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ON THE LERAY–HIRSCH THEOREM

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Abstract. In [7], E. Spanier directly proved that for the total pair $(E, \dot{E})$ of a fiber-bundle pair with base B and fiber pair $(F, \dot{F})$ such that $H_*(F, \dot{F}, R)$ is free and finitely generated over R and θ is a cohomology extension of the fiber, the homomorphism

$$\Phi_* : H_*(E, \dot{E}, G) \longrightarrow H_*(B, G) \otimes H_*(F, \dot{F}, R),$$

where H_* is the singular homology, is an isomorphism for all R modules G ([7, Theorem 5.7.9]), where R is a commutative ring with a unit.

About the homomorphism

$$\Phi^* : H^*(B, G) \otimes H^*(F, \dot{F}, R) \longrightarrow H^*(E, \dot{E}, R),$$

where H^* is the singular cohomology, he said that a similar argument does not appear possible, because it is not true that $H^*(B, R)$ is isomorphic to the inverse limit $\varprojlim \{H^*(U, R)\}_{U \in \mathcal{U}}$.

In [8], R. Switzer, using the spectral sequence of Serre, proved that the homomorphism Φ^* is an isomorphism ([8, Theorem 15.47]).

In [1], the Leray–Hirsch theorem (Theorem 4D.1) is proved, not using the spectral sequence, however, the base B is an infinite-dimensional CW complex.

In this paper, we give another proof of the fact that the homomorphism Φ^* is an isomorphism not using the spectral sequence of Serre.

Below, we give the brief summaries of some results used in the paper.

Let Ab be the category of abelian groups and homomorphisms.

Lemma 1 ([7, Lemma 5.5.6]). *If B is a finitely generated free abelian group, then for arbitrary abelian groups A and G, μ is an isomorphism*

$$\mu : \mathrm{Hom}(A, G) \otimes \mathrm{Hom}(B, \mathbb{Z}) \approx \mathrm{Hom}(A \otimes B, G).$$

Lemma 2 ([7, Corollary 5.5.4]). *If (X, A) is a topological pair such that $H_*(X, A)$ is finitely generated, then the free subgroups of $H^*(X, A)$ and $H_*(X, A)$ are isomorphic and the torsion subgroups of $H^*(X, A)$ and $H_{*-1}(X, A)$ are isomorphic, where H_* (H^*) is the integral singular homology (cohomology) theory.*

Lemma 3 ([6, Lemma 5.2]). *Given a short exact sequence of abelian groups*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

and an abelian group B, if A'' or B is torsion free (where being torsion free is equivalent to being free), there is a short exact sequence

$$0 \longrightarrow A' \otimes B \longrightarrow A \otimes B \longrightarrow A'' \otimes B \longrightarrow 0.$$

Lemma 4 ([3, V.1]). *If A and B are free abelian groups, then $A \otimes B$ is a free abelian group.*

Lemma 5 ([8, 10.36]). *Let $\{X^\alpha,\ \alpha \in \Lambda\}$ be a directed set ($\alpha \le \beta \Rightarrow X^\alpha \subset X^\beta$) of subspaces of topological space X such that for any compact $C \subset X$ there exists $\alpha \in \Lambda$ with $C \subset X^\alpha$. The inclusions $i_\alpha : X^\alpha \to X$, $\alpha \in \Lambda$, induce an isomorphism*

$$\{i_{\alpha,*}\} : \varinjlim H_*(X^\alpha, G) \xrightarrow{\sim} H_*(X, G).$$

Theorem 1 ([4, Theorem 11.32]). *Let*

$$0 \longrightarrow \underline{X}' \longrightarrow \underline{X} \longrightarrow \underline{X}'' \longrightarrow 0$$

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be a short exact sequence of inverse systems. Then there exists an exact sequence

$$0 \longrightarrow \varprojlim \underline{X}' \longrightarrow \varprojlim \underline{X} \longrightarrow \varprojlim \underline{X}'' \longrightarrow \varprojlim{}^{(1)} \underline{X}' \longrightarrow \cdots \longrightarrow \varprojlim{}^{(n)} \underline{X}' \longrightarrow \varprojlim{}^{(n)} \underline{X} \longrightarrow \varprojlim{}^{(n)} \underline{X}'' \longrightarrow \cdots,$$

where $\varprojlim{}^{(i)}$, $i \geq 1$, *is a derived functor.*

Lemma 6. *If B is a free and finitely generated abelian group and $\{A_\alpha\}$ is an inverse system of abelian groups A_α, then there is an isomorphism*

$$\varprojlim{}^{(i)} \{A_\alpha\} \otimes B \approx \varprojlim{}^{(i)} \{A_\alpha \otimes B\}, \quad i \geq 0.$$

