(\frac{\partial}{\partial x}\widetilde{V}_j(t,x)\right)^T g_j(x)R_{jj}^{-1}R_{jj}R_{jj}^{-1}g_j^T(x)\frac{\partial}{\partial x}\widetilde{V}_j(t,x),\\ &k=1,2,\dots,n, \end{aligned}$$

where $\mathrm{Res}_{V_k}(t,x)$, $k=1,2,\dots,n$ are the residual equations error. To find the coefficients v_{ij}^ks, the method of weighted residuals is used.

Consider the expression

$$< \mathrm{Res}_{V_k}(t,x), W_{r,s} >= \left[\int_{t_0}^{T}\int_{\Omega} \mathrm{Res}_{V_k}(t,x).W_{r,s}d\Omega dt\right], \tag{14}$$

where $W_{r,s}$, $r = 0,1,\ldots,N_1$, $s = 0,1,\ldots,N_2$ are the suitable functions.

The coefficients v_{ij}^ks will be selected to minimize residual equations error in a collocation sense over a set of points sampled from a compact set $[t_0,T]\times\Omega$.

To this end, the coefficients v_{ij}^ks are determined by projecting the residual errors onto the Dirac delta function and setting the results to zero $\forall x \in \Omega$ and $t \in [0,T]$.

Setting $P^{rs} = (t_r, x_s)$, we define:

$$W_{r,s} = \delta(P^{rs}), \quad r = 0,1,\ldots,N_1, \quad s = 0,1,\ldots,N_2, \tag{15}$$

where $\delta(P^{rs})$ is the Dirac delta function. By substituting (15) into (14), the coefficients v_{ij}^ks are obtained from equalizing $\mathrm{Res}_{V_k}(t_r,x_s)$ to zero at the collocation points as follows:

$$\begin{aligned}&< \mathrm{Res}_{V_k}(t,x), \delta(P^{rs}) >= \mathrm{Res}_{V_k}(t_r,x_s) = 0,\\ &r = 0,1,\ldots,N_1, \quad s = 0,1,\ldots,N_2, \quad k = 1,2,\ldots,n.\end{aligned} \tag{16}$$

Equations (16) generate a set of $n(N_1+1)(N_2+1)$ nonlinear algebraic equations that can be solved by the Newton method for the unknown coefficients v_{ij}^ks. Consequently, the cost functions $\widetilde{V}_k(t,x)$, $k = 1,2,\ldots,n$ can be calculated.

From (5), the corresponding Nash equilibrium solutions as a function of the time and the state are approximated as:

$$\widetilde{u}_k(t,x) = -\frac{1}{2}R_{kk}^{-1}g_k^T(x)\frac{\partial}{\partial x}\widetilde{V}_k(t,x), \quad k = 1,2,\ldots,n.$$

5. Illustrative Example

To demonstrate the application of the shifted Chebyshev-Gauss collocation method (SC-GCM) and its performance for finding feedback Nash equilibrium solution of nonzero-sum dynamic games, several examples are examined in this section. Example 5.1 is a linear-quadratic dynamic game that can be solved analytically. This allows one to verify the validity of the method by comparing with the results of exact solution. The analytic solution for Examples 5.2 and 5.3 is unachievable. It should be noted that for Example 5.2, the results obtained by the proposed method coincide with those obtained by the variables separation method.

Example 5.1. Consider the linear-quadratic nonzero-sum differential game defined by the system [4]

$$\dot{x}(t) = \sqrt{2}u_1(t) - u_2(t), \quad x(0) = 1, \quad 0 \le t \le T = 2,$$

and the performance criteria of Players 1 and 2 as follows:

$$\begin{aligned}\min_{u_1} J_1 &= \int_0^T (u_1^2(t) - u_2^2(t))dt + \frac{1}{2}x(T)^2,\\ \min_{u_2} J_2 &= \int_0^T (u_2^2(t) - u_1^2(t))dt - \frac{1}{2}x(T)^2.\end{aligned}$$

The exact solution for the feedback Nash equilibrium of this problem is

$$\begin{aligned}V_1(t,x) &= -V_2(t,x) = \frac{x^2}{2(3-t)},\\ u_1^*(t,x) &= \frac{-\sqrt{2}x}{3-t},\\ u_2^*(t,x) &= \frac{-x}{3-t}.\end{aligned}$$

TABLE 1. The numerical optimal value of cost functionals $J_i, i = 1, 2$ obtained by using the SC-GCM as compared with the exact solutions for Example 5.1

$(N_1,\ N_2)$	J_1	J_2	$\mid J_i - J_i^* \mid, i = 1, 2$
(2, 2)	0.1708622317	-0.1708622317	0.0041955650
(4, 2)	0.1666009734	-0.1666009734	0.0000656933
(6, 2)	0.1666658930	-0.1666658930	7.73×10^{-7}
(8, 2)	0.1666666471	-0.1666666471	1.96×10^{-8}
(10, 2)	0.1666666660	-0.1666666660	7.00×10^{-10}

