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STEADY VIBRATIONS PROBLEMS IN THE THEORY OF THERMOVISCOELASTIC POROUS MIXTURES

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Abstract. In this paper, the linear theory of thermoviscoelastic binary porous mixtures is considered and the basic boundary value problems (BVPs) of steady vibrations are investigated. Namely, the fundamental solution of the system of equations of steady vibrations is constructed explicitly and its basic properties are established. Green's identities are obtained and the uniqueness theorems for classical solutions of the internal and external basic BVPs of steady vibrations are proved. The surface and volume potentials are constructed and their basic properties are given. The determinants of symbolic matrices of the singular integral operators are calculated explicitly and the BVPs are reduced to the always solvable singular integral equations for which Fredholm's theorems are valid. Finally, the existence theorems for classical solutions of the internal and external BVPs of steady vibrations are proved by means of the potential method and the theory of singular integral equations.

1. Introduction

The prediction of the mechanical properties of viscoelastic materials has been one of hot topics of continuum mechanics for more than 100 years. The construction of mathematical models of viscoelastic continua arise by an extensive use of viscous materials in many branches of engineering, technology and biomechanics (see Lakes [19], Brinson and Brinson [5] and references therein).

In the past two decades there has been much effort to develop mathematical models of thermoviscoelastic mixtures. Indeed, Ieşan [12] has presented the theory of thermoelasticity of binary porous mixtures in Lagrangian description, and the classical Kelvin–Voigt viscoelastic model is generalized by using a mixture theory. The existence and exponential decay of a solution in the linear variant of this theory is studied by Quintanilla [23]. The theory of thermoviscoelastic composites modelled as interacting Cosserat continua is introduced by Ieşan [14]. A mathematical model of porous thermoviscoelastic binary mixtures is presented by Ieşan and Quintanilla [16], where the individual components are modelled as Kelvin–Voigt viscoelastic materials. In [15], a nonlinear theory of heat conducting mixtures is introduced. A mixture theory for microstretch thermoviscoelastic solids is developed by Chiriţă and Galeş [6]. The theory of microstretch thermoviscoelastic composite materials is constructed by Passarella et al. [21]. A continuum theory for a thermoviscoelastic composite with the help of an entropy production inequality proposed by Green and Laws is presented by Ieşan and Scalia [17]. Recently, a nonlinear theory is derived for a thermoviscoelastic diffusion composite which is modeled as a binary mixture consisting of two Kelvin–Voigt viscoelastic materials by Aouadi et al. [2].

The basic problems of these theories are intensively investigated by scientists of several research groups in the series of papers [1,3,7–11,13,22]. Moreover, in [25,26], the basic properties of plane waves are established, the uniqueness and existence theorems are proved in the theories of viscoelasticity and thermoviscoelasticity for binary mixtures without pores. Recently, the potential method is developed in the theory of viscoelastic binary porous mixtures by Svanadze [27].

For an extensive review of the works and basic results in the theory of mixtures see the books of Bowen [4] and Rajagopal and Tao [24].

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In this paper, the linear theory of thermoviscoelastic binary porous mixtures (see Ieşan [12]) is considered and the basic BVPs of steady vibrations are investigated. Indeed, the fundamental solution of the system of equations of steady vibrations in the considered theory is constructed explicitly and its basic properties are established. Green's identities are obtained and the uniqueness theorems for classical solutions of the internal and external basic BVPs of steady vibrations are proved. The surface and volume potentials are constructed and their basic properties are given. The determinants of symbolic matrices are calculated explicitly. The BVPs are reduced to the always solvable singular integral equations for which Fredholm's theorems are valid. Finally, the existence theorems for classical solutions of the internal and external BVPs of steady vibrations are proved by means of the potential method and the theory of singular integral equations.

2. Basic Equations

We consider a thermoelastic binary porous mixture of constituents $s^{(1)}$ and $s^{(2)}$ that occupies the region Ω of the Euclidean three-dimensional space $\mathbb{R}^3$, where $s^{(1)}$ and $s^{(2)}$ are a Kelvin–Voigt material and an isotropic elastic solid, respectively. Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of $\mathbb{R}^3$ and let t denote the time variable. We assume that subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range (1,2,3) and the dot denotes differentiation with respect to t.

