Transactions of A. Razmadze
Mathematical Institute
Vol. **175** (2021), issue 1, 69–74

MATHEMATICAL MODELING OF STOCHASTIC SYSTEMS USING THE GENERALIZED NORMAL SOLUTION METHOD

ALEXANDER MADERA[1], HAMLET MELADZE[2], MINDIA SURGULADZE[1] AND EKATERINA GREBENNIKOVA[1]

Abstract. Operation of complex engineering systems gives rise to various physical processes, including thermal, electrical, hydrodynamic, mechanical, electromagnetic, etc. The parameters of the elements of an engineering system and the processes going on in the same are stochastic, which results both from a stochastic nature of the parameters of the elements and from a random nature of the parameters of the environment and external influencing factors. Mathematical modeling of stochastic engineering systems developed in this paper relies on a universal structural conceptual model of an engineering system represented as a directed graph which reflects the structure of the engineering system and the modeled physical processes. State variables in a structural conceptual model of a system are the potentials in the graph nodes and flows at the graph edges, which edges may contain elements modeling the processes of energy dissipation, potential energy accumulation and kinetic energy storage, and also independent sources such as potential and physical quantity flow with the a priori known value. Stochastic processes in a graph of an engineering system model for each elementary event ω from the space of elementary events Ω are described through the mathematical model $H(\omega)X(\omega) = Y(\omega)$, $\omega \in \Omega$ with a stochastic matrix $H(\omega) = AG(\omega)A^T$, where A is an incidence matrix and $G(\omega)$ is a stochastic diagonal matrix of such parameters of the elements of a graph as conductance. The present paper offers a method based on the generalized normal solution concept, known also as pseudosolution, pseudoinverse matrix and generalized inverse matrix method allowing one to determine an equation for statistical measures (expectations, covariances, dispersions, standard deviations) of the stochastic solution $X(\omega)$ of the mathematical model of a stochastic engineering system under the a priori known statistical measures of the matrix of system elements $G(\omega)$ and the stochastic right-hand side vector $Y(\omega)$. Utilization of the method in modeling of stochastic thermal processes and statistical measures for complex electronic systems has shown that the method is adequate and efficient.

Various physical processes going on in complex engineering systems are stochastic in the majority of practically important cases resulting both from a random nature of the internal parameters of the elements and structure of the system and from the external influencing factors. To allow mathematical modeling of physical processes, the design of an engineering system should, first of all, be represented as a structural conceptual model (SC model) constructed as a graph, which replaces the real design of the engineering system by a simplified, but still sufficiently adequate model reflecting both the structure of the engineering system and the physical processes going on in the same [3, 6, 14]. State variables in the SC model graph are the quantities such as potentials in the graph nodes and flows at the graph edges. The graph nodes are connected to each other by edges including such elements as R, where energy is dissipated; C accumulates potential energy; L stores kinetic energy, and also the elements that determine independent sources of state variables potential or physical quantity flow with the a priori known values [3, 6, 13, 14]. The SC models of engineering systems are sufficiently universal and allow to get efficient and highly adequate modeling of complex engineering systems and various physical processes (thermal, hydrodynamic, mechanical, electrical, etc.) going on in the same.

A mathematical model of a stationary stochastic physical process in a SC model of an engineering system is represented by a stochastic matrix equation [1, 6]

$$H(\omega)X(\omega) = Y(\omega), \quad \omega \in \Omega, \tag{1}$$

2020 *Mathematics Subject Classification.* 60H25, 60-08, 60H35, 65C20, 93-10.

Key words and phrases. Stochastic; Engineering system; Mathematical model; Matrix equation; Generalized normal solution; Pseudosolution; Pseudoinverse matrix; Directed graph.

where $H(\omega)$ is a stochastic $n \times n$-square matrix reflecting the structure of the SC model graph and component relations between state variables and graph elements; $X(\omega)$ is an n-column vector of stochastic state variables; $Y(\omega)$ is an n-column vector of independent stochastic sources of state variables; ω are elementary events from the space of elementary events Ω in the probability space $\{\Omega, U, P\}$, U is the σ-algebra, P is probability in U. It should be noted that random elements of the stochastic matrix $H(\omega)$ and vector $X(\omega)$ are stochastically interdependent and statically independent of the elements of the stochastic vector $Y(\omega)$.

