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A SIMPLE DERIVATION OF THE KEY EQUATION IN JANASHIA–LAGVILAVA METHOD

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Abstract. We provide a simple derivation of the key system of equations for the corresponding boundary value problem in the Janashia–Lagvilava matrix spectral factorization method.

1. Introduction

Let

$$S(t)=\begin{pmatrix} s_{11}(t) & s_{12}(t) & \cdots & s_{1r}(t)\\ s_{21}(t) & s_{22}(t) & \cdots & s_{2r}(t)\\ \vdots & \vdots & \vdots & \vdots\\ s_{r1}(t) & s_{r2}(t) & \cdots & s_{rr}(t)\end{pmatrix}, \tag{1}$$

$|t|=1$, be a positive definite (a.e.) matrix function with integrable entries, $s_{ij}\in L^1(\mathbb{T})$, defined on the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$.

Wiener's matrix spectral factorization theorem [9] asserts that if

$$\int\limits_{\mathbb{T}} \log\det S(t)\,dt > -\infty, \tag{2}$$

then S admits the factorization

$$S(t)=S_+(t)S_+^*(t), \tag{3}$$

where S_+ can be analytically extended inside the unit disk $\mathbb{D}$, and $S_+^*(t)$ is the Hermitian conjugate to $S_+(t)$. Furthermore, the entries of S_+ are the square integrable functions and, actually, belong to the Hardy space $H^2=H^2(\mathbb{D})$ (as usual, the functions from the Hardy space and their boundary values are identified). Representation (3) is unique (up to a constant unitary factor) under the additional requirement that the analytic function S_+ is *outer* (for the definition, see §2). Condition (2) is necessary and sufficient for the spectral factorization (3) to exist.

An approximate computation of the factor S_+ for the given matrix function (1) is an important challenging problem due to its practical applications. Therefore, different authors have developed dozens of methods for such factorization as the Levinson–Durbin algorithm, Bauer method (by Toeplitz matrix decomposition), Wilsons algorithm (based on Newton–Raphson iterations), symmetric factor extraction, solutions via algebraic Riccati equation, etc. (see [7,8]).

The Janashia–Lagvilava algorithm [4,5] is a relatively new method of a matrix spectral factorization which proved to be effective [3].

In this algorithm, the computational complexity of the problem is reduced to the minimum by intelligent manipulations. The algorithm starts with the LU triangular factorization

$$S(t)=M(t)M^*(t),$$

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with

$$M(t)=\begin{pmatrix} f_1^+(t) & 0 & \cdots & 0 & 0\\ \xi_{21}(t) & f_2^+(t) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \xi_{r-1,1}(t) & \xi_{r-1,2}(t) & \cdots & f_{r-1}^+(t) & 0\\ \xi_{r1}(t) & \xi_{r2}(t) & \cdots & \xi_{r,r-1}(t) & f_r^+(t)\end{pmatrix},$$

where f_j^+, $j=1,2,\dots,r$, are outer analytic functions in H^2 (denoted as $f_j^+\in H^2_O$) and $\xi_{ij}\in L^2(\mathbb{T})$, $2\le i\le r$, $1\le j<j$. Then the algorithm performs step-by-step spectral factorization of principal leading submatrices of S (see [5]).

A key component of this scheme is the constructive proof of the following

Theorem 1. *Let*

$$F(t)=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0\\ \zeta_1(t) & \zeta_2(t) & \zeta_3(t) & \cdots & \zeta_{m-1}(t) & f^+(t)\end{pmatrix} \tag{4}$$

be an $m\times m$ matrix, where $f^+\in H^2_O$ and $\zeta_j\in L^2(\mathbb{T})$, $j=1,2,\dots,m-1$. Then, there exists an $m\times m$ unitary matrix function U of the special structure