Let $\hat{\mathbf{u}}(\mathbf{x}, t)$ and $\hat{\mathbf{w}}(\mathbf{x}, t)$ be the partial displacements of constituents $s^{(1)}$ and $s^{(2)}$, respectively; $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$, $\hat{\mathbf{w}} = (\hat{w}_1, \hat{w}_2, \hat{w}_3)$. We denote by $\hat{\varphi}(\mathbf{x}, t)$ and $\hat{\psi}(\mathbf{x}, t)$ the changes of volume fraction fields from the reference configuration for the constituents $s^{(1)}$ and $s^{(2)}$, respectively. Let $\hat{\theta}(\mathbf{x})$ be the temperature measured from some constant absolute temperature T_0 ($T_0 > 0$).

The governing system of field equations of motion in the linear theory of thermoviscoelastic binary porous mixtures consists of the following equations (see Ieşan [12]):

1. *The constitutive equations*

$$
\begin{gathered}
t_{jl} = (\lambda + \nu) e_{rr} \delta_{jl} + 2(\mu + \zeta) e_{jl} + (\alpha + \nu) g_{rr} \delta_{jl} + (2\kappa + \zeta) g_{jl} + (2\gamma + \zeta) g_{lj} \\
+ (m^{(1)} + l^{(1)}) \hat{\varphi}\, \delta_{jl} + (m^{(2)} + l^{(2)}) \hat{\psi}\, \delta_{jl} - (\beta^{(1)} + \beta^{(2)})\, \hat{\theta}\, \delta_{jl} + \lambda^* \dot{e}_{rr} \delta_{jl} + 2\mu^* \dot{e}_{jl}, \\
s_{jl} = \nu e_{rr} \delta_{jl} + 2\zeta e_{lj} + \alpha g_{rr} \delta_{jl} + 2\kappa g_{lj} + 2\gamma g_{jl} + (l^{(1)} \hat{\varphi} + l^{(2)} \hat{\psi})\, \delta_{jl} - \beta^{(2)}\, \hat{\theta}\, \delta_{jl}, \\
h_l^{(1)} = \alpha^{(1)} \hat{\varphi}_{,l} + \alpha^{(3)} \hat{\psi}_{,l} + b\, d_l, \qquad h_l^{(2)} = \alpha^{(3)} \hat{\varphi}_{,l} + \alpha^{(2)} \hat{\psi}_{,l} + c_0 d_l, \\
g^{(1)} = -m^{(1)} e_{rr} - l^{(1)} g_{rr} - \zeta^{(1)} \hat{\varphi} - \zeta^{(3)} \hat{\psi} + b^{(1)}\, \hat{\theta}, \\
g^{(2)} = -m^{(2)} e_{rr} - l^{(2)} g_{rr} - \zeta^{(3)} \hat{\varphi} - \zeta^{(2)} \hat{\psi} + b^{(2)}\, \hat{\theta}, \\
p_l = \xi d_l + \xi^* \dot{d}_l + b \hat{\varphi}_{,l} + c_0 \hat{\varphi}_{,l} + b^* \hat{\theta}_{,l}, \qquad \rho \eta = \beta^{(1)} e_{rr} + \beta^{(2)} g_{rr} + b^{(1)}\, \hat{\varphi} + b^{(2)}\, \hat{\psi} + a \hat{\theta}, \\
q_l = k \theta_{,l} + f^* \dot{d}_l, \qquad l, j = 1, 2, 3,
\end{gathered}
\tag{1}
$$

where t_{jl} and s_{jl} are the components of the partial stresses of the constituents $s^{(1)}$ and $s^{(2)}$, respectively; λ, μ, α, γ, ζ, ν, κ, ξ, $\beta^{(1)}$, $\beta^{(2)}$, a, b, c_0, k, $b^{(1)}$, $b^{(2)}$, $m^{(1)}$, $m^{(2)}$, $l^{(1)}$, $l^{(2)}$, $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$, $\zeta^{(1)}$, $\zeta^{(2)}$, $\zeta^{(3)}$, λ^*, μ^*, ξ^*, b^*, f^* are the constitutive coefficients and $a \neq 0$, δ_{jl} is the Kronecker delta and

$$
e_{lj} = \frac{1}{2}(\hat{u}_{l,j} + \hat{u}_{j,l}), \quad g_{lj} = \hat{u}_{j,l} + \hat{w}_{l,j}, \quad d_l = \hat{u}_l - \hat{w}_l, \quad l, j = 1, 2, 3. \tag{2}
$$