The stochastic vectors $X(\omega)$, $Y(\omega)$ and the matrix $H(\omega)$ in equation (1) are the interval stochastic [7–9] quantities $\xi(\omega)$, whose values are evenly distributed within the interval of values $[\xi_{down}, \xi_{up}]$ with a density $p_\xi = \Delta_\xi^{-1}$, $\xi(\omega) \in [\xi_{down}, \xi_{up}]$ and $p_\xi = 0$, $\xi(\omega) \notin [\xi_{down}, \xi_{up}]$, where $\Delta_\xi = \xi_{up} - \xi_{down}$ is the length of the interval $[\xi_{down}, \xi_{up}]$; ξ_{up} and ξ_{down} are the upper and lower interval limits.

A random process is fully characterized by the sequence of all its distribution laws of various order over time [1, 11]. At the same time, it is impossible to determine the laws for the stochastic vector $X(\omega)$, which is a solution to the matrix equation (1), as the task is extremely difficult. However, modeling of stochastic processes going on in engineering systems does not require any knowledge of distribution laws, as the most informative and most important ones in the engineering practice are statistical measures of the vector of stochastic state variables $X(\omega)$, in particular:

–n-column vector of expectations $\bar{X} = E\{X(\omega)\}$ with elements $\bar{x}_i = E\{x_i(\omega)\}$, $i = 1, 2, \ldots, n$, where $E\{\cdot\}$ is the expectation operator;

– covariance $n \times n$-matrix $K_{XX} = E\{\mathring{X}(\omega)\mathring{X}^T(\omega)\}$ with elements ij, equal to $k_{ij} = E\{\mathring{x}_i(\omega)\mathring{x}_j(\omega)\}$, $i, j = 1, 2, \ldots, n$, where $\mathring{x}_i(\omega) = x_i(\omega) - \bar{x}_i$ is a centered stochastic quantity with a zero expectation, $(\cdot)^T$ is the operation of transposition;

– n-column vector of dispersions D_X, equal to diagonal elements $d_{x,i}$ of the correlation matrix K_{XX}, i.e., $d_{x,i} = k_{ii} = E\{(\mathring{x}_i(\omega))^2\}$, $i = 1, 2, \ldots, n$;

– n-column vector of standard deviations σ_X with elements $\sigma_{x,i} = \sqrt{d_{x,i}}$, $i = 1, 2, \ldots, n$.

The determined vectors of statistical measures $\bar{X} = \{\bar{x}_i\}_1^n$ and $\sigma_X = \{\sigma_{x,i}\}_1^n$ of the stochastic vector $X(\omega) = \{x_i\}_1^n$ allow to determine the vectors of the lower $X_{down} = \{x_{i,down}\}_1^n$ and upper $X_{up} = \{x_{i,up}\}_1^n$ interval limits $[x_{i,down}, x_{i,up}]$, $i = 1, 2, \ldots, n$, which will contain real values of the interval stochastic quantities $x_i(\omega) \in [x_{i,down}, x_{i,up}]$.

In the simplest case, where the matrix H of the set of equations [6] is deterministic, while external perturbations being a part of the right-hand side vector $Y(\omega)$ are stochastic only, the statistical measures $\bar{X}$ and K_{XX} of the stochastic vector $X(\omega)$ are determined by using the equations $\bar{X} = H^{-1}\bar{Y}$ and $K_{XX} = H^{-1}K_{YY}H^{-1}$, where H^{-1} is the deterministic inverse of the matrix H, $\bar{Y} = E\{Y(\omega)\}$ is the vector of expectations of the stochastic vector $Y(\omega)$, $K_{YY} = E\{\mathring{Y}(\omega)\mathring{Y}^T(\omega)\}$ is the covariance matrix of the stochastic vector $Y(\omega)$.

