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ON OSCILLATIONS OF REAL-VALUED FUNCTIONS

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Abstract. We consider the question whether a given real-valued non-negative upper semi-continuous function on a topological space E is the oscillation function of a Borel real-valued function defined on the same space E.

Let E be a topological space, let $\mathbf{R}$ denote the real line and let $f : E \to \mathbf{R}$ be a function. Suppose that f is locally bounded at each point x of E, i.e., there exists a neighborhood $U(x)$ of x such that the restriction $f|U(x)$ is bounded. Then there exist two values

$$f^*(x) = \limsup_{y \to x} f(y), \quad f_*(x) = \liminf_{y \to x} f(y),$$

and the difference

$$O_f(x) = \limsup_{y \to x} f(y) - \liminf_{y \to x} f(y)$$

is called the oscillation of f at x. As is known, the real-valued function

$$f^*(x) = \limsup_{y \to x} f(y) \ (x \in E)$$

is upper semi-continuous on E and the real-valued function

$$f_*(x) = \liminf_{y \to x} f(y) \ (x \in E)$$

is lower semi-continuous on E (see, e.g., [1], [3], [5]). Consequently, the produced function

$$O_f(x) = f^*(x) - f_*(x) \ (x \in E)$$

is non-negative and upper semi-continuous on E.

Let us mention some facts concerning the behavior of the oscillations of real-valued functions.

(a) f is continuous at a point $x \in E$ if and only if $O_f(x) = 0$.

In particular, if x is an isolated point of E, then $O_f(x) = 0$ for an arbitrary $f : E \to \mathbf{R}$.

(b) $O_{tf}(x) = |t|O_f(x)$ for any real number t and for each point $x \in E$;

(c) $O_{f_1+f_2} \le O_{f_1} + O_{f_2}$.

Actually, (c) implies the finite sub-additivity of the operator $O : f \to O_f$.

(d) If a series $\sum\{f_n : n \in \mathbf{N}\}$ of real-valued locally bounded functions on E converges uniformly to f, then $O_f \le \sum\{O_{f_n} : n \in \mathbf{N}\}$.

(e) If a sequence $\{f_n : n \in \mathbf{N}\}$ of real-valued locally bounded functions on E converges uniformly to f, then the corresponding sequence of oscillations $\{O_{f_n} : n \in \mathbf{N}\}$ converges uniformly to the oscillation O_f.

Notice that (e) is a generalization of the well-known theorem of mathematical analysis, according to which the limit of a uniformly convergent sequence of real-valued continuous functions is also continuous.

In connection with the above facts, there arises the following natural question:

For a given real-valued non-negative upper semi-continuous function g on E, is it true that there exists a locally bounded function $f : E \to \mathbf{R}$ such that $O_f = g$?

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In case the answer to this question is positive, as far as g has a good descriptive structure (namely, g is upper semi-continuous), it is natural to try to find an f satisfying $O_f = g$ and also having good descriptive properties (e.g., a real-valued Borel measurable function f on E for which $O_f = g$).

Exemple 1. Let $E = \mathbf{R}$ and $g : \mathbf{R} \to \{1\}$. The widely known Dirichlet function $\chi : \mathbf{R} \to \{0, 1\}$ satisfies the equality $O_\chi = g$. Recall that f takes value 1 at all rational points of $\mathbf{R}$ and takes value 0 at all irrational points of $\mathbf{R}$. Obviously, χ is a Borel function. Denoting by $\mathbf{c}$ the cardinality of the continuum, there are $2^{\mathbf{c}}$ many functions $f : \mathbf{R} \to \mathbf{R}$ such that $O_f = g$. Clearly, most of such f are not Borel functions.

The main goal of the present communication is to consider the above-formulated question and to give its solution for some classes of topological spaces E.

First of all, let us remark that the trivial necessary condition for the existence of f is the equality $g(x) = 0$ for all isolated points x in E.

Suppose that this condition is satisfied and denote by E' the closure of the set of all isolated points in E. Further, put $U = E \setminus E'$ and observe that the open set U does not contain isolated points. Denote by $g|U$ the restriction of g to U.

Lemma 1. *Assume that there exists a function $\phi : U \to \mathbf{R}$ such that $O_\phi = g|U$ and the relation $0 \le \phi \le g|U$ holds true.*

Then there exists a function $f : E \to [0, +\infty[$ such that $O_f = g$. Moreover, if ϕ is Borel, then f can be chosen to be Borel, too.

Proof. We define the required f as follows:

$f(x) = g(x)$ if x belongs to the set E';

$f(x) = \phi(x)$ if x belongs to the set U.

Let us verify that $O_f(x) = g(x)$ for each point $x \in E$.