As is discussed in Section 2, the HJB equations system for this problem has the following form:

$$
\begin{aligned}
V_{1,t}(t,x) + \min_{u_1} \left\{ \frac{1}{2}(u_1(t)^2 - u_2(t)^2) + V_{1,x}(t,x)(\sqrt{2}u_1(t) - u_2(t)) \right\} = 0, \\
V_{2,t}(t,x) + \min_{u_2} \left\{ \frac{1}{2}(u_2(t)^2 - u_1(t)^2) + V_{2,x}(t,x)(\sqrt{2}u_1(t) - u_2(t)) \right\} = 0,
\end{aligned}
\tag{17}
$$

with the boundary conditions

$$
V_1(2, x(2)) = -V_2(2, x(2)) = \frac{1}{2}x^2(2).
$$

The corresponding Hamiltonian functions are given in the form

$$
\begin{aligned}
H_1(t, x, u_1, u_2, V_{1,x}) &= \frac{1}{2}(u_1(t)^2 - u_2(t)^2) + V_{1,x}(t,x)(\sqrt{2}u_1(t) - u_2(t)), \\
H_2(t, x, u_1, u_2, V_{2,x}) &= \frac{1}{2}(u_2(t)^2 - u_1(t)^2) + V_{2,x}(t,x)(\sqrt{2}u_1(t) - u_2(t)).
\end{aligned}
$$

Differentiating $H_1(t, x, u_1, u_2, V_{1,x})$ and $H_2(t, x, u_1, u_2, V_{2,x})$ with respect to u_1 and u_2, respectively, and by finding the functions u_1 and u_2, where these derivatives tend to zero, we have

$$
\begin{aligned}
u_1^*(t,x) &= -\sqrt{2}V_{1,x}(t,x), \\
u_2^*(t,x) &= V_{2,x}(t,x).
\end{aligned}
$$

Now, by substituting u_1^* and u_2^* into HJB equations system (17), we have the following partial differential equations:

$$
\begin{cases}
V_{1,t}(t,x) - V_{1,x}(t,x)^2 - V_{2,x}(t,x)^2 - V_{1,x}(t,x)V_{2,x}(t,x) = 0, \\
V_{2,t}(t,x) - V_{1,x}(t,x)^2 - V_{2,x}(t,x)^2 - V_{1,x}(t,x)V_{2,x}(t,x) = 0, \\
V_{1,t}(t,x) = -V_{2,t}(t,x) = \frac{1}{2}x^2(2).
\end{cases}
\tag{18}
$$

We intend to solve the PDEs system (18) using the SC-GCM (as discussed in section 4). The numerical approximation of optimal value functions, the control solutions and state trajectory are plotted using SC-GCM for $N_1 = 10$ and $N_2 = 2$ on the computational domain $[0, 2] \times [-2, 2]$ in Figure 1. The graphs of the absolute error are also show in the same Figure 1. The exact optimal value cost functionals are $J_1^* = -J_2^* = 0.1666666667$.

Comparison of the optimal cost functionals $J_i,\ i = 1, 2$ for the SC-GCM with the exact solutions are shown in Table 1.

Example 5.2. In this example, we consider the application of differential games in competitive advertising in Sorger. There are two firms in a market and the profit of firm1 and that of 2 are respectively [49]:

$$
\max_{u_1} J_1(u_1, u_2) = \int_0^T e^{-r_1 t}[q_1 x(t) - \frac{c_1}{2}u_1^2(t)]dt + e^{-r_1 T} S_1 x(T),
\tag{19}
$$