If the matrix $H(\omega)$ in equation (1) is stochastic, it is impossible to determine statistical measures of the vector $X(\omega)$ by a direct impact of the expectation operator on the both sides of equation [6] in view of the statistical relationship between the stochastic elements of the matrix $H(\omega)$ and vector $X(\omega)$; so, $E\{H(\omega)X(\omega)\} \neq E\{H(\omega)\} \cdot E\{X(\omega)\}$. In this case, to determine statistical measures of the stochastic vector $X(\omega)$, papers [1, 6] represent the stochastic matrix of the system $H(\omega)$ as $H(\omega) = \bar{H}(I + \bar{H}^{-1}\mathring{H}(\omega))$, and the stochastic inverse of the matrix $H^{-1}(\omega)$ for each $\omega \in \Omega$ is expanded along an infinite almost surely uniformly convergent series [1]

$$H^{-1}(\omega) = (I + \bar{H}^{-1}\mathring{H}(\omega))^{-1} \cdot \bar{H}^{-1} = \sum_{k=1}^{\infty}(-1)^k(\bar{H}^{-1}\mathring{H}(\omega))^k \cdot \bar{H}^{-1}, \tag{2}$$

provided the condition $||\bar{H}^{-1}\mathring{H}(\omega)|| < 1$ is satisfied for all realizations of $\omega \in \Omega$. Here, $||\cdot||$ is the matrix norm [5]; $\mathring{H}(\omega) = H(\omega) - \bar{H}$ is the centered stochastic $n \times n$-matrix with a zero expectation; $\bar{H} = E\{H(\omega)\}$ is the expectation of the stochastic matrix $H(\omega)$; $\bar{H}^{-1}$ is the inverse of the deterministic matrix $\bar{H}$, which can be easily determined.

Then, the statistical measures $\bar{X}$ and K_{XX} of the stochastic vector $X(\omega) = H^{-1}(\omega) \cdot Y(\omega)$, being a solution to equation (1), will be determined by using the following equations:

$$\bar{X} = E\{H^{-1}(\omega)\} \cdot \bar{Y} \quad \text{and} \quad K_{XX} = E\{H^{-1}(\omega)Y(\omega)Y^T(\omega)(H^T(\omega))^{-1}\},$$

where $H^{-1}(\omega)$ is the stochastic inverse matrix to be determined by equation (2).

Practical calculations are limited to the terms of the infinite series (2) containing a matrix $\mathring{H}(\omega)$ of degree max. 2. The above method, known also as a stochastic inverse matrix method, allows to obtain the results with errors, sufficient to be used in practice and not exceeding $5-7\%$ [1,6]. At the same time, the range of applicability of the method is subject to the condition $||\bar{H}^{-1}\mathring{H}(\omega)|| < 1$, which imposes significant limitations on the allowable values of the parameters of the engineering system.

It should be also noted that the use of the perturbation and hierarchy methods [1] described in literature to determine the statistical measures $\bar{X}$ and K_{XX} of the stochastic processes in engineering systems have not found practical use. The reason is that the first of the above methods is good only for extremely small perturbations, which do not occur in real practice, while the second one is heuristic and does not have mathematical justification. The assumptions concerning special types of random processes, such as Wiener or white noise, presented in a great number of papers, allow to obtain final solutions for statistical measures in many cases, but are unrealistic and cannot exist in practice of engineering systems operation. The use of the statistical test method [12] may not be recommended for designing engineering systems, as far as it requires a huge input of machine time and memory caused by the necessity to solve multiple (up to several tens of thousands) simultaneous equations (1) to achieve an acceptable accuracy [11].