If $x \in E'$, then it is easy to see that $f_*(x) = 0$ and $f^*(x) \ge g(x)$. At the same time, keeping in mind the relation $0 \le \phi \le g|U$ and the upper semi-continuity of g, we infer that $f^*(x) \le g(x)$, which implies

$$f^*(x) = g(x), \ O_f(x) = f^*(x) - f_*(x) = g(x) - 0 = g(x).$$

If $x \in U$, then using the equality $O_\phi = g|U$ and taking into account that U is an open set, we conclude that $O_f(x) = g(x)$, which completes the proof. □

In many cases, the above lemma enables one to reduce the formulated problem to those topological spaces E which do not contain isolated points.

Lemma 2. *Let E be a topological space, let $g : E \to \mathbf{R}$ be a non-negative upper semi-continuous function, and let $\{U_i : i \in I\}$ be a disjoint family of nonempty open subsets of E such that the union $\cup\{U_i : i \in I\}$ is everywhere dense in E. Suppose also that for each index $i \in I$, there exists a function $\phi_i : U_i \to \mathbf{R}$ satisfying these two conditions:*

(1) *$0 \le \phi_i \le g|U_i$ and the set $\{x \in U_i : \phi_i(x) = 0\}$ is everywhere dense in U_i;*

(2) *$g|U_i = O_{\phi_i}$.*

Let a function $f : E \to \mathbf{R}$ be defined by the formula

$f(x) = \phi_i(x)$ if $x \in U_i$, and $f(x) = g(x)$ if $x \in E \setminus \cup\{U_i : i \in I\}$.

Then the equality $g = O_f$ holds true.

The proof of Lemma 2 is similar to that of Lemma 1.

Using the above lemmas, one can deduce the following statement.

Theorem 1. *Let E be a locally compact metric space and let $g : E \to \mathbf{R}$ be a non-negative upper semi-continuous function such that $g(x) = 0$ for any isolated point x of E.*

Then there exists a Borel function $f : E \to \mathbf{R}$ for which $g = O_f$.

Theorem 2. *Let E be a topological space satisfying the following condition:*

There exists an infinite base $\mathcal{B}$ of E such that the cardinality of any nonempty set $U \in \mathcal{B}$ is strictly greater than $\mathrm{card}(\mathcal{B})$.

Then for every non-negative upper semi-continuous function $g : E \to \mathbf{R}$, there exists a function $f : E \to \mathbf{R}$ such that $O_f = g$.

The proof of Theorem 2 essentially uses one auxiliary notion and Lemma 3 presented below.

Let $\mathbf{b}$ be an infinite cardinal and let E be a topological space.

A point $x \in E$ is called a $\mathbf{b}$-point in E if there exists a neighborhood $U(x)$ of x whose cardinality does not exceed $\mathbf{b}$.

Lemma 3. *If E is a topological space with a base whose cardinality does not exceed $\mathbf{b}$, then the cardinality of the set of all $\mathbf{b}$-points in E does not exceed $\mathbf{b}$.*

Lemma 3 enables one to make appropriate changes in the graph of a given real-valued non-negative upper semi-continuous function $g : E \to \mathbf{R}$ in order to obtain a function $f : E \to \mathbf{R}$ such that $O_f = g$.

In general, those changes produce a function f with bad descriptive properties. However, if E fulfils certain additional assumptions, then the required f can be chosen to be Borel.

Theorem 3. *Let E be a metric space satisfying the condition of Theorem 2.*

Then for every non-negative upper semi-continuous function $g : E \to \mathbf{R}$, there exists a Borel function $f : E \to \mathbf{R}$ such that $O_f = g$.

The proof of Theorem 3 is based on the following fact which is valid for any metric space E satisfying the condition of Theorem 2:

If $X \subset E$ has cardinality, strictly less than $\mathrm{card}(E)$, then there exists an everywhere dense set $Y \subset E$ of type F_σ in E such that

$$X \cap Y = \emptyset, \quad \mathrm{card}(Y) < \mathrm{card}(E).$$

For certain topological groups, we have the next statement.

Theorem 4. *Let E be a non-discrete locally compact σ-compact topological group and let $g : E \to \mathbf{R}$ be a non-negative upper semi-continuous function.*

Then there exists a function $f : E \to \mathbf{R}$ such that $O_f = g$.

The proof of Theorem 4 is based on the following important equality

$$\mathrm{card}(E) = 2^{w(E)},$$

where $w(E)$ denotes the topological weight of E (see, e.g., [2]). This equality implies that the assumption of Theorem 2 is automatically satisfied.

Recall that a topological space E is resolvable (in the sense of E. Hewitt) if there exists a partition $\{A, B\}$ of E such that both sets A and B are everywhere dense in E (see [4]). Otherwise, E is called an irresolvable space. Resolvable spaces have a number of interesting properties. For instance, the following assertions are valid.