2. *The equations of motion*

$$
\begin{gathered}
t_{jl,j} - p_l = \rho_1 \left(\ddot{\hat{u}}_l - \hat{F}_l^{(1)} \right), \qquad s_{jl,j} + p_l = \rho_2 \left(\ddot{\hat{w}}_l - \hat{F}_l^{(2)} \right), \qquad l = 1, 2, 3, \\
h_{j,j}^{(1)} + g^{(1)} = \rho_1 \left(\kappa_1 \ddot{\hat{\varphi}} - \hat{L}^{(1)} \right), \qquad h_{j,j}^{(2)} + g^{(2)} = \rho_2 \left(\kappa_2 \ddot{\hat{\psi}} - \hat{L}^{(2)} \right),
\end{gathered}
\tag{3}
$$

where $\hat{L}^{(r)}$, κ_r, ρ_r and $\hat{\mathbf{F}}^{(r)} = (\hat{F}_1^{(r)}, \hat{F}_2^{(r)}, \hat{F}_3^{(r)})$ are the extrinsic equilibrated body force, the coefficient of the equilibrated inertia, the mass density and the partial body force of the constituent $s^{(r)}$, respectively; $\rho_r > 0$, $\kappa_r > 0$ and $r = 1, 2$.

3. *The heat transfer equation*

$$\rho T_0 \dot{\eta} = q_{l,l} + \rho \hat{s}, \tag{4}$$

where $\rho = \rho_1 + \rho_2$ and $\hat{s}$ is the heat source.

Substituting equations (1) and (2) into (3) and (4), we obtain the following system of equations of motion in the linear theory of thermoviscoelastic binary porous mixtures expressed in terms of the partial displacement vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{w}}$, the changes of volume fractions $\hat{\varphi}(\mathbf{x},t)$, $\hat{\psi}(\mathbf{x},t)$ and the change of temperature $\hat{\theta}$:

$$\begin{gathered}
\hat{\alpha}_1 \Delta \hat{\mathbf{u}} + \hat{\alpha}_2 \nabla \mathrm{div}\, \hat{\mathbf{u}} + \beta_1 \Delta \hat{\mathbf{w}} + \beta_2 \nabla \mathrm{div}\, \hat{\mathbf{w}} - \hat{\xi}(\hat{\mathbf{u}} - \hat{\mathbf{w}}) + \sigma_1 \nabla \hat{\varphi} + \sigma_2 \nabla \hat{\psi} - m_1 \nabla \hat{\theta} = \rho_1 \left(\ddot{\hat{\mathbf{u}}} - \hat{\mathbf{F}}^{(1)} \right),\\
\beta_1 \Delta \hat{\mathbf{u}} + \beta_2 \nabla \mathrm{div}\, \hat{\mathbf{u}} + \gamma_1 \Delta \hat{\mathbf{w}} + \gamma_2 \nabla \mathrm{div}\, \hat{\mathbf{w}} + \hat{\xi}(\hat{\mathbf{u}} - \hat{\mathbf{w}}) + \tau_1 \nabla \hat{\varphi} + \tau_2 \nabla \hat{\psi} - m_2 \nabla \hat{\theta} = \rho_2 \left(\ddot{\hat{\mathbf{w}}} - \hat{\mathbf{F}}^{(2)} \right),\\
\alpha^{(1)} \Delta \hat{\varphi} + \alpha^{(3)} \Delta \hat{\psi} - \sigma_1 \mathrm{div}\, \hat{\mathbf{u}} - \tau_1 \mathrm{div}\, \hat{\mathbf{w}} - \zeta^{(1)} \hat{\varphi} - \zeta^{(3)} \hat{\psi} + b^{(1)} \hat{\theta} = \rho_1 \left(\kappa_1 \ddot{\hat{\varphi}} - \hat{L}^{(1)} \right),\\
\alpha^{(3)} \Delta \hat{\varphi} + \alpha^{(2)} \Delta \hat{\psi} - \sigma_2 \mathrm{div}\, \hat{\mathbf{u}} - \tau_2 \mathrm{div}\, \hat{\mathbf{w}} - \zeta^{(3)} \hat{\varphi} - \zeta^{(2)} \hat{\psi} + b^{(2)} \hat{\theta} = \rho_2 \left(\kappa_2 \ddot{\hat{\varphi}} - \hat{L}^{(2)} \right),\\
k \Delta \hat{\theta} - a T_0 \dot{\hat{\theta}} - a_1 \mathrm{div}\, \dot{\hat{\mathbf{u}}} - a_2 \mathrm{div}\, \dot{\hat{\mathbf{w}}} - b^{(1)} T_0 \hat{\varphi} - b^{(2)} T_0 \hat{\psi} = -\rho \hat{s},
\end{gathered} \tag{5}$$