Proof. Let $A = \varprojlim \{A_\alpha\}$ be an inverse limit of abelian groups A_α. Since B is a free and finitely generated abelian group, there is an isomorphism

$$B \approx \mathbb{Z}^n.$$

Hence, for all α, we have an isomorphism

$$A_\alpha \otimes B \approx A_\alpha \otimes \mathbb{Z}^n \approx (A_\alpha \otimes \mathbb{Z})^n \approx (A_\alpha)^n.$$

a) By Lemma 11.24 [4], the functor $\varprojlim$ preserves finite products. Therefore there is

$$\varprojlim \{A_\alpha \otimes B\} \approx \varprojlim (A_\alpha)^n = \big(\varprojlim A_\alpha\big)^n = A^n$$
$$\approx (A \otimes \mathbb{Z})^n \approx A \otimes \mathbb{Z}^n \approx A \otimes B = \varprojlim \{A_\alpha\} \otimes B.$$

b) By Corollary 12.15 [4], for $i \geq 1$, we have

$$\varprojlim{}^{(i)} \{A_\alpha \otimes B\} \approx \varprojlim{}^{(i)} \{A_\alpha \otimes \mathbb{Z}^n\} \approx \varprojlim{}^{(i)} \{(A_\alpha \otimes \mathbb{Z})^n\} \approx \varprojlim{}^{(i)} \{A_\alpha\}^n$$
$$\approx \big(\varprojlim{}^{(i)} \{A_\alpha\}\big)^n \approx \big(\varprojlim{}^{(i)} \{A_\alpha\} \otimes \mathbb{Z}\big)^n \approx \varprojlim{}^{(i)} \{A_\alpha\} \otimes \mathbb{Z}^n \approx \varprojlim{}^{(i)} \{A_\alpha\} \otimes B. \qquad \square$$

Lemma 7 ([2, Proposition 1.2])**.** *For any direct system $\{A_\alpha\}$ of abelian groups A_α, there are an exact sequence*

$$\text{a)} \quad 0 \longrightarrow \varprojlim{}^{(1)} \operatorname{Hom}(A_\alpha, G) \longrightarrow \operatorname{Ext}(\varinjlim A_\alpha, G) \longrightarrow \varprojlim \operatorname{Ext}(A_\alpha, G) \longrightarrow \varprojlim{}^{(2)} \operatorname{Hom}(A_\alpha, G) \longrightarrow 0$$

and an isomorphism

$$\text{b)} \quad \varprojlim{}^{(i)} \operatorname{Ext}(A_\alpha, G) \approx \varprojlim{}^{(i+2)} \operatorname{Hom}(A_\alpha, G), \quad i \geq 1.$$

Lemma 8 ([7, Theorem 5.1.9])**.** *The tensor-product functor commutes with direct limits, i.e., there is an isomorphism*

$$\varinjlim \{A_\alpha\} \otimes B \approx \varinjlim \{A_\alpha \otimes B\}.$$

Lemma 9 ([5, Exercise 3, §A.3])**.** *If $\{A_\alpha\}$ is a direct system of abelian groups A_α, then there is an isomorphism*

$$\operatorname{Hom}\big(\varinjlim \{A_\alpha\}, B\big) \approx \varprojlim \operatorname{Hom}(A_\alpha, B).$$

A fiber-bundle pair with the base space B consists of a total pair $(E, \dot{E})$, a fiber pair $(F, \dot{F})$ and a projection $p : E \to B$ such that there exist an open covering $\{V\}$ of B and, for each $V \in \{V\}$, a homeomorphism $\varphi_V : V \times (F, \dot{F}) \to (p^{-1}(V), p^{-1}(V) \cap \dot{E})$ such that the composite

$$V \times F \xrightarrow{\varphi_V} p^{-1}(V) \xrightarrow{p} V$$

is the projection to the first factor. If $A \subset B$, we suppose $E_A = p^{-1}(A)$ and $\dot{E}_A = p^{-1}(A) \cap \dot{E}$, and if $b \in B$, then $(E_b, \dot{E}_b)$ is the fiber pair over b.