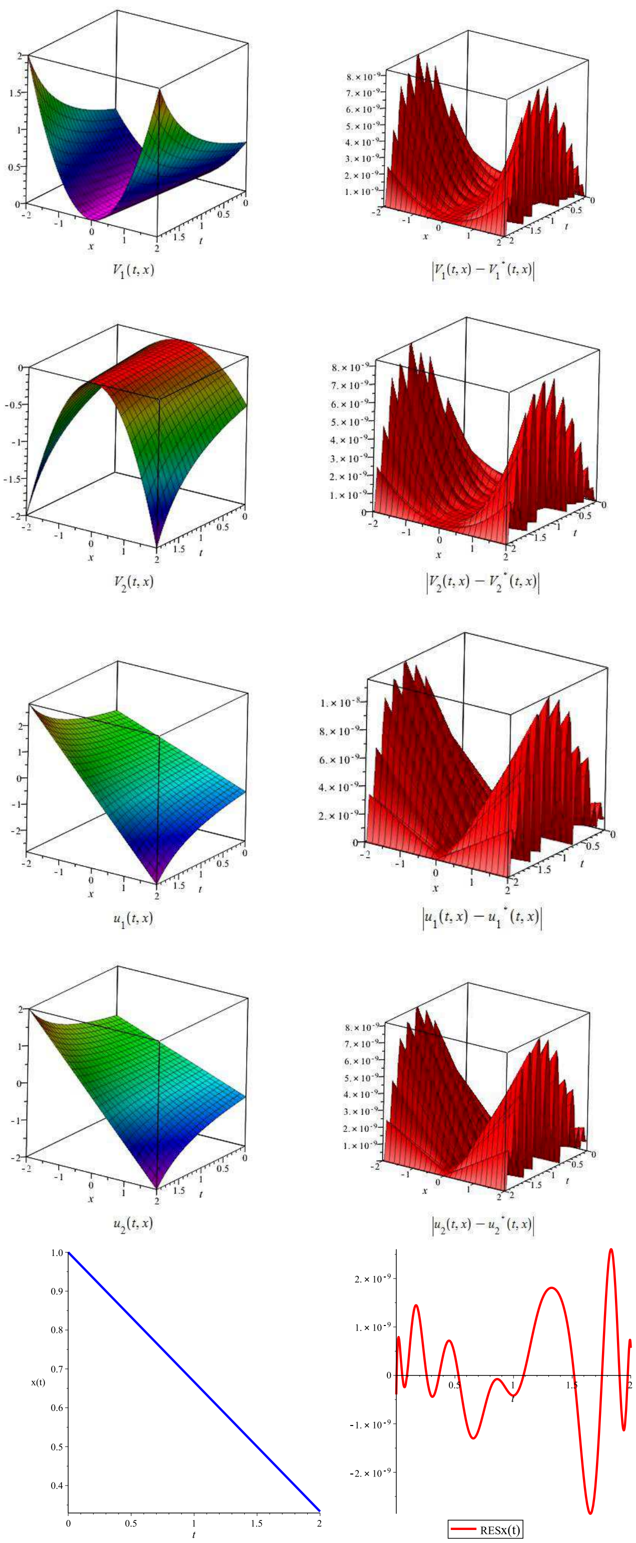

FIGURE 1. The numerical approximation of optimal value functions $V_i(t,x), i = 1,2$, control solutions $u_i(t,x), i = 1,2$, state trajectory $x(t)$ and absolute error functions, using the SC-GCM for $N_1 = 10$ and $N_2 = 2$ on the domain $[0,2] \times [-2,2]$ for Example 5.1.

and

$$\max_{u_2} J_2(u_1,u_2) = \int_0^T e^{-r_2 t}[q_2(1-x(t)) - \frac{c_2}{2}u_2^2(t)]dt + e^{-r_2 T}S_2(1-x(T)),$$

where r_i, q_i, c_i and S_i for $i=1,2$, are the positive constants. The dynamics of firms market share is governed by

$$\dot{x}(t) = u_1(t)\sqrt{1-x(t)} - u_2(t)\sqrt{x(t)}, \quad x(0)=1, \quad 0 \le x \le 1, \tag{20}$$

where $x(t)$ is the market share of firm1 at time t, $[1-x(t)]$ is that of firm2, $u_i(t)$ is advertising rate for firm $i=1,2$. A feedback solution which allows the firm to choose its advertising rates contingent upon the state of the game is a realistic approach to this problem. Invoking the dynamic programming principle, a feedback Nash equilibrium solution to the game (19)–(20) has to satisfy the following conditions:

$$\begin{aligned} V_{1,t}(t,x) + &\max_{u_1}\{e^{-r_1 t}[q_1 x(t) - \frac{c_1}{2}u_1^2] \\ &+ V_{1,x}(t,x)(u_1\sqrt{1-x(t)} - u_2^*(t,x)\sqrt{x(t)})\} = 0, \\ V_{2,t}(t,x) + &\max_{u_2}\{e^{-r_2 t}[q_2(1-x(t)) - \frac{c_2}{2}u_2^2(t)] \\ &+ V_{2,x}(t,x)(u_1^*(t,x)\sqrt{1-x(t)} - u_2\sqrt{x(t)})\} = 0, \\ &V_1(T,x) = e^{-r_1 T}S_1 x(T), \\ &V_2(T,x) = e^{-r_2 T}S_2(1-x(T)). \end{aligned} \tag{21}$$

Performing the indicated maximization in (21) yields

$$u_1^*(t,x) = \frac{V_{1,x}(t,x)}{c_1}\sqrt{1-x(t)}\exp(rt), \tag{22}$$

$$u_2^*(t,x) = \frac{-V_{2,x}(t,x)}{c_2}\sqrt{x(t)}\exp(rt). \tag{23}$$

If $q_i = S_i = 1, i=1,2$ and $T = r_1 = r_2 = 2$, upon substituting $u_1^*(t,x)$ and $u_2^*(t,x)$ from (22) and (23) into (21), we have the following system of PDEs:

$$\begin{aligned} V_{1,t}(t,x) + &x\exp(-2t) \\ &+ \frac{1}{2}(1-x)\exp(2t)V_{1,x}(t,x)^2 + x\exp(2t)V_{1,x}(t,x)V_{2,x}(t,x) = 0, \\ V_{2,t}(t,x) + &(1-x)\exp(-2t) \\ &+ \frac{1}{2}x\exp(2t)V_{2,x}(t,x)^2 + (1-x)\exp(2t)V_{1,x}(t,x)V_{2,x}(t,x) = 0, \\ V_1(2,x) =& e^{-4}x \\ V_2(2,x) =& e^{-4}(1-x). \end{aligned} \tag{24}$$