where Δ is the Laplacian operator,

$$\hat{\alpha}_1 = \alpha_1 + \mu^* \frac{\partial}{\partial t}, \quad \hat{\alpha}_2 = \alpha_2 + (\lambda^* + \mu^*) \frac{\partial}{\partial t}, \quad \hat{\xi} = \xi + \xi^* \frac{\partial}{\partial t}$$

and

$$\begin{gathered}
\alpha_1 = \mu + 2\kappa + 2\zeta, \quad \alpha_2 = \lambda + \mu + \alpha + 2\nu + 2\gamma + 2\zeta, \quad \beta_1 = 2\gamma + \zeta,\\
\beta_2 = \alpha + \nu + 2\kappa + \zeta, \quad \gamma_1 = 2\kappa, \quad \gamma_2 = \alpha + 2\gamma, \quad m_1 = \beta^{(1)} + \beta^{(2)} + b^*,\\
m_2 = \beta^{(2)} - b^*, \qquad \sigma_1 = m^{(1)} + l^{(1)} - b, \quad \sigma_2 = m^{(2)} + l^{(2)} - c_0,\\
\tau_1 = l^{(1)} + b, \quad \tau_2 = l^{(2)} + c_0, \quad a_1 = T_0(\beta^{(1)} + \beta^{(2)}) - f^*, \quad a_2 = T_0 \beta^{(2)} + f^*.
\end{gathered} \tag{6}$$

If the functions $\hat{\mathbf{u}}$, $\hat{\mathbf{w}}$, $\hat{\varphi}$, $\hat{\psi}$, $\hat{\theta}$, $\hat{\mathbf{F}}^{(1)}$, $\hat{\mathbf{F}}^{(2)}$, $\hat{L}^{(1)}$, $\hat{L}^{(2)}$ and $\hat{s}$ are postulated to have a harmonic time variation, that is,

$$\{\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\varphi}, \hat{\psi}, \hat{\theta}, \hat{\mathbf{F}}^{(1)}, \hat{\mathbf{F}}^{(2)}, \hat{L}^{(1)}, \hat{L}^{(2)}, \hat{s}\}(\mathbf{x},t) = \mathrm{Re}\left[\{\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta, \mathbf{F}^{(1)}, \mathbf{F}^{(2)}, L^{(1)}, L^{(2)}, s\}(\mathbf{x})\, e^{-i\omega t} \right],$$

then from the system of equations of motion (5), we obtain the following system of equations of steady vibrations in the theory under consideration:

$$\begin{gathered}
(\alpha_1' \Delta + \eta_1') \mathbf{u} + \alpha_2' \nabla \mathrm{div}\, \mathbf{u} + (\beta_1 \Delta + \xi') \mathbf{w} + \beta_2 \nabla \mathrm{div}\, \mathbf{w} + \sigma_1 \nabla \varphi + \sigma_2 \nabla \psi - m_1 \nabla \theta = -\rho_1 \mathbf{F}^{(1)},\\
(\beta_1 \Delta + \xi') \mathbf{u} + \beta_2 \nabla \mathrm{div}\, \mathbf{u} + (\gamma_1 \Delta + \eta_2') \mathbf{w} + \gamma_2 \nabla \mathrm{div}\, \mathbf{w} + \tau_1 \nabla \varphi + \tau_2 \nabla \psi - m_2 \nabla \theta = -\rho_2 \mathbf{F}^{(2)},\\
(\alpha^{(1)} \Delta + \eta_1) \varphi + (\alpha^{(3)} \Delta - \zeta^{(3)}) \psi - \sigma_1 \mathrm{div}\, \mathbf{u} - \tau_1 \mathrm{div}\, \mathbf{w} + b^{(1)} \theta = -\rho_1 L^{(1)},\\
(\alpha^{(3)} \Delta - \zeta^{(3)}) \varphi + (\alpha^{(2)} \Delta + \eta_2) \psi - \sigma_2 \mathrm{div}\, \mathbf{u} - \tau_2 \mathrm{div}\, \mathbf{w} + b^{(2)} \theta = -\rho_2 L^{(2)},\\
(k \Delta + a')\, \theta + i\omega a_1 \mathrm{div}\, \mathbf{u} + i\omega a_2 \mathrm{div}\, \mathbf{w} + i\omega b^{(1)} T_0 \varphi + i\omega b^{(2)} T_0 \psi = -\rho s,
\end{gathered} \tag{7}$$