This paper offers a method to determine the statistical measures of the stochastic vector $X(\omega)$, which is a solution to the matrix equation (1) describing physical processes in SC models of engineering systems. The method is based on the generalized normal solution concept and allows to get final closed-form equations for statistical measures of stochastic processes in engineering (electronic) systems of any complexity, which are adequately simulated by SC models, being free from the above defects. The developed method is used in modeling of stochastic thermal processes in real electronic systems and has proven to be adequate and efficient.

The method developed in this paper is based on the generalized normal solution concept, which is also known in relation to the matrix equation as pseudosolution, pseudoinverse matrix method, Moore–Penrose generalized inverse matrix method [4,5,10]. The essence of the method consists in the following.

If the matrix $A = \{a_{ij}\}_1^n$ in the matrix equation $Ax = y$ is square and nonsingular, then an inverse matrix A^{-1} and a unique solution to the equation $x = A^{-1}y$ exist. If the matrix A is square, but singular, or the matrix $A = \{a_{ij}\}_{(n\times m)}$ is a rectangular $n \times m$-matrix (n and m are the number of lines and columns), then the matrix A is known to have no inverse. At the same time, a unique so-called pseudoinverse matrix A^+ can be constructed for such matrix, which pseudoinverse matrix allows one to obtain the best approximate solution $x^0 = A^+y$, $x^0 = (x_1^0, x_2^0, \ldots, x_m^0)^T$ to the equation $Ax = y$ in terms of the minimum value of the residual norm (Euclidian l_2-norm) square achieved for $x = x^0$, in particular [4],

$$\min_x ||y - Ax||^2 = \min_x \sum_{i=1}^{n} \left| y_i - \sum_{j=1}^{m} a_{ij}x_j \right|^2. \tag{3}$$

In this case, the vector of the best approximate solution x^0 has the lowest length, i.e., $||x^0||^2 = (x^0)^T \cdot x^0 = \min$, where $x^T \cdot x$ is the scalar product of the vector x. It should be noted that if the matrix A is square and nonsingular, then the inverse A^{-1} is the same as the pseudoinverse matrix A^+. Further, it may be shown [4,5] that a rectangular $n \times m$-matrix of the A range $r = \min\{n, m\}$ can always be represented as the so-called skeleton decomposition $A = BC$, i.e., as a product of two rectangular matrices, in particular, the $n \times r$-matrix B and the $r \times m$-matrix C. In this case, the pseudoinverse matrix A^+ is determined by using the equation $A^+ = C^+B^+$, where $C^+ = C^T(CC^T)^{-1}$ and $B^+ = (B^TB)^{-1}B^T$ [4]. Despite the fact that the skeleton decomposition $A = BC$ provides no unambiguous determination of the multiplier matrices B and C, the equation $A^+ = C^+B^+$ determines the unique pseudoinverse matrix with any skeleton decompositions [4].

Let us apply the generalized normal solution method to the stochastic matrix equation (1) describing the physical processes in a SC model of an engineering system. Toward this end, let us represent the $n \times n$-matrix of the $H(\omega)$ graph of the SC model for each $\omega \in \Omega$ as a product of three matrices $H(\omega) = AG(\omega)A^T$, in particular, as a deterministic incidence $n \times m$-matrix of the A graph of the SC model ($n,\ m$ are the number of nodes and edges, respectively), which contains elements 0 and 1 only, and a stochastic diagonal $m \times m$-matrix of elements $G(\omega)$ such as random conductance in the graph edges. The decomposition $H(\omega) = AG(\omega)A^T$ can be always performed for a random graph in a single way only [2,6]. Then the stochastic equation (1) can be written as follows:

$$AG(\omega)A^T X(\omega) = Y(\omega), \quad \omega \in \Omega, \tag{4}$$

where $G(\omega) = \mathrm{diag}(g_1(\omega), g_2(\omega), \ldots, g_m(\omega))$ is the stochastic diagonal $m \times m$-matrix with stochastic elements $g_i(\omega),\ i = 1, 2, \ldots, m$ at m edges of the graph of a SC model of an engineering system, which elements are expressed through physical stochastic parameters of the engineering system and the process going on in the same [2,6].