Given a fiber-bundle pair with a total pair $(E, \dot{E})$ and a fiber pair $(F, \dot{F})$, a cohomology extension of the fiber is a homomorphism $\theta : H^*(F, \dot{F}) \to H^*(E, \dot{E})$ of graded abelian groups (of degree 0) such that for each $b \in B$ the composite

$$H^*(F, \dot{F}) \xrightarrow{\theta} H^*(E, \dot{E}) \longrightarrow H^*(E_b, \dot{E}_b)$$

is an isomorphism, where H^* is the integral singular cohomology.

Let $\overline{p} : B \times (F, \dot{F}) \to (F, \dot{F})$ be the projection to the second factor. Then

$$\theta = \overline{p}^* : H^*(F, \dot{F}) \longrightarrow H^*(B \times (F, \dot{F}))$$

is a cohomology extension of the fiber of the product-bundle pair.

Theorem of Leray–Hirsch. *Let $(E,\dot E)$ be the total pair of a fiber-bundle pair with the base B and fiber pair $(F,\dot F)$. Assume that $H_*(F,\dot F)$ is free and finitely generated over $\mathbb{Z}$ and that θ is a cohomology extension of the fiber. Then the homomorphism*

$$\Phi^* : H^*(B,C)\otimes H^*(F,\dot F)\longrightarrow H^*(E,\dot E,G)$$

is an isomorphism for all abelian groups G, where $\Phi^*(u\otimes v)=p^*(u)\smile\theta(v)$, $\smile$ is the cup-product homomorphism.

Proof. By Lemma 5.7.1 [7], it suffices to prove the result for the map Φ^* in the case $G=\mathbb{Z}$.

For any subset $A\subset B$, let θ_A be the composite

$$H^*(F,\dot F)\xrightarrow{\theta} H^*(E,\dot E)\longrightarrow H^*(E_A,\dot E_A).$$

Then θ_A is a cohomology extension of the fiber in the induced bundle over A. It follows from Lemma 5.7.8 [7] that if the induced bundle over A is homeomorphic to the product-bundle pair $A\times(F,\dot F)$, then

$$\Phi^*_A : H^*(A)\otimes H^*(F,\dot F)\xrightarrow{\sim} H^*(E_A,\dot E_A).$$

Hence Φ^*_A is a cohomology extension of the fiber in the induced bundle over A.

Using the exact Mayer–Vietoris sequences, property 5.6.20 [7] and also the fact that $H^*(F,\dot F)$ is a free and finitely generated abelian group, we find that Φ^*_U is an isomorphism for any U which is a finite union of sufficiently small open sets. Let $\mathcal{U}=\{U\}$ be the collection of these sets. Since any compact subset of B lies in some element of $\mathcal{U}$, by Lemma 5, there is an isomorphism

$$H_*(B)\approx \varinjlim_{U\in\mathcal{U}} H_*(U).$$

Also, any compact subset of E lies in some element of $E_{\mathcal{U}}=\{E_U\}$, where $E_U=p^{-1}(U)$, $U\in\mathcal{U}$. Therefore, by Lemma 5, there is an isomorphism

$$H_*(E,\dot E)\approx \varinjlim H_*(E_U,\dot E_U). \tag{1}$$

Since $C_*(E_U,\dot E_U)$ is a subcomplex of the free chain complex $C_*(E_U)$, for the pair $(E_U,\dot E_U)$ there is an exact sequence

$$0\longrightarrow \operatorname{Ext}(H_{*-1}(E_U,\dot E_U),\mathbb{Z})\longrightarrow H^*(E_U,\dot E_U)\longrightarrow \operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z})\longrightarrow 0. \tag{2}$$

The collection $\mathcal{U}=\{U\}$ generates the collection $E_{\mathcal{U}}=\{(E_U,\dot E_U)\}$ directed by inclusions. Hence the exact sequence (2) induces an exact sequence of inverse systems

$$0\longrightarrow \{\operatorname{Ext}(H_{*-1}(E_U,\dot E_U),\mathbb{Z})\}\longrightarrow \{H^*(E_U,\dot E_U)\}\longrightarrow \{\operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z})\}\longrightarrow 0.$$