where

$$\begin{gathered}
\alpha_1' = \alpha_1 - i\omega \mu^*, \quad \alpha_2' = \alpha_2 - i\omega(\lambda^* + \mu^*), \quad \xi' = \xi - i\omega\, \xi^*,\\
\eta_1' = \rho_1 \omega^2 - \xi', \quad \eta_2' = \rho_2 \omega^2 - \xi', \quad \eta_1 = \rho_1 \kappa_1 \omega^2 - \zeta^{(1)},\\
\eta_2 = \rho_2 \kappa_2 \omega^2 - \zeta^{(2)}, \quad a' = i\omega a T_0
\end{gathered} \tag{8}$$

and ω is the oscillation frequency ($\omega > 0$).

We introduce the matrix differential operator $\mathbf{A}(\mathbf{D_x}) = (A_{rq}(\mathbf{D_x}))_{9\times 9}$, where

$$A_{lj}(\mathbf{D_x}) = (\alpha_1'\Delta + \eta_1')\delta_{lj} + \alpha_2'\frac{\partial^2}{\partial x_l \partial x_j},$$

$$A_{l;j+3}(\mathbf{D_x}) = A_{l+3;j}(\mathbf{D_x}) = (\beta_1\Delta + \xi')\delta_{lj} + \beta_2\frac{\partial^2}{\partial x_l \partial x_j}, \quad A_{l;r+6}(\mathbf{D_x}) = -A_{r+6;l}(\mathbf{D_x}) = \sigma_r\frac{\partial}{\partial x_l},$$

$$A_{l9}(\mathbf{D_x}) = -m_1\frac{\partial}{\partial x_l}, \quad A_{l+3;j+3}(\mathbf{D_x}) = (\gamma_1\Delta + \eta_2')\delta_{lj} + \gamma_2\frac{\partial^2}{\partial x_l \partial x_j},$$

$$A_{l+3;r+6}(\mathbf{D_x}) = -A_{r+6;l+3}(\mathbf{D_x}) = \tau_r\frac{\partial}{\partial x_l}, \quad A_{l+3;9}(\mathbf{D_x}) = -m_2\frac{\partial}{\partial x_l},$$

$$A_{77}(\mathbf{D_x}) = \alpha^{(1)}\Delta + \eta_1, \quad A_{78}(\mathbf{D_x}) = A_{87}(\mathbf{D_x}) = \alpha^{(3)}\Delta + \zeta^{(3)},$$

$$A_{88}(\mathbf{D_x}) = \alpha^{(1)}\Delta + \eta_1, \quad A_{9l}(\mathbf{D_x}) = i\omega a_1\frac{\partial}{\partial x_l}, \quad A_{9;l+3}(\mathbf{D_x}) = i\omega a_2\frac{\partial}{\partial x_l},$$

$$A_{9;r+6}(\mathbf{D_x}) = i\omega b^{(r)}T_0, \quad A_{99}(\mathbf{D_x}) = k\Delta + a', \quad l,j = 1,2,3, \quad r = 1,2.$$

Obviously, system (7) can be written as follows:

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{9}$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta)$, $\mathbf{F} = (-\rho_1\mathbf{F}^{(1)}, -\rho_2\mathbf{F}^{(2)}, -\rho_1 L^{(1)}, -\rho_2 L^{(2)}, -\rho s)$ and $\mathbf{x} \in \Omega$.

3. Fundamental Solution

In this section, the fundamental solution of system (7) is constructed explicitly and its basic properties are established.

Definition 1. The fundamental solution of system (7) is the matrix $\mathbf{\Gamma}(\mathbf{x}) = (\Gamma_{lj}(\mathbf{x}))_{9\times 9}$ satisfying the following equation in the class of generalized functions:

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J},$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = (\delta_{lj})_{9\times 9}$ is the unit matrix and $\mathbf{x} \in \mathbb{R}^3$.