The numerical approximation of optimal value functions, residual errors and numerical approximation of control solutions for Player1 and Player2 are plotted by using the SC-GCM for $N_1 = 10$ and $N_2 = 2$ on the computational domain $[0,2]\times[0,1]$ in Figure 2. To solve the partial differential equations system (24) by separation variables method, we try a solution of the form

$$\begin{aligned} V_1(t,x) =& \exp(-2t)[A_1(t)x + B_1(t)], \\ V_2(t,x) =& \exp(-2t)[A_2(t)x + B_2(t)], \end{aligned}$$

where $A_1(t), B_1(t), A_2(t)$ and $B_2(t)$ satisfy

$$\dot{A}_1(t) = \frac{1}{2}A_1(t)^2 - A_1(t)A_2(t) - 2A_1(t) + 1, \ A_1(2) = 1, \tag{25}$$

$$\dot{B}_1(t) = -\frac{1}{2}A_1(t)^2 + B_1(t), \ B_1(2) = 0, \tag{26}$$

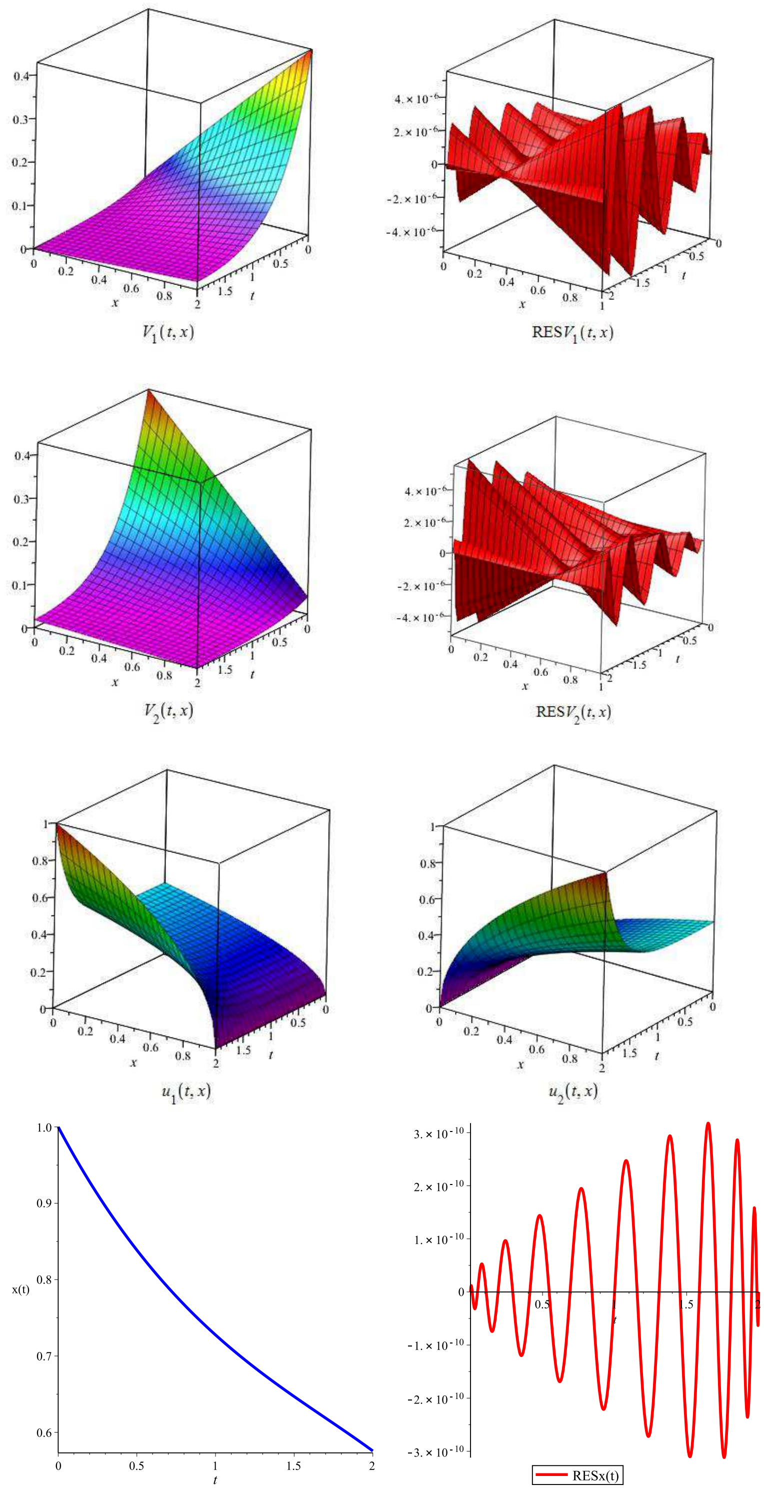

FIGURE 2. The numerical approximation of optimal value functions $V_i(t,x), i = 1, 2,$ the residual errors $RES_{V_i}(t,x), i = 1, 2,$ control solutions $u_i(t,x), i = 1, 2,$ state trajectory $x(t)$ and $RES_x(t)$, using the SC-GCM for $N_1 = 10$ and $N_2 = 2$ on the domain $[0, 2] \times [0, 1]$ for Example 5.2.