By Theorem 1, there is an exact sequence

$$0\longrightarrow \varprojlim \operatorname{Ext}(H_{*-1}(E_U,\dot E_U),\mathbb{Z})\longrightarrow \varprojlim H^*(E_U,\dot E_U)\longrightarrow \varprojlim \operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z})\longrightarrow$$
$$\longrightarrow \varprojlim{}^{(1)} \operatorname{Ext}(H_{*-1}(E_U,\dot E_U),\mathbb{Z})\longrightarrow\cdots.$$

Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ext}(H_{*-1}(E,\dot E),\mathbb{Z}) & \longrightarrow & H^*(E,\dot E) & \longrightarrow & \operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z}) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi''} & & \\
0 & \longrightarrow & \varprojlim \operatorname{Ext}(H_{*-1}(E,\dot E),\mathbb{Z}) & \longrightarrow & \varprojlim H^*(E,\dot E) & \longrightarrow & \varprojlim \operatorname{Hom}(H_*(E_U,\dot E_U),\mathbb{Z}) & \longrightarrow &
\end{array}$$
$$\longrightarrow \varprojlim{}^{(1)} \operatorname{Ext}(H_{*-1}(E,\dot E),\mathbb{Z})\longrightarrow\cdots. \tag{3}$$

Since there is the isomorphism (1), by Lemma 9, using the connection between the functors $\operatorname{Hom}(-,\mathbb{Z})$ and $\varinjlim$, we have an isomorphism

$$\operatorname{Hom}(H_*(E,\dot E),\mathbb{Z})\approx \operatorname{Hom}(\varinjlim H_*(U,\dot U),\mathbb{Z})\approx \varprojlim \operatorname{Hom}(H_*(U,\dot U),\mathbb{Z}).$$

Hence in diagram (3), the homomorphism φ'' is an isomorphism, and also, the isomorphisms

$$\operatorname{Ker}\varphi' \approx \operatorname{Ker}\varphi, \quad \operatorname{Coker}\varphi' \approx \operatorname{Coker}\varphi.$$

By Lemma 7 a), there are isomorphisms

$$\operatorname{Ker}\varphi \approx \operatorname{Ker}\varphi' \approx \varprojlim{}^{(1)} \operatorname{Hom}(H_{*-1}(E_U, \dot{E}_U), \mathbb{Z}), \tag{4}$$

$$\operatorname{Coker}\varphi \approx \operatorname{Coker}\varphi' \approx \varprojlim{}^{(2)} \operatorname{Hom}(H_{*-1}(E_U, \dot{E}_U), \mathbb{Z}). \tag{5}$$

Using isomorphisms (4) and (5), we have an exact sequence

$$\begin{gathered} 0 \longrightarrow \varprojlim{}^{(1)} \operatorname{Hom}(H_{*-1}(E_U, \dot{E}_U), \mathbb{Z}) \longrightarrow H^*(E, \dot{E}) \longrightarrow \varprojlim H^*(E_U, \dot{E}_U) \longrightarrow \\ \longrightarrow \varprojlim{}^{(2)} \operatorname{Hom}(H_{*-1}(E_U, \dot{E}_U), \mathbb{Z}) \longrightarrow 0. \end{gathered} \tag{6}$$

Using Lemma 2 [6], for each $U \in \mathcal{U}$, there is the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}(B_{*-1}, \mathbb{Z}) & \longrightarrow & Z_U^* & \longrightarrow & \operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \operatorname{Ext}(H_{*-1}(E_U, \dot{E}_U), \mathbb{Z}) & \longrightarrow & H_U^* & \longrightarrow & \operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z}) & \longrightarrow & 0, \end{array}$$

where $B_{*-1} = B_{*-1}(E_U, \dot{E}_U)$, $Z_U^* = Z^*(E_U, \dot{E}_U)$, $H_U^* = H^*(E_U, \dot{E}_U)$, which induces, by Theorem 1, a long commutative diagram with exact sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \varprojlim{}^{(i)} \operatorname{Hom}(B_{*-1}, \mathbb{Z}) & \longrightarrow & \varprojlim{}^{(i)} Z_U^* & \longrightarrow & \varprojlim{}^{(i)} \operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \varprojlim{}^{(i)} \operatorname{Ext}(H_{*-1}, \mathbb{Z}) & \longrightarrow & \varprojlim{}^{(i)} H_U^* & \longrightarrow & \varprojlim{}^{(i)} \operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z}) & \longrightarrow & \cdots, \end{array} \tag{7}$$

where $H_{*-1} = H_{*-1}(E_U, \dot{E}_U)$.