We denote by

$$\begin{gathered}\alpha_0' = \alpha_1' + \alpha_2', \quad \beta_0 = \beta_1 + \beta_2, \quad \gamma_0 = \gamma_1 + \gamma_2, \\ k_0 = \alpha_0'\gamma_0 - \beta_0^2, \quad k_1 = \alpha_1'\gamma_1 - \beta_1^2, \quad \alpha_0 = \alpha^{(1)}\alpha^{(2)} - \left(\alpha^{(3)}\right)^2.\end{gathered} \tag{10}$$

In this section, we assume that

$$\alpha_0 k k_0 k_1 \neq 0. \tag{11}$$

We introduce the following notation:

i)

$$\mathbf{B}(\Delta) = (B_{lj}(\Delta))_{5\times 5}$$

$$= \begin{pmatrix} \alpha_0'\Delta + \eta_1' & \beta_0\Delta + \xi' & -\sigma_1\Delta & -\sigma_2\Delta & i\omega a_1\Delta \\ \beta_0\Delta + \xi' & \gamma_0\Delta + \eta_2' & -\tau_1\Delta & -\tau_2\Delta & i\omega a_2\Delta \\ \sigma_1 & \tau_1 & \alpha^{(1)}\Delta + \eta_1 & \alpha^{(3)}\Delta - \zeta^{(3)} & i\omega b^{(1)}T_0 \\ \sigma_2 & \tau_2 & \alpha^{(3)}\Delta - \zeta^{(3)} & \alpha^{(2)}\Delta + \eta_2 & i\omega b^{(2)}T_0 \\ -m_1 & -m_2 & b^{(1)} & b^{(2)} & k\Delta + a' \end{pmatrix}_{5\times 5}.$$

ii)

$$\Lambda_1(\Delta) = \frac{1}{\alpha_0 k k_0}\,\det\mathbf{B}(\Delta) = \prod_{j=1}^{5}(\Delta + \lambda_j^2),$$

where $\lambda_j^2 (j = 1, 2, \dots, 5)$ are the roots of the equation $\Lambda_1(-\tilde{\lambda}) = 0$ (with respect to $\tilde{\lambda}$).

iii)

$$\Lambda_2(\Delta)=\frac{1}{k_1}\det\begin{pmatrix}\alpha_1'\Delta+\eta_1' & \beta_1\Delta+\xi'\\ \beta_1\Delta+\xi' & \gamma_1\Delta+\eta_2'\end{pmatrix}_{2\times 2}=(\Delta+\lambda_6^2)(\Delta+\lambda_7^2),$$

where λ_6^2 and λ_7^2 are the roots of the equation $\Lambda_2(-\tilde{\lambda})=0$ (with respect to $\tilde{\lambda}$). We assume that $\operatorname{Im}\lambda_l>0$ and $\lambda_l\neq\lambda_j$ $(l,j=1,2,\dots,7)$.

iv)

$$n_{l1}(\Delta)=\frac{1}{\alpha_0 kk_0k_1}\sum_{j=1}^{5}C_jB_{lj}^*(\Delta),\quad n_{l2}(\Delta)=\frac{1}{\alpha_0 kk_0k_1}\sum_{j=1}^{5}C_{j+5}B_{lj}^*(\Delta),$$
$$n_{lr}(\Delta)=\frac{1}{\alpha_0 kk_0}B_{lr}^*(\Delta),\quad l=1,2,\dots,5,\quad r=3,4,5,$$

where B_{lj}^* is the cofactor of element B_{lj} of the matrix $\mathbf{B}$ and

$$C_1=\beta_2(\beta_1\Delta+\xi')-\alpha_2'(\gamma_1\Delta+\eta_2'),\quad C_2=\gamma_2(\beta_1\Delta+\xi')-\beta_2(\gamma_1\Delta+\eta_2'),$$
$$C_3=\sigma_1(\gamma_1\Delta+\eta_2')-\tau_1(\beta_1\Delta+\xi'),\quad C_4=\sigma_2(\gamma_1\Delta+\eta_2')-\tau_2(\beta_1\Delta+\xi'),$$
$$C_5=i\omega[a_2(\beta_1\Delta+\xi')-a_1(\gamma_1\Delta+\eta_2')],\quad C_6=\alpha_2'(\beta_1\Delta+\xi')-\beta_2(\alpha_1'\Delta+\eta_1'),$$
$$C_7=\beta_2(\beta_1\Delta+\xi')-\gamma_2(\alpha_1'\Delta+\eta_1'),\quad C_8=\tau_1(\alpha_1'\Delta+\eta_1')-\sigma_1(\beta_1\Delta+\xi'),$$
$$C_9=\tau_2(\alpha_1'\Delta+\eta_1')-\sigma_2(\beta_1\Delta+\xi'),\quad C_{10}=i\omega[a_1(\beta_1\Delta+\xi')-a_2(\alpha_1'\Delta+\eta_1')].$$