TABLE 2. Optimal value of cost functionals $J_i, i = 1, 2$ obtained using the SC-GCM as compared with that of obtained by the SVM for Example 5.2

(N_1, N_2)	$J_{1SC-GCM}$	$J_{2SC-GCM}$
(2, 1)	0.4288819515	0.0406674857
(4, 1)	0.4285865391	0.0402552676
(6, 1)	0.4285683563	0.0402927736
(8, 1)	0.4285702876	0.0402936397
(10, 2)	0.4285703954	0.0402937017

$J_{1SVM} = 0.4285704042$, $J_{2SVM} = 0.0402937073$

$$\dot{A}_2(t) = -\frac{1}{2}A_2(t)^2 + A_1(t)A_2(t) + 2A_2(t) + 1, \ A_2(2) = -1, \tag{27}$$

$$\dot{B}_2(t) = -A_1(t)A_2(t) + 2B_2(t) - 1, \ B_2(2) = 1. \tag{28}$$

If ordinary differential equations system (25)–(28) has a solution, then the optimal control strategies as a function of the time and the current state are given in the form

$$u_1^*(t, x) = A_1(t)\sqrt{1 - x}\exp(2t),$$
$$u_2^*(t, x) = -A_2(t)\sqrt{x}\exp(2t).$$

The SC-GCM method is also applied to solve the ordinary differential equations system (25)–(28) for $N = 10$ on the domain $[0, 2]$. Comparison of the optimal cost functionals $J_i, i = 1, 2$ obtained by the SC-GCM and the separation variables method (SVM) is shown in Table 2.

Example 5.3. The following example corresponds to a nonlinear electrical circuit managed by two electric companies, which employ different costs for the consumed electric energy. The purpose of the game problem is to minimize the energy cost for each company [36].

Consider the following nonlinear polynomial game:

$$\dot{x}_1(t) = x_2(t), \ x_1(0) = 1,$$
$$\dot{x}_2(t) = x_1^2(t) + u_1(t) + u_2(t), \ x_2(0) = 1, \tag{29}$$

with the finite-time quadratic cost functions

$$\min_{u_1} J_1(u_1, u_2) = \frac{1}{2}(0.1x_1^2(T) + x_2^2(T)) + \frac{1}{2}\int_0^T (0.1x_1^2(t) + x_2^2(t) + u_1^2(t) + u_2^2(t))dt, \tag{30}$$

$$\min_{u_2} J_2(u_1, u_2) = \frac{1}{2}(x_1^2(T) + 0.1x_2^2(T)) + \frac{1}{2}\int_0^T (x_1^2(t) + 0.1x_2^2(t) + u_1^2(t) + u_2^2(t))dt. \tag{31}$$

Invoking dynamic programming principle, a feedback Nash equilibrium solution to the game (29)–(31) has to satisfy the following conditions:

$$V_{1,t}(t, x_1, x_2) + \min_{u_1}\Big\{\frac{1}{2}(0.1x_1^2 + x_2^2 + u_1^2 + (u_2^*(t, x_1, x_2)^2) + V_{1,x_1}(t, x_1, x_2)(x_2) + V_{1,x_2}(t, x_1, x_2)\left(x_1^2 + u_1 + u_2^*(t, x_1, x_2)\right)\Big\} = 0,$$

$$V_{2,t}(t, x_1, x_2) + \min_{u_2}\Big\{\frac{1}{2}\left(x_1^2 + 0.1x_2^2 + (u_1^*(t, x_1, x_2)^2 + u_2^2\right) + V_{2,x_1}(t, x_1, x_2)(x_2) + V_{2,x_2}(t, x_1, x_2)\left(x_1^2 + u_1^*(t, x_1, x_2 + u_2)\right)\Big\} = 0,$$