By Lemma 7 b), for $i \geq 1$, there is an isomorphism

$$\varprojlim{}^{(i)} \operatorname{Ext}(B_{*-1}, \mathbb{Z}) \approx \varprojlim{}^{(i+2)} \operatorname{Hom}(B_{*-1}, \mathbb{Z}). \tag{8}$$

Since B_{*-1} is a free abelian group, there is the equality [3, Theorem 3.5]

$$\operatorname{Ext}(B_{*-1}, \mathbb{Z}) = 0. \tag{9}$$

Using isomorphism (8) and equality (9), for $k \geq 3$, we have the equality

$$\varprojlim{}^{(k)} \operatorname{Hom}(B_{*-1}, \mathbb{Z}) = 0. \tag{10}$$

By Lemma 7 a) and equality (9), there is the equality

$$\varprojlim{}^{(2)} \operatorname{Hom}(B_{*-1}, \mathbb{Z}) = 0. \tag{11}$$

From the commutative diagram (7) and equalities (10), (11), for $i \geq 2$, we have an isomorphism

$$\varprojlim{}^{(i)} Z_U^* \approx \varprojlim{}^{(i)} \operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z})$$

and a split exact sequence

$$0 \longrightarrow \varprojlim{}^{(i)} \operatorname{Ext}(H_{*-1}(E_U, \dot{E}_U), \mathbb{Z}) \longrightarrow \varprojlim{}^{(i)} H^*(E_U, \dot{E}_U) \longrightarrow \varprojlim{}^{(i)} \operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z}) \longrightarrow 0. \tag{12}$$

Using the exact sequence (6), the split exact sequence (12) for $i \geq 2$, the isomorphism φ'' from the commutative diagram (3), we have an exact sequence

$$0 \longrightarrow \varprojlim{}^{(1)} \operatorname{Ext}(H_{*-1}(E_U, \dot{E}_U), \mathbb{Z}) \longrightarrow \varprojlim{}^{(1)} H^*(E_U, \dot{E}_U) \longrightarrow \varprojlim{}^{(1)} \operatorname{Hom}(H_*(E_U, \dot{E}_U), \mathbb{Z}) \longrightarrow 0,$$

and using the Yoneda method, we also have a finite exact sequence

$$\begin{aligned} 0 \longrightarrow \varprojlim{}^{(2*-3)} H_U^1 \longrightarrow \cdots \longrightarrow \varprojlim{}^{(1)} H_U^{*-1} \longrightarrow H^*(E,\dot{E}) \longrightarrow \varprojlim H_U^* \longrightarrow \\ \longrightarrow \varprojlim{}^{(2)} H_U^{*-1} \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2*-2)} H_U^1 \longrightarrow 0, \end{aligned} \tag{13}$$

where $H_U^* = H^*(E_U, \dot{E}_U)$.

For the base B, there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{*-1}(B),\mathbb{Z}) \longrightarrow H^*(B) \longrightarrow \operatorname{Hom}(H_*(B),\mathbb{Z}) \longrightarrow 0.$$

Since $H_*(F,\dot{F})$ is free and finitely generated over $\mathbb{Z}$, there is an isomorphism $\operatorname{Hom}(H_*(F,\dot{F}),\mathbb{Z}) \approx H^*(F,\dot{F})$, and, by Lemma 3, there is a short exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Ext}(H_{*-1}(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot{F}),\mathbb{Z}) \longrightarrow H^*(B) \otimes H^*(F,\dot{F}) \xrightarrow{\xi} \\ \xrightarrow{\xi} \operatorname{Hom}(H_*(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot{F}),\mathbb{Z}) \longrightarrow 0. \end{aligned}$$

Denote $\operatorname{Ker}\xi = \operatorname{Ext}(H_{*-1}(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot{F}),\mathbb{Z})$. For each $U \in \mathcal{U} = \{U\}$, there is an exact sequence

$$0 \longrightarrow \operatorname{Ker}\xi_U \longrightarrow H^*(U) \otimes H^*(F,\dot{F}) \longrightarrow \operatorname{Hom}(H_*(U),\mathbb{Z}) \otimes H^*(F,\dot{F}) \longrightarrow 0,$$

where $\operatorname{Ker}\xi_U = \operatorname{Ext}(H_{*-1}(U),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot{F}),\mathbb{Z})$.