Let us apply the generalized normal solution method to the stochastic equation (4). To do this, let us represent equation (4) as $AZ(\omega) = Y(\omega)$, $\omega \in \Omega$ with a stochastic column vector $Z(\omega) = G(\omega)A^T X(\omega)$ and multiply both right-hand sides by the transposed incidence matrix A^T. We obtain the equation $A^T AZ(\omega) = A^T Y(\omega)$ with a singular square $n \times n$-matrix $A^T A = B$, for which there exists no inverse matrix. At the same time, the product $A^T A$ is, in fact, a skeleton decomposition of the matrix $B = A^T A$, thus we can build a pseudoinverse deterministic matrix B^+ [4]

$$B^+ = A^T (AA^T)^{-1} (AA^T)^{-1} A, \tag{5}$$

and use the pseudoinverse matrix method to get the best approximate solution to the equation $A^T AZ(\omega) = BZ(\omega) = A^T Y(\omega)$, in particular,

$$Z^0(\omega) = B^+ A^T Y(\omega), \quad \omega \in \Omega, \tag{6}$$

which is understood as the minimum residual norm square $\min_Z ||Y(\omega) - AZ(\omega)||^2$ (3) for each realization of $\omega \in \Omega$ and has the lowest length $||Z^0||^2$.

If we write equation [8] as $G(\omega)A^T X^0(\omega) = B^+ A^T Y(\omega)$ considering that $Z^0(\omega) = G(\omega)A^T X^0(\omega)$ and successively multiply both its right-hand sides by the stochastic inverse matrix $G^{-1}(\omega)$ and then by the deterministic incidence matrix A, we get

$$AA^T X^0(\omega) = AG^{-1}(\omega)B^+ A^T Y(\omega), \quad \omega \in \Omega. \tag{7}$$

Note that the stochastic inverse $m \times m$-matrix $G^{-1}(\omega)$ is diagonal and easily determinable for each $\omega \in \Omega$; in particular, $G^{-1}(\omega) = \mathrm{diag}(g_1^{-1}(\omega), g_2^{-1}(\omega), \ldots, g_m^{-1}(\omega))$, $i = 1, 2, \ldots, m$.

Considering that the matrix AA^T is square, symmetrical and, hence, has an inverse matrix $(AA^T)^{-1}$, we get the final stochastic solution $X^0(\omega)$ to the stochastic equation (7) understood in terms of the generalized normal solution (3) as:

$$X^0(\omega) = (AA^T)^{-1} AG^{-1}(\omega)A^T (AA^T)^{-1} Y(\omega), \quad \omega \in \Omega, \tag{8}$$

which can after the introduction of the deterministic matrix

$$C = (AA^T)^{-1} A, \quad C^T = A^T ((AA^T)^{-1})^T \tag{9}$$

be written more compactly as follows:

$$X^0(\omega) = CG^{-1}(\omega)C^T Y(\omega), \quad \omega \in \Omega. \tag{10}$$

The statistical measures of the stochastic vector $X^0(\omega)$, in particular, the expectation vector $\bar{X^0}$ and the covariance matrix $K_{X^0 X^0}$, are determined from the stochastic solution (10) considering the stochastic independence of the elements of the stochastic vector $Y(\omega)$ and stochastic matrix $G(\omega)$, and appear to be presented as follows:

– the expectation vector $\bar{X^0} = E\{X^0(\omega)\}$

$$\bar{X^0} = C\bar{G^{-1}}C^T \bar{Y},$$

where $\bar{G^{-1}} = E\{G^{-1}(\omega)\} = \mathrm{diag}(E\{g_1^{-1}(\omega)\}, E\{g_2^{-1}(\omega)\}, \ldots, E\{g_m^{-1}(\omega)\})$ is the diagonal matrix of expectations with elements $E\{g_i^{-1}(\omega)\}$, $i = 1, 2, \ldots, m$; $\bar{Y} = E\{Y(\omega)\}$ is the vector of expectations of

the stochastic vector $Y(\omega)$. As the elements $g_i(\omega)$ are interval stochastic ones, i.e., evenly distributed within the intervals $[g_{down,i}, g_{up,i}]$ with the length $\Delta_{g_i} = g_{up,i} - g_{down,i}$, $E\{g_i^{-1}(\omega)\} = \frac{1}{\Delta_{g_i}} \ln \frac{g_{up,i}}{g_{down,i}}$;

– the covariance matrix $K_{X^0X^0} = E\{(\mathring{X}^0)(\omega)\mathring{X}^{0T}(\omega)\}$

$$K_{X^0X^0} = CE\{G^{-1}(\omega)C^T M_{YY} CG^{-1}(\omega)\}C^T - \bar{X}^0\bar{X}^{0T}$$

where $M_{YY} = E\{Y(\omega)Y^T(\omega)\}$ is the matrix of moments about the origin of the stochastic vector $Y(\omega)$. Note that the diagonal structure of the matrix G allows easy calculation of the equation $E\{G^{-1}(\omega)C^T M_{YY} CG^{-1}(\omega)\}$ in its final form.

Let us estimate the relative error of the stochastic generalized normal solution $X^0(\omega)$ of (8), (10) relatively to the accurate stochastic solution $X(\omega)$ of equation (4). We determine the relative error δ as an expectation of the stochastic error $\delta(\omega)$, equal to the difference ratio between the norms $||X^0(\omega)||$ and $||X(\omega)||$ of the compared stochastic solutions and the norm $||X(\omega)||$ of the accurate solution to equation (4), i.e.,

$$|\bar{\delta}| = |E\{\delta(\omega)\}| = \Big|E\Big\{\frac{||X(\omega)|| - ||X^0(\omega)||}{||X(\omega)||}\Big\}\Big|, \tag{11}$$

where $||\Theta(\omega)||$ is the stochastic $l_2(\omega)$-norm determined for each realization of $\omega \in \Omega$, for the stochastic vector $\Theta(\omega) = (\Theta_1(\omega), \Theta_2(\omega), \ldots, \Theta_n(\omega))^T$ or the stochastic diagonal matrix $\Theta(\omega) = \{_{ij}(\omega)\}_{i,j=1}^{n,m}$ according to the equations

$$l_2^{vector}(\omega) = \Big(\sum_{i=1}^{n} \Theta_i^2(\omega)\Big)^{1/2}, \quad l_2^{matrix}(\omega) = \Big(\sum_{i,j=1}^{nm} \Theta_{ij}^2(\omega)\Big)^{1/2}.$$

It can be shown that the estimate of the stochastic error $\delta(\omega)$ (11) satisfies the inequality

$$\Big|1 - \frac{||I||_G^2 \cdot ||I||_{AA^T}^2}{||G^{-1}(\omega)|| \cdot ||G(\omega)||}\Big| \le |\delta(\omega)|, \quad \omega \in \Omega,$$

where $||I||_G^2$ and $||I||_{AA^T}^2$ are Euclidean l_2-norms of single matrices I, one of which has the shape of the matrix G, and the other has the shape of the matrix AA^T.