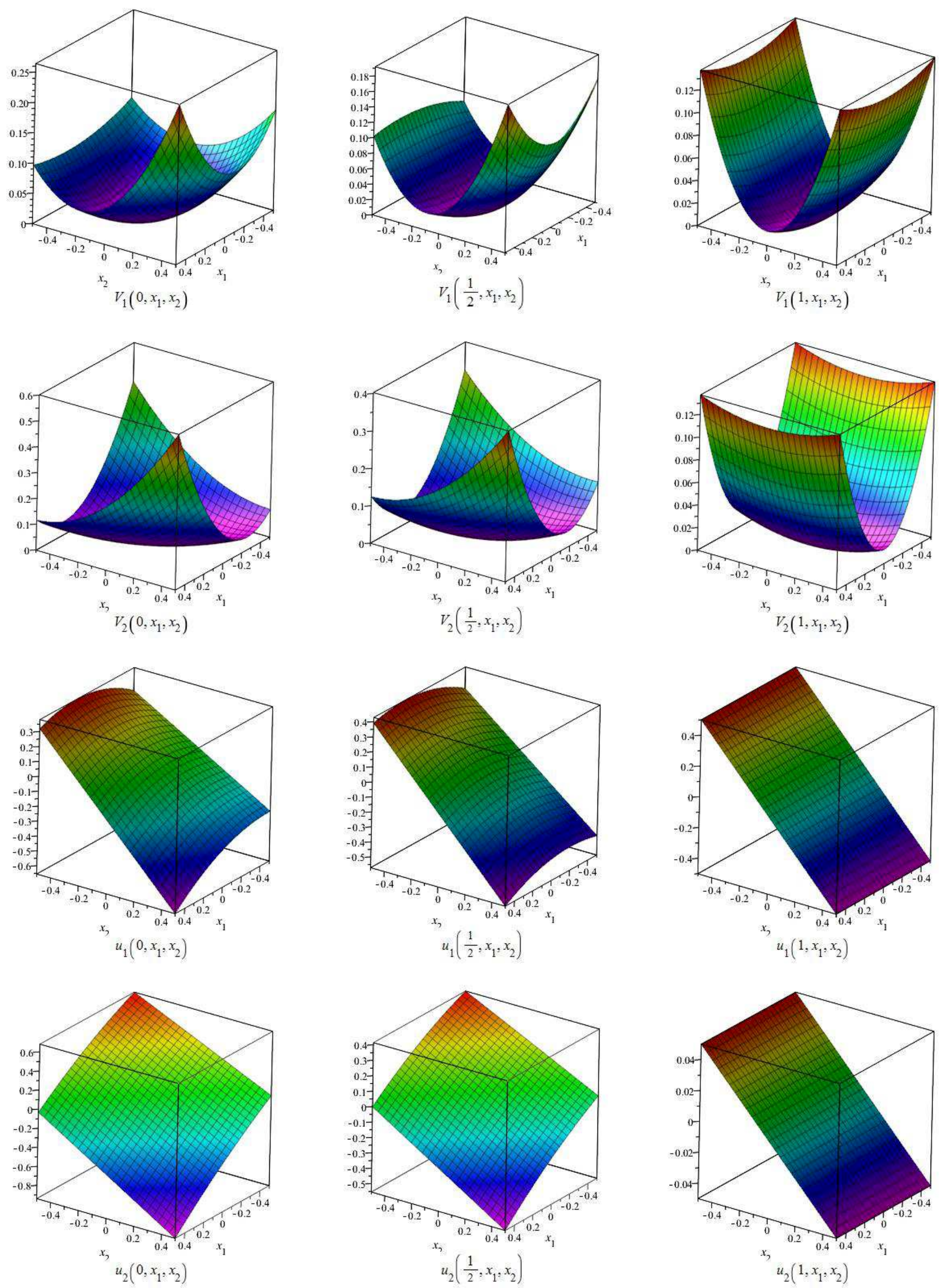

FIGURE 3. The numerical approximation of optimal value functions $V_i(t, x_1, x_2), i = 1, 2,$ control solutions $u_i(t, x_1, x_2),\ i = 1, 2$ using the SC-GCM for $t = 0, \frac{1}{2}, 1,$ and $N_1 = 6, N_2 = 4, N_3 = 4$ on the domain $[0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ for Example 5.3.

$$
\begin{aligned}
V_1(T, x_1, x_2) =& \frac{1}{2}\left(0.1x_1^2 + x_2^2\right), \\
V_2(T, x_1, x_2) =& \frac{1}{2}\left(x_1^2 + 0.1x_2^2\right).
\end{aligned} \tag{32}
$$

Performing the indicated minimization in (32) yields:

$$u_1^*(t, x_1, x_2) = -V_{1,x_2}(t, x_1, x_2),$$

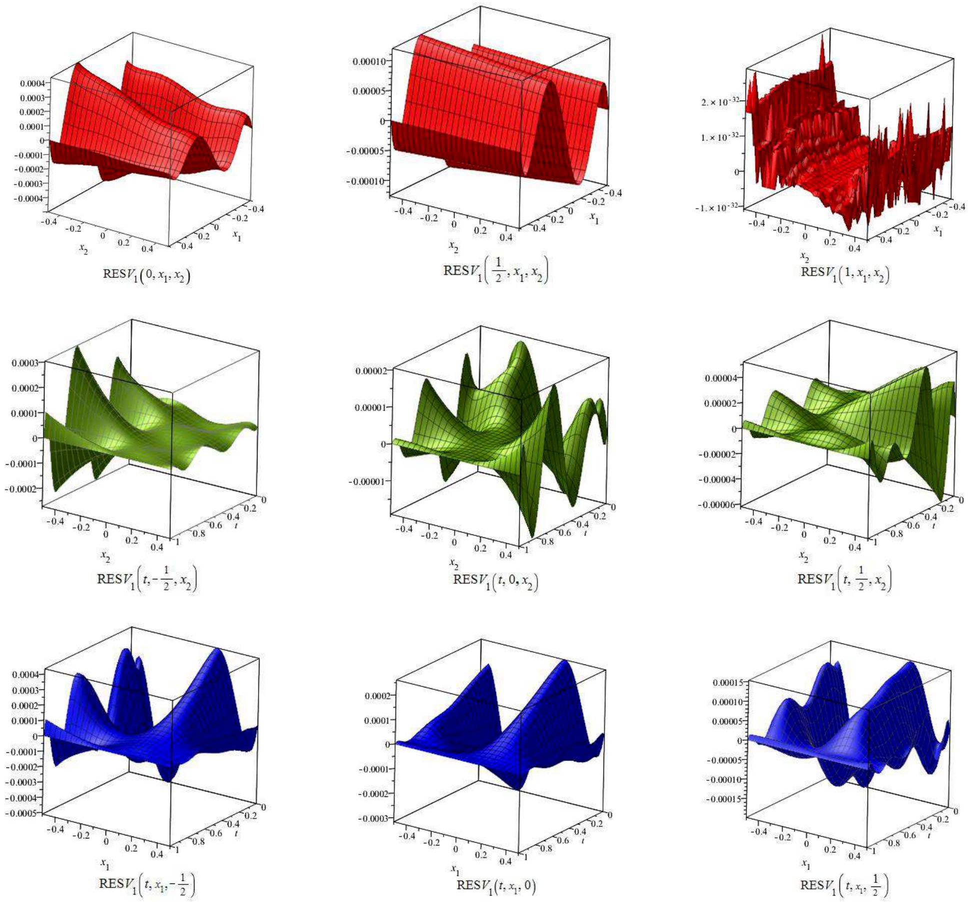

FIGURE 4. The residual errors $RES_{V_1}(t, x_1, x_2)$ for $t = 0, \frac{1}{2}, 1,\ x_1 = -\frac{1}{2}, 0, \frac{1}{2},\ x_2 = -\frac{1}{2}, 0, \frac{1}{2}$ using the SC-GCM at $N_1 = 6, N_2 = 4, N_3 = 4$ on the domain $[0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ for Example 5.3.

$$u_2^*(t, x_1, x_2) = -V_{2,x_2}(t, x_1, x_2). \tag{33}$$

Upon substituting $u_1^*(t, x_1, x_2)$ and $u_2^*(t, x_1, x_2)$ into (32), we have the following system of PDEs:

$$\begin{aligned}
V_{1,t}(t, x_1, x_2)+&\frac{1}{20}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}\left(V_{2,x_2}(t, x_1, x_2)^2 - V_{1,x_2}(t, x_1, x_2)^2\right)\\
&+V_{1,x_1}(t, x_1, x_2)x_2 + V_{1,x_2}(t, x_1, x_2)x_1^2\\
&-V_{2,x_2}(t, x_1, x_2)V_{1,x_2}(t, x_1, x_2) = 0,\\
V_{2,t}(t, x_1, x_2)+&\frac{1}{2}x_1^2 + \frac{1}{20}x_2^2 + \frac{1}{2}\left(V_{1,x_2}(t, x_1, x_2)^2 - V_{2,x_2}(t, x_1, x_2)^2\right)\\
&+V_{2,x_1}(t, x_1, x_2)x_2 + V_{2,x_2}(t, x_1, x_2)x_1^2\\
&-V_{2,x_2}(t, x_1, x_2)V_{1,x_2}(t, x_1, x_2) = 0,\\
V_1(T, x_1, x_2) =&\frac{1}{2}\left(0.1x_1^2 + x_2^2\right),\\
V_2(T, x_1, x_2) =&\frac{1}{2}\left(x_1^2 + 0.1x_2^2\right).
\end{aligned} \tag{34}$$

Substituting the relevant partial derivatives of $V_1(t, x_1, x_2)$ and $V_2(t, x_1, x_2)$ from (34) into (33), we get the feedback Nash equilibrium strategies $u_1^*(t, x_1, x_2) = \phi_1^*(t, x_1, x_2)$ and $u_2^*(t, x_1, x_2) = \phi_2^*(t, x_1, x_2)$.