By Lemma 1, there is an isomorphism

$$\operatorname{Hom}(H_*(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot{F}),\mathbb{Z}) \approx \operatorname{Hom}(H_*(B) \otimes H_*(F,\dot{F}),\mathbb{Z}). \tag{14}$$

The family $\mathcal{U} = \{U\}$ induces an exact sequence of inverse systems

$$0 \longrightarrow \{\operatorname{Ker}\xi_U\} \longrightarrow \{H^*(U) \otimes H^*(F,\dot{F})\} \longrightarrow \{\operatorname{Hom}(H_*(U) \otimes H_*(F,\dot{F}),\mathbb{Z})\} \longrightarrow 0,$$

which, by Theorem 1, generate an exact sequence

$$\begin{aligned} 0 \longrightarrow \varprojlim \{\operatorname{Ker}\xi_U\} \longrightarrow \varprojlim \{H^*(U) \otimes H^*(F,\dot{F})\} \longrightarrow \varprojlim \{\operatorname{Hom}(H_*(U) \otimes H_*(F,\dot{F}),\mathbb{Z})\} \longrightarrow \\ \longrightarrow \varprojlim{}^{(1)} \{\operatorname{Ker}\xi_U\} \longrightarrow \cdots . \end{aligned}$$

Using isomorphism (14) and Lemma 8, we have an isomorphism

$$\begin{aligned} \operatorname{Hom}(H_*(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot{F}),\mathbb{Z}) \approx \operatorname{Hom}(H_*(B) \otimes H_*(F,\dot{F}),\mathbb{Z}) \\ \approx \operatorname{Hom}(\varinjlim \{H_*(U)\} \otimes H_*(F,\dot{F}),\mathbb{Z}) \approx \varprojlim \operatorname{Hom}(H_*(U) \otimes H_*(F,\dot{F}),\mathbb{Z}). \end{aligned} \tag{15}$$

Hence, using isomorphism (15) and Lemma 9, there is a commutative diagram with exact rows

$$\begin{array}{ccccc} 0 \longrightarrow & \operatorname{Ext}(H_{*-1}(B),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot{F}),\mathbb{Z}) & \longrightarrow & H^*(B) \otimes H^*(F) & \longrightarrow \\ & \downarrow \psi' & & \downarrow \psi & \\ 0 \longrightarrow & \varprojlim \{\operatorname{Ext}(H_*(U),\mathbb{Z}) \otimes \operatorname{Hom}(H_*(F,\dot{F}),\mathbb{Z})\} & \longrightarrow & \varprojlim \{H^*(U) \otimes H^*(F,\dot{F})\} & \longrightarrow \end{array}$$

$$\begin{array}{c} \longrightarrow \operatorname{Hom}(H_*(B) \otimes \operatorname{Hom}(F,\dot{F}),\mathbb{Z}) \longrightarrow 0 \\ \approx \downarrow \psi'' \\ \longrightarrow \varprojlim \{\operatorname{Hom}(H_*(U) \otimes H_*(F,\dot{F}),\mathbb{Z})\} \longrightarrow 0. \end{array} \tag{16}$$

Since ψ'' is an isomorphism, by the commutative diagram (16), there are isomorphisms

$$\operatorname{Ker}\psi' \approx \operatorname{Ker}\psi, \tag{17}$$

$$\operatorname{Coker}\psi' \approx \operatorname{Coker}\psi. \tag{18}$$

Using Lemmas 6 a) and 9, we have an exact sequence

$$0 \longrightarrow \big(\varprojlim{}^{(1)}\{\operatorname{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow \operatorname{Ext}(H_{*-1}(B),\mathbb{Z})\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow$$
$$\longrightarrow \big(\varprojlim\{\operatorname{Ext}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow$$
$$\longrightarrow \big(\varprojlim{}^{(2)}\{\operatorname{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow 0.$$