$$U(t)=\begin{pmatrix} u_{11}^+(t) & u_{12}^+(t) & \cdots & u_{1,m-1}^+(t) & u_{1m}^+(t)\\ u_{21}^+(t) & u_{22}^+(t) & \cdots & u_{2,m-1}^+(t) & u_{2m}^+(t)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ u_{m-1,1}^+(t) & u_{m-1,2}^+(t) & \cdots & u_{m-1,m-1}^+(t) & u_{m-1,m}^+(t)\\ \overline{u_{m1}^+(t)} & \overline{u_{m2}^+(t)} & \cdots & \overline{u_{m,m-1}^+(t)} & \overline{u_{mm}^+(t)}\end{pmatrix},\quad u_{ij}^+\in H^\infty, \tag{5}$$

with

$$\det U(t)=1 \ \text{ for a.a. } t\in\mathbb{T} \tag{6}$$

such that the entries of the product FU are analytic functions in H^2, i.e.,

$$FU\in H^2(\mathbb{D})^{m\times m}. \tag{7}$$

The existence of such a unitary matrix function U follows from the general existence theorem of the matrix spectral factorization and is demonstrated in [1]. The most important finding of Janashia and Lagvilava was, however, the observation that the columns of U can be constructed separately, independently of each other. In particular, the following theorem holds.

Theorem 2. *Let F and U be as in Theorem 1. Then, the columns of U (more specifically, taking $x_i^+=u_{ij}^+$, $i=1,2,\dots,m$, for each $j=1,2,\dots,m$), are the solutions of the following multi-dimensional boundary value problem*

$$\begin{cases} \zeta_1(t)x_m^+(t)-f^+(t)\overline{x_1^+(t)}=\varphi_1^+(t),\\ \zeta_2(t)x_m^+(t)-f^+(t)\overline{x_2^+(t)}=\varphi_2^+(t),\\ \vdots\\ \zeta_{m-1}(t)x_m^+(t)-f^+(t)\overline{x_{m-1}^+(t)}=\varphi_{m-1}^+(t),\\ \zeta_1(t)x_1^+(t)+\zeta_2(t)x_2^+(t)+\ldots+\zeta_{m-1}(t)x_{m-1}^+(t)+f^+(t)\overline{x_m^+(t)}=\varphi_m^+(t),\end{cases} \tag{8}$$

where ζ_i and f^+ are the entries of F, and $x_i^+\in H^\infty$ and $\varphi_i^+\in H^2$ are the unknowns.

Actually, the Janashia–Lagvilava algorithm approximates the solution of the above system for the given matrix function F. This task is not anymore as difficult as the discovery of system (8) itself.

A long sequence of transformations which derives system (8) from condition (6) is presented in [1]. In the present paper, we deduce the same system much easier by using a more transparent way.

The paper is organized as follows. In the next section we introduce the necessary notation and formulate the well-known theorems used afterwards. Although the proof of Theorem 1 based on the Wiener's existence theorem of the matrix spectral factorization is outlined in [1], for the readers convenience, we present the detailed proof of this theorem in Section 3. This makes the paper more self-contained. The proof of Theorem 2 is given in Section 4.

2. Notation and Preliminary Observations

Let $L^p(\mathbb{T})$, $p>0$, be the Lebesgue space of p integrable functions on the unit circle $\mathbb{T}:=\{z\in\mathbb{C}: |z|=1\}$ and

$$H^p=H^p(\mathbb{D}):=\left\{f\in\mathcal{A}(\mathbb{D}):\sup_{r<1}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta<\infty\right\}$$

be the Hardy space of analytic functions on the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$.