Considering that the product of the norms $||G^{-1}(\omega)|| \cdot ||G(\omega)||$ is equal to the stochastic conditioning number $\mu_G(\omega)$ of the stochastic matrix $G(\omega)$, and the l_2-norms of single matrices I are equal to $||I||_G^2 = m$ and $||I||_{AA^T}^2 = n$ (n, m are the number of nodes and edges of the graph of the SC model of the system), we get the following estimate of the stochastic error $\delta(\omega)$ between the generalized normal and the accurate solutions $|1 - m \cdot n/\mu_G(\omega)| \le |\delta(\omega)|$, $\omega \in \Omega$. If we expand the equation $1/\mu_G(\omega)$ along the Taylor series retaining the first-order terms only and applying the expectation operator to the resulting equation, we find that the expectation of the relative error $\bar{\delta}$ satisfies the inequality $|1 - m \cdot n/\mu_{\bar{G}}| \le |\bar{\delta}|$, where $\mu_{\bar{G}} = ||\bar{G}^{-1}|| \cdot ||\bar{G}||$ is the conditioning number of the matrix of expectations $\bar{G}$ with all elements being equal to their expectations $\bar{G} = \mathrm{diag}(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_m)$ and the l_2-norms $||\bar{G}||$ and $||\bar{G}^{-1}||$ being equal to $||\bar{G}|| = (\sum_{i=1}^{m} \bar{g}_i^2)^{1/2}$ and $||\bar{G}^{-1}|| = (\sum_{i=1}^{m} \bar{g}_i^{-2})^{1/2}$, respectively. The equation for the expectation of the stochastic estimate of the error shows that $\bar{\delta}$ depends on the conditioning number $\mu_{\bar{G}}$ of the matrix $\bar{G}$ and the number of edges (m) and nodes (n) in the graph of the SC model of the engineering system.

The developed method is used in modeling of thermal processes in complex electronic systems and has proven to be efficient.

Acknowledgement

Publication is performed as a part of the national assignment for SRISA RAS (fundamental scientic research 47 GP) on the topic No.0065-2019-0001 (AAAA-A19-119011790077-1).

References

1. G. Adomian, *Stochastic Systems*. Mathematics in Science and Engineering, 169. Academic Press, Inc., Orlando, FL, 1983.

2. L. O. Chua, *Computer-aided Analysis of Electronic Circuits*. Algorithms and computational techniques. Prentice-Hall, Englwood Cliffs, New Jersey, 1975.
3. G. N. Ellison, *Thermal Computations for Electronics*. Conductive, radiative, and convective air cooling. CRC Press, New York, 2011.
4. F. R. Gantmacher, *The Theory of Matrices*. (Russian) Fizmatlit. Moscow, 2010.
5. R. A. Horn, C. R. Johnson, *Matrix Analysi*s. Cambridge University Press, Cambridge, 2013.
6. A. G. Madera, *Modeling of Heat Exchange in Technical Systems*. (Russian) Scientific Foundation named after academician VA. Melnikov, Moscow, 2005.
7. A. G. Madera, Interval-stochastic thermal processes in electronic systems: analysis and modeling. *J. Eng. Thermophys* **26** (2017), 17–28.
8. A. G. Madera, P. I. Kandalov, Modeling of temperature fields of technical systems under interval uncertainty. (Russian) *Tepl. Prots. Tekhn.* **6** (2014), 225–229.
9. A. G. Madera, P. I. Kandalov, Analysis of interval-stochastic temperature fields of technical systems. (Russian) *Prog. Prod. Sist.* **4** (2014), 41–45.
10. R. Penrose, A generalized inverse for matrices. *Proc. Cambridge Philos. Soc.* **51** (1955), 406–413.
11. V. S. Pugachev, *Probability Theory and Mathematical Statistics for Engineers*. Pergamon Press, Oxford, 1984.
12. R. Y. Rubinstein, D. P. Kroese, *Simulation and Monte Carlo Method*. 3rd edition. Wiley, New Jersey, 2016.
13. P. E. Wellstead, *Introduction to Physical System Modeling*. Hamilton Institute, 2005. Electronically published by: www.control-systems-principles.co.uk
14. V. S. Zarubin, *Mathematical Modeling in Engineering*. (Russian) MSTU, Moscow, 2003.

(Received 05.07.2020)

[1]Scientific Research Institute for System Analysis of the Russian Academy of Sciences, Moscow, Russia

[2]Muskhelishvili Institute of Computational Mathematics, Georgian Technical University, Tbilisi, Georgia

E-mail address: sfm12@mail.ru
E-mail address: h_meladze@hotmail.com