Since

$$\operatorname{Ker}\psi' = \big(\varprojlim{}^{(1)}\{\operatorname{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})$$

and

$$\operatorname{Coker}\psi' = \big(\varprojlim{}^{(2)}\{\operatorname{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}),$$

using isomorphisms (17) and (18), there is an exact sequence

$$0 \longrightarrow \big(\varprojlim{}^{(1)}\{\operatorname{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow H^*(B)\otimes H^*(F,\dot F) \longrightarrow$$
$$\longrightarrow \varprojlim\{H^*(U)\otimes H_*(F,\dot F)\} \longrightarrow \big(\varprojlim{}^{(2)}\{\operatorname{Hom}(H_{*-1}(U),\mathbb{Z})\}\big)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow 0. \tag{19}$$

By Lemma 1 [6], for each $U\in\mathcal{U}=\{U\}$, there is an exact sequence

$$0 \longrightarrow \operatorname{Hom}(B_{*-1}(U),\mathbb{Z}) \longrightarrow Z^*(U) \longrightarrow \operatorname{Hom}(H_*(U),\mathbb{Z}) \longrightarrow 0.$$

Since $\operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})$ is free and finitely generated, by Lemma 3, we have an exact sequence

$$0 \longrightarrow \operatorname{Hom}(B_{*-1}(U),\mathbb{Z})\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow Z^*(U)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow$$
$$\longrightarrow \operatorname{Hom}(H_*(U),\mathbb{Z})\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z}) \longrightarrow 0. \tag{20}$$

Using Lemma 1, for the exact sequence (20), there is an exact sequence of inverse systems

$$0 \longrightarrow \Big\{\operatorname{Hom}(B_{*-1}(U))\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow \Big\{Z^*(U)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow$$
$$\longrightarrow \Big\{\operatorname{Hom}(H_*(U))\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow 0,$$

which, by Theorem 1, generates an exact sequence

$$0 \longrightarrow \varprojlim \Big\{\operatorname{Hom}(B_{*-1}(U))\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow \varprojlim \Big\{Z^*(U)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow$$
$$\longrightarrow \varprojlim \Big\{\operatorname{Hom}(H_*(U))\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\Big\} \longrightarrow \cdots . \tag{21}$$

Since $B_{*-1}(U)$ and $H_*(F,\dot F)$ are free abelian groups, by Lemma 4, $B_{*-1}(U)\otimes H_C(F,\dot F)$ is a free abelian group. Using Lemma 7, we have

a) an epimorphism

$$\varprojlim \operatorname{Ext}(B_{*-1}(U)\otimes H_*(F,\dot F),\mathbb{Z}) \longrightarrow \varprojlim{}^{(2)} \operatorname{Hom}(B_{*-1}(U)\otimes H_*(F,\dot F),\mathbb{Z}) \longrightarrow 0;$$

b) an isomorphism

$$\varprojlim{}^{(i)} \operatorname{Ext}(B_{*-1}(U)\otimes H_*(F,\dot F),\mathbb{Z}) \approx \varprojlim{}^{(i+2)} \operatorname{Hom}(B_{*-1}(U)\otimes H_*(F,\dot F),\mathbb{Z}) \quad \text{for } \ i\geq 1;$$

Therefore, there is the equality

$$\varprojlim{}^{(i)} \operatorname{Hom}(B_{*-1}(U)\otimes H_*(F,\dot F),\mathbb{Z}) = 0, \quad i\geq 2. \tag{22}$$

From the exact sequence (21) and equality (22) it follows that for $i\geq 2$, there is an isomorphism

$$\varprojlim{}^{(i)}\{Z^*(U)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\} \approx \varprojlim{}^{(i)}\{\operatorname{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z})\}.$$

Consider the commutative diagram

$$\begin{array}{ccc}
\varprojlim{}^{(i)}\{Z^*(U)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\} & \approx & \varprojlim{}^{(i)}\{\operatorname{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z})\} \\
\downarrow & & \| \\
\varprojlim{}^{(i)}\{H^*(U)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\} & \longrightarrow & \varprojlim{}^{(i)}\{\operatorname{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z})\}.
\end{array}$$