(1) Any open subspace of a resolvable space is resolvable.

(2) The topological product of a family $\{E_i : i \in I\}$ of nonempty topological spaces is resolvable whenever at least one E_i is resolvable.

(3) The topological sum of a family $\{E_i : i \in I\}$ of nonempty topological spaces is resolvable if and only if all E_i $(i \in I)$ are resolvable.

(4) If E possesses a pseudo-base all members of which are resolvable, then E itself is resolvable.

(5) Any nonempty locally compact space without isolated points is resolvable.

(6) Any metric space without isolated points is resolvable.

(7) If E is resolvable, then for each F_σ-subset X of E there exists a function $f : E \to \mathbf{R}$ such that X coincides with the set of all points of discontinuity of f.

In this context, it makes sense to notice that the topological product of a family of irresolvable spaces can be resolvable, a closed subspace of a resolvable space can be irresolvable, and a continuous image of a resolvable space can be irresolvable.

Exemple 2. Let E be a topological space and let $g : E \to \mathbf{R}$ be a real-valued non-negative upper semi-continuous function. Suppose that the graph of g is a resolvable subspace of the product space $E \times \mathbf{R}$. Then there exists a function $f : E \to \mathbf{R}$ such that $g = O_f$. In particular, if E is a resolvable space, then for any real-valued non-negative constant function $g : E \to \mathbf{R}$, there exists a function $f : E \to \mathbf{R}$ such that $O_f = g$.

Theorem 5. *Let E be a metric space and let $g : E \to \mathbf{R}$ be a real-valued non-negative upper semi-continuous function.*

If the graph of g considered as a subspace of $E \times \mathbf{R}$ does not contain isolated points, then there exists a Borel function $f : E \to \mathbf{R}$ such that $O_f = g$.

Exemple 3. Let E be an infinite set, let $\mathcal{J}$ be a σ-ideal of subsets of E, and let $\mathcal{F} = \{X \subset E : E \setminus X \in \mathcal{J}\}$ be the dual filter of $\mathcal{J}$. Suppose that the following two conditions are fulfilled:

($*$) $\mathrm{card}(X) = \mathrm{card}(E)$ for each set $X \in \mathcal{F}$;

($**$) there exists a base $\mathcal{B}$ of $\mathcal{J}$ with $\mathrm{card}(\mathcal{B}) \leq \mathrm{card}(E)$.

Denote $\mathcal{T} = \{\emptyset\} \cup \mathcal{F}$. Then $\mathcal{T}$ is a topology on E such that:

(i) the space $(E, \mathcal{T})$ is resolvable;

(ii) for a function $g : E \to \{1\}$, there exists a function $f : E \to \mathbf{R}$ satisfying $O_f = g$;

(iii) for the same function $g : E \to \{1\}$, there exists no Borel function $h : E \to \mathbf{R}$ satisfying $O_h = g$.

Exemple 4. Take an infinite set E equipped with a nontrivial ω_1-complete ultrafilter Φ of subsets of E (this condition is equivalent to the existence of a two-valued measurable cardinal number). Equip E with the topology

$$\mathcal{T} = \{\emptyset\} \cup \Phi.$$

The obtained topological space $(E, \mathcal{T})$ has the following property:

For any function $f : E \to \mathbf{R}$, there exists a set $X \in \Phi$ such that the restriction $f|X$ is constant.

Therefore, for every function $f : E \to \mathbf{R}$, there are points x in E at which f is continuous and, consequently, $O_f(x) = 0$.

The latter implies that if g is a real-valued strictly positive constant function on E, then there is no $f : E \to \mathbf{R}$ such that $O_f = g$.

So, Example 4 shows us that certain restrictions on a general topological space E are necessary if one wants to have a positive solution to the question discussed in this note.

Let E be a topological space, (M, ρ) be a bounded metric space and let $f : E \to M$ be a function. For each point $x \in E$, one can define

$$O_f(x) = \inf\{\mathrm{diam}(f(U(x))) : U(x) \in \mathcal{F}(x)\},$$

where $\mathcal{F}(x)$ is the filter of all neighborhoods of x and $\mathrm{diam}(f(U(x)))$ denoting the diameter of the set $f(U(x))$. The obtained function $O_f : E \to \mathbf{R}$ called also the oscillation of f, is non-negative and upper semi-continuous.

Notice that the assertions (a) and (e) remain true for this more general concept of O_f.

The question analogous to the considered above can be formulated in terms of the pair (E, M).

Namely, one can ask about a characterization of all those pairs (E, M) for which any non-negative upper semi-continuous function $g : E \to \mathbf{R}$ admits a (Borel) function $f : E \to M$ such that $O_f = g$.

This question seems to be of interest from the viewpoint of mathematical analysis and general topology.

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