For $f\in H^p$ and $t=e^{i\theta}\in\mathbb{T}$, we assume that

$$f(t)=f(z)|_{z=t}:=\lim\nolimits_{r\to1}f(re^{i\theta})$$

(which is defined a.e. on $\mathbb{T}$); the class of the boundary value functions of all functions from H^p is denoted by L^p_+. It is well known that $L^p_+\subset L^p$ and, for $p\geq 1$,

$$L_p^+=\{f\in L^p(\mathbb{T}):c_k\{f\}=0 \text{ for } k<0\},$$

where $c_k\{f\}$ stands for the k-th Fourier coefficient of f. Furthermore, there is a one-to-one correspondence

$$L^p_+\longleftrightarrow H^p,\quad p>0, \tag{9}$$

which allows these two classes to be naturally identified. In particular, one can speak about the values of $f\in L^p_+$ inside the unit disk. The relation (9) can be strengthened by claiming that the function $f\in L^p_+$ cannot be equal to zero on a subset of $\mathbb{T}$ of positive measure and, furthermore, for each $f\in L^p_+$, we have

$$\int\limits_{\mathbb{T}}\log|f(t)|\,dt>-\infty.$$

That is why condition (2) is necessary for the existence of factorization (3) and Wiener proved its sufficiency, as well.

We use Smirnov's theorem (see, e.g., [6]) which claims that if a function $f\in H^p$ and its boundary values function belongs to L^q $(q>p)$, then $f\in H^q$. This theorem can be briefly formulated as

$$f\in H^p\cap L^q_+\Longrightarrow f\in H^q. \tag{10}$$

A nonzero function f is called *outer* if it can be reconstructed from the absolute values of its boundary values, namely,

$$f(z)=c\cdot\exp\left(\frac{1}{2\pi}\int_{\mathbb{T}}\frac{t+z}{t-z}\log\big|f(t)\big|\,dt\right),\qquad |c|=1. \tag{11}$$

The class of outer functions in H^p is denoted by H^p_O. Formula (11) implies that if $f,g\in H^p_O$ and $|f(t)|=|g(t)|$ for a.a. $t\in\mathbb{T}$, then $f=cg$ for some constant c with absolute value 1. The product of two outer functions is again outer and Hölder's inequality guarantees that if $f\in H^p_O$ and $g\in H^q_O$, then $fg\in H_O^{pq/(p+q)}$.

For any set $\mathcal{S}$, we denote by $\mathcal{S}^{m\times n}$ the set of $m\times n$ matrices with entries from $\mathcal{S}$.

A matrix function $G\in H^2(\mathbb{D})^{m\times m}$ is called *outer*, and we write $G\in H^2(\mathbb{D})^{m\times m}_O$, if the determinant of G is outer, i.e., $\det G\in H_O^{2/m}$ (cf. [2]).

For any matrix $M\in\mathbb{C}^{m\times m}$, we use the standard notation M^T, $M^*:=\overline{M}^T$, $\mathrm{Cof}(M)$, and $\mathrm{Adj}(M):=\mathrm{Cof}(M)^T$ for the transpose, the Hermitian conjugate, the cofactor matrix and the adjugate. The same notation is used for the matrix functions, as well.

A matrix function $U\in L^\infty(\mathbb{T})^{m\times m}$ is called *unitary* if

$$U(t)U^*(t)=I_m\quad\text{a.e.},$$

where I_m stands for the $m\times m$ unit matrix.

3. Proof of Theorem 1

Since $F\in L^2(\mathbb{T})^{m\times m}$ and $\det F=f^+\in H^2_O$, we have $FF^*\in L^1(\mathbb{T})^{m\times m}$ and

$$\int\limits_{\mathbb{T}}\log\det F(t)F^*(t)\,dt=2\int\limits_{\mathbb{T}}\log|f^+(t)|\,dt>-\infty.$$