v)

$$\mathbf{\Lambda}(\Delta)=\left(\Lambda_{lj}\left(\Delta\right)\right)_{9\times 9},\quad \Lambda_{11}(\Delta)=\Lambda_{22}(\Delta)=\cdots=\Lambda_{66}(\Delta)=\Lambda_2(\Delta),$$
$$\Lambda_{77}(\Delta)=\Lambda_{88}(\Delta)=\Lambda_{99}(\Delta)=\Lambda_1(\Delta),\quad \Lambda_{lj}(\Delta)=0,$$
$$l\neq j,\quad l,j=1,2,\dots,9.$$

vi)

$$\begin{gathered}
\mathbf{L}\left(\mathbf{D_x}\right)=\left(L_{lj}\left(\mathbf{D_x}\right)\right)_{9\times 9},\\
L_{lj}\left(\mathbf{D_x}\right)=\frac{1}{k}(\gamma_1\Delta+\eta_2')\Lambda_1(\Delta)\,\delta_{lj}+n_{11}(\Delta)\frac{\partial^2}{\partial x_l\partial x_j},\\
L_{l;j+3}\left(\mathbf{D_x}\right)=-\frac{1}{k}(\beta_1\Delta+\xi')\Lambda_1(\Delta)\,\delta_{lj}+n_{12}(\Delta)\frac{\partial^2}{\partial x_l\partial x_j},\\
L_{l+3;j}\left(\mathbf{D_x}\right)=-\frac{1}{k}(\beta_1\Delta+\xi')\Lambda_1(\Delta)\,\delta_{lj}+n_{21}(\Delta)\frac{\partial^2}{\partial x_l\partial x_j},\\
L_{l+3;j+3}\left(\mathbf{D_x}\right)=\frac{1}{k}(\alpha_1'\Delta+\eta_1')\Lambda_1(\Delta)\,\delta_{lj}+n_{22}(\Delta)\frac{\partial^2}{\partial x_l\partial x_j},\\
L_{lr}\left(\mathbf{D_x}\right)=n_{1;r-4}(\Delta)\frac{\partial}{\partial x_l},\qquad L_{l+3;r}\left(\mathbf{D_x}\right)=n_{2;r-4}(\Delta)\frac{\partial}{\partial x_l},\\
L_{rl}\left(\mathbf{D_x}\right)=n_{r-4;1}(\Delta)\frac{\partial}{\partial x_l},\qquad L_{r;l+3}\left(\mathbf{D_x}\right)=n_{r-4;2}(\Delta)\frac{\partial}{\partial x_l},\\
L_{rm}\left(\mathbf{D_x}\right)=n_{r-4;m-4}(\Delta),\qquad l,j=1,2,3,\quad r,m=7,8,9.
\end{gathered}\tag{12}$$

vii)

$$\begin{aligned} &\mathbf{Y}(\mathbf{x}) = (Y_{lj}(\mathbf{x}))_{9\times 9}, \\ &Y_{11}(\mathbf{x}) = Y_{22}(\mathbf{x}) = \cdots = Y_{66}(\mathbf{x}) = \sum_{j=1}^{7} \eta_{2j}\gamma^{(j)}(\mathbf{x}), \\ &Y_{77}(\mathbf{x}) = Y_{88}(\mathbf{x}) = Y_{88}(\mathbf{x}) = \sum_{j=1}^{5} \eta_{1j}\gamma^{(j)}(\mathbf{x}), \\ &Y_{lj}(\mathbf{x}) = 0, \quad l \neq j, \quad l,j = 1,2,\dots,9, \end{aligned} \tag{13}$$

where

$$\gamma^{(j)}(\mathbf{x}) = -\frac{e^{i\lambda_j|\mathbf{x}|}}{4\pi\,|\mathbf{x}|}$$

and

$$\eta_{1m} = \prod_{l=1,\,l\neq m}^{5} (\lambda_l^2 - \lambda_m^2)^{-1}, \quad \eta_{2j} = \prod_{l=1,\,l\neq j}^{7} (\lambda_l^2 - \lambda_j^2)^{-1},$$
$$m = 1,2,\dots,5, \quad j = 1,2\dots,7.$$