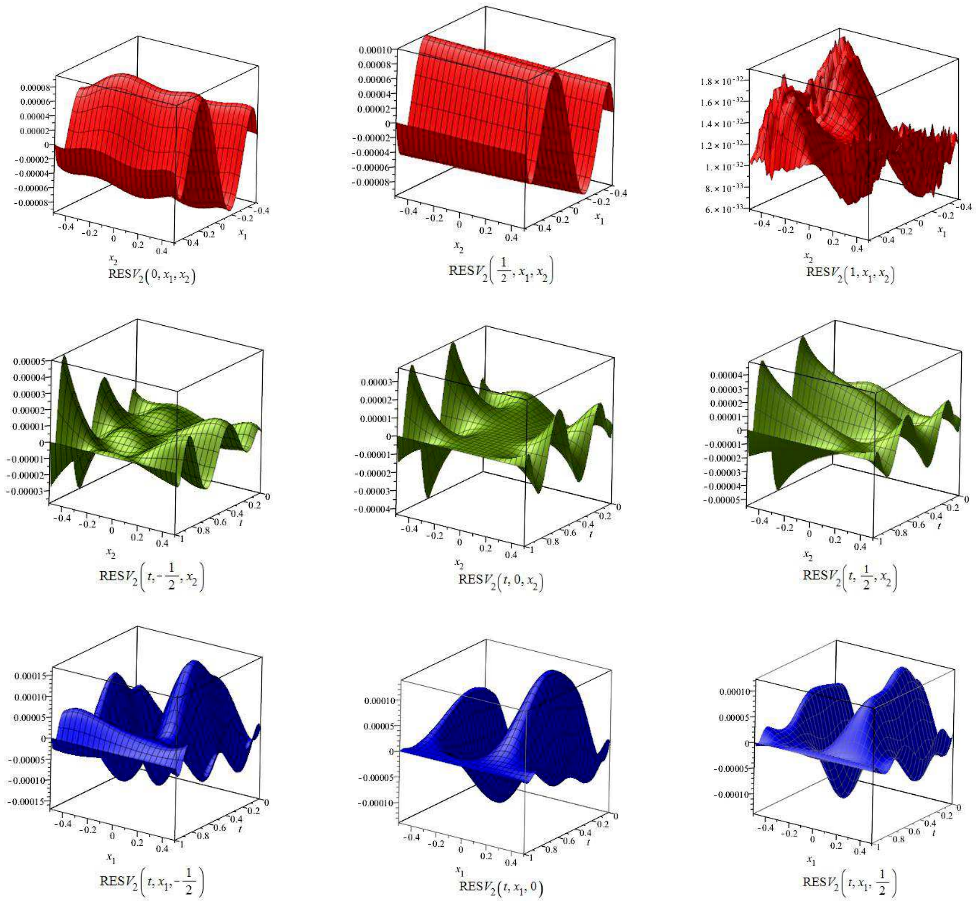

FIGURE 5. The residual errors $RES_{V_2}(t, x_1, x_2)$ for $t = 0, \frac{1}{2}, 1,\ x_1 = -\frac{1}{2}, 0, \frac{1}{2},\ x_2 = -\frac{1}{2}, 0, \frac{1}{2}$ using the SC-GCM at $N_1 = 6, N_2 = 4, N_3 = 4$ on the domain $[0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ for Example 5.3.

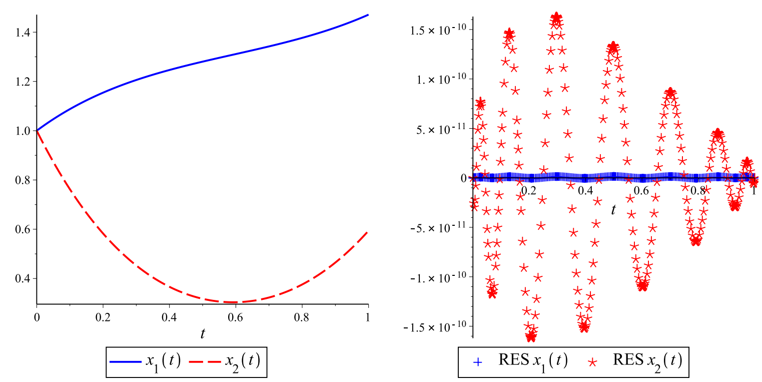

FIGURE 6. The numerical approximation of optimal state trajectories $x_i^*(t), i = 1, 2$ and the residual errors $RES_{x_i}(t), i = 1, 2,$ using the SC-GCM by $N = 10$ on the domain $[0, 1]$ for Example 5.3.

Table 3. Optimal value of cost functionals J_i, $i = 1, 2$ is obtained by using the SC-GCM, for Example 5.3.

(N_1, N_2, N_3)	J_1	J_2
(4, 2, 2)	1.32637334	2.47275222
(4, 3, 3)	1.379273134	2.884931842
(6, 4, 4)	1.694880607	3.001789476
(8, 4, 4)	1.694872219	3.001803382

After substituting $\phi_1^*(t, x_1(t), x_2(t))$ and $\phi_2^*(t, x_1(t), x_2(t))$ into the system of differential equations (29) and solving, we obtain the optimal state trajectories $x_1^*(t)$ and $x_2^*(t)$.

The SC-GCM method is applied to obtain the numerical approximation of optimal value functions $V_i(t, x_1, x_2)$ and the Nash equilibrium strategies $u_i(t, x_1, x_2)$ for $i = 1, 2$, for $t = 0, \frac{1}{2}, 1$, using a $6\times4\times4$ grid discreditization scheme on the computational domain $[0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, the obtained results are shown in Figure 3. The residual errors $RES_{V_1}(t, x_1, x_2)$ for $t = 0, \frac{1}{2}, 1,\ x_1 = \frac{-1}{2}, 0, \frac{1}{2}$, and $x_2 = -\frac{1}{2}, 0, \frac{1}{2}$ using SC-GCM for $N_1 = 6$, $N_2 = 4$, $N_3 = 4$ on the domain $[0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ are plotted in Figures 4 and 5.

The SC-GCM method is also applied to obtain the numerical optimal state trajectories $x_1^*(t)$ and $x_2^*(t)$, using $M = 10$ on the computational domain $[0, 1]$ (see Figure 6).

In Table 3, the computational results of the performance index of Player1 and Player2 for different values of N_1, N_2 and N_3 are reported. It should be noted that small values for $N_i, i = 1, 2, 3$ are needed to obtain a satisfactory convergence.

6. Conclusion

In this paper, we have proposed the SC-GCM to solve the HJB equations system of nonlinear nonzero-sum differential games for finding the feedback Nash equilibrium solution of these games. The main advantage of this method is that the boundary conditions of the system of HJB PDEs can be included implicitly in the chosen approximations of value functions. The majority of numerical methods are grid based suffer from the so-called "curse-of-dimensionality". However, the SC-GCM is also a grid based method, but with the Chebyshev–Gauss–Lobatto nodes the results show that selecting a limited number of collocation points, excellent numerical results are obtained.

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