Therefore, by virtue of the matrix spectral factorization theorem,

$$F(t)F^*(t)=G_+(t)G_+^*(t),$$

where $G_+\in H^2(\mathbb{D})^{m\times m}_O$. Since $\det G_+\in H_O^{2/m}$ and $|\det G_+(t)|=|\det F(t)|$ for a. a. $t\in\mathbb{T}$, we have $\det G_+(z)=c\,(\det F)(z)=cf^+(z)$ for $z\in\mathbb{D}$, with $|c|=1$ and it can be assumed that $c=1$, i.e.,

$$\det G_+=f^+. \tag{12}$$

Let

$$U(t)=F^{-1}(t)G_+(t). \tag{13}$$

We have

$$UU^*=F^{-1}G_+G_+^*(F^{-1})^*=F^{-1}FF^*(F^*)^{-1}=I_m \text{ a.e. on } \mathbb{T},$$

which implies that U is a unitary matrix function, and therefore,

$$U\in L^\infty(\mathbb{T})^{m\times m}. \tag{14}$$

We also know that (6) holds because of equations (13) and (12).

Note that

$$F^{-1}=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0\\ -\zeta_1/f^+ & -\zeta_2/f^+ & -\zeta_3/f^+ & \cdots & -\zeta_{m-1}/f^+ & 1/f^+ \end{pmatrix}. \tag{15}$$

Therefore, it follows from (13) that the entries in the first $m-1$ rows of U and G_+ coincide. Since we know that these entries belong to H^2 and also (14) holds, it follows from Smirnov's theorem that

$$u_{ij}\in H^\infty,\quad 1\le i\le m-1,\;\; 1\le j\le m.$$

For the entries of the last row of U, we have

$$\overline{u_{mj}}=\operatorname{cof}(u_{mj})\in H^\infty,$$

since $U^*=U^{-1}=\operatorname{Adj}(U)=\operatorname{Cof}(U)^T$. Hence, the structure of U has the form (5), and Theorem 1 is proved.

4. Proof of Theorem 2

Assume

$$F(t)U(t)=\Phi_+, \tag{16}$$

where F is the matrix function (4), U is the unitary matrix function (5) satisfying (6) and

$$\Phi_+\in H^2(\mathbb{D})^{m\times m}_O$$

(the determinant of Φ^+ is outer because $f^+\in H^2_O$ and (6) holds). Then the last equation in (8) follows immediately from (16). It also follows from (16) that

$$U^*(t)F^{-1}(t)=\Phi_+^{-1}(t)=\frac{1}{f^+}\operatorname{Adj}\Phi_+,$$

i.e.,

$$\begin{pmatrix} \overline{u_{11}^+} & \overline{u_{21}^+} & \dots & \overline{u_{m-1,\,1}^+} & u_{m1}^+ \\ \overline{u_{12}^+} & \overline{u_{22}^+} & \dots & \overline{u_{m-1,\,2}^+} & u_{m2}^+ \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{u_{1m}^+} & \overline{u_{2m}^+} & \dots & \overline{u_{m-1,\,m}^+} & u_{mm}^+ \end{pmatrix} \begin{pmatrix} f^+ & 0 & \cdots & 0 & 0 \\ 0 & f^+ & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & f^+ & 0 \\ -\zeta_1 & -\zeta_2 & \cdots & -\zeta_{m-1} & 1 \end{pmatrix} = \operatorname{Adj}\Phi_+.$$

Then, we conclude that for each $j = 1, 2, \dots, m$,

$$\begin{cases} f^+\overline{u_{1j}^+} - \zeta_1\, u_{mj}^+ = \phi_{j1}^+ \\ f^+\overline{u_{2j}^+} - \zeta_2\, u_{mj}^+ = \phi_{j2}^+ \\ \vdots \\ f^+\overline{u_{m-1,\,j}^+} - \zeta_{m-1}\, u_{mj}^+ = \phi_{j,m-1}^+, \end{cases} \tag{17}$$

where we know that each ϕ_{jk}^+ belongs to $H^{2/(m-1)}$ as they are the entries of $\operatorname{Adj}\Phi_+$. However, equations (17) suggest that $\phi_{jk}^+ \in L^2(\mathbb{T})$ and applying Smirnov's theorem, we can conclude that $\phi_{jk}^+ \in H^2$.

Thus Theorem 2 is proved.

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