For $i\geq 2$, there is a split exact sequence

$$0\longrightarrow \varprojlim{}^{(i)}\{\operatorname{Ext}(H_{*-1}(U),\mathbb{Z})\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\}\longrightarrow \varprojlim{}^{(i)}\{H^*(U)\otimes \operatorname{Hom}(H_*(F,\dot F),\mathbb{Z})\}\longrightarrow$$
$$\longrightarrow \varprojlim{}^{(i)}\{\operatorname{Hom}(H_*(U)\otimes H_*(F,\dot F),\mathbb{Z})\}\longrightarrow 0. \tag{23}$$

Using the exact sequence (19), the split exact sequence (23), Lemma 6 b), Lemma 7 b) and the Yoneda method, we have a finite exact sequence

$$0\longrightarrow \varprojlim{}^{(2*-3)}\{H^1(U)\otimes H^*(F,\dot F)\}\longrightarrow\cdots\longrightarrow \varprojlim{}^{(1)}\{H^{*-1}(U)\otimes H^*(F,\dot F)\}\longrightarrow$$
$$\longrightarrow H^*(B)\otimes H^*(F,\dot F)\longrightarrow \varprojlim\{H^*(U)\otimes H_*(F,\dot F)\}\longrightarrow \varprojlim{}^{(2)}\{H^{*-1}(U)\otimes H^*(F,\dot F)\}\longrightarrow$$
$$\longrightarrow\cdots\longrightarrow \varprojlim{}^{(2*-2)}\{H^1(U)\otimes H^*(F,\dot F)\}\longrightarrow 0. \tag{24}$$

Exact sequences (13), (24) and the homomorphisms Φ^* and $\{\Phi^*_U\}$ induce a commutative diagram

$$\begin{array}{ccccccc}
\cdots\longrightarrow & \varprojlim{}^{(3)}\Big\{H^{*-2}(U)\otimes H^*(F,\dot F)\Big\} & \longrightarrow & \varprojlim{}^{(1)}\Big\{H^{*-1}(U)\otimes H^*(F,\dot F)\Big\} & \longrightarrow & H^*(B)\otimes H^*(F,\dot F) & \longrightarrow \\
 & \Big\downarrow{\scriptstyle \varprojlim{}^{(3)}\Phi^*_U} & & \Big\downarrow{\scriptstyle \varprojlim{}^{(1)}\Phi^*_U} & & \Big\downarrow{\scriptstyle \Phi^*} & \\
\cdots\longrightarrow & \varprojlim{}^{(3)}H^{*-2}(E_U,\dot E_U) & \longrightarrow & \varprojlim{}^{(1)}H^{*-1}(E_U,\dot E_U) & \longrightarrow & H^*(E,\dot E) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \varprojlim\Big\{H^*(U)\otimes H^*(F,\dot F)\Big\} & \longrightarrow & \varprojlim{}^{(2)}\Big\{H^{*-1}(U)\otimes H^*(F,\dot F)\Big\} & \longrightarrow\cdots \\
 & \Big\downarrow{\scriptstyle \varprojlim\Phi^*_U} & & \Big\downarrow{\scriptstyle \varprojlim{}^{(2)}\Phi^*_U} & \\
\longrightarrow & \varprojlim H^*(E,\dot E) & \longrightarrow & \varprojlim{}^{(2)}H^{*-1}(E,\dot E) & \longrightarrow\cdots.
\end{array} \tag{25}$$

By Theorem 5.7.10 [7], for each $U\in\mathcal{U}=\{U\}$, there is an isomorphism

$$\Phi^*_U: H^*(U)\otimes H^*(F,\dot F)\xrightarrow{\sim} H^*(E_U,\dot E_U).$$

Hence the homomorphism $\{\Phi^*_U\}$ of inverse systems

$$\{\Psi^*_U\}:\{H^*(U)\otimes H^*(F,\dot F)\}\longrightarrow\{H^*(E_U,\dot E_U)\}$$

is an isomorphism and for $i\geq 0$ induces an isomorphism

$$\varprojlim{}^{(i)}\{H^*(U)\otimes H^*(F,\dot F)\}\xrightarrow{\sim}\varprojlim{}^{(i)}\{H^*(E_U,\dot E_U)\}. \tag{26}$$

By five Lemma [7, Lemma 4.5.11] and isomorphisms (26), from the commutative diagram (25) it follows that Φ^* is an isomorphism. □

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