It is not difficult to prove

Lemma 1. *If the condition* (11) *is satisfied, then:*
a) *the following identity*

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{L}(\mathbf{D_x}) = \mathbf{\Lambda}(\Delta)$$

is valid;
b) *the matrix* $\mathbf{Y}(\mathbf{x})$ *is the fundamental solution of the operator* $\mathbf{\Lambda}(\Delta)$*, i.e.,*

$$\mathbf{\Lambda}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}.$$

Lemma 1 leads to the following

Theorem 1. *If the condition* (11) *is satisfied, then the matrix* $\mathbf{\Gamma}(\mathbf{x}) = (\Gamma_{lj}(\mathbf{x}))_{9\times 9}$ *defined by*

$$\mathbf{\Gamma}(\mathbf{x}) = \mathbf{L}(\mathbf{D_x})\,\mathbf{Y}(\mathbf{x}) \tag{14}$$

is the fundamental solution of system (7) (*the fundamental matrix of the operator* $\mathbf{A}(\mathbf{D_x})$)*, where the matrices* $\mathbf{L}(\mathbf{D_x})$ *and* $\mathbf{Y}(\mathbf{x})$ *are given by* (12) *and* (13)*, respectively.*

We now formulate the basic properties of the matrix $\mathbf{\Gamma}(\mathbf{x})$. Theorem 1 has the following consequences.

Theorem 2. *Each column of the matrix* $\mathbf{\Gamma}(\mathbf{x})$ *is a solution of the homogeneous equation*

$$\mathbf{A}(\mathbf{D_x})\mathbf{\Gamma}(\mathbf{x}) = \mathbf{0}$$

at every point $\mathbf{x} \in \mathbb{R}^3$*, except the origin.*

Theorem 3. *The relations*

$$\Gamma_{lj}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right), \quad \Gamma_{rm}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right), \quad \Gamma_{99}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right),$$
$$\Gamma_{le}(\mathbf{x}) = O(1), \quad \Gamma_{el}(\mathbf{x}) = O(1), \quad \Gamma_{r9}(\mathbf{x}) = O(1),$$
$$\Gamma_{9r}(\mathbf{x}) = O(1), \quad l,j = 1,2,\dots,6, \quad r,m = 7,8, \quad e = 7,8,9$$

hold in the neighborhood of the origin.

We introduce the notation:

i)

$$\mathbf{A}^{(0)}(\mathbf{D_x}) = \left(A_{lj}^{(0)}(\mathbf{D_x})\right)_{9\times 9}, \quad A_{lj}^{(0)}(\mathbf{D_x}) = \alpha_1'\Delta\delta_{lj} + \alpha_2'\frac{\partial^2}{\partial x_l \partial x_j},$$
$$A_{l;j+3}^{(0)}(\mathbf{D_x}) = A_{l+3;j}^{(0)}(\mathbf{D_x}) = \beta_1\Delta\delta_{lj} + \beta_2\frac{\partial^2}{\partial x_l \partial x_j},$$
$$A_{l+3;j+3}^{(0)}(\mathbf{D_x}) = \gamma_1\Delta\delta_{lj} + \gamma_2\frac{\partial^2}{\partial x_l \partial x_j}, \quad A_{77}^{(0)}(\mathbf{D_x}) = \alpha^{(1)}\Delta,$$
$$A_{78}^{(0)}(\mathbf{D_x}) = A_{87}^{(0)}(\mathbf{D_x}) = \alpha^{(3)}\Delta, \quad A_{88}^{(0)}(\mathbf{D_x}) = \alpha^{(2)}\Delta, \quad A_{99}^{(0)}(\mathbf{D_x}) = k\Delta,$$
$$A_{mr}^{(0)}(\mathbf{D_x}) = A_{rm}^{(0)}(\mathbf{D_x}) = A_{e9}^{(0)}(\mathbf{D_x}) = A_{9e}^{(0)}(\mathbf{D_x}) = 0.$$

ii)

$$\mathbf{\Gamma}^{(0)}(\mathbf{x}) = \left(\Gamma_{lj}^{(0)}(\mathbf{x})\right)_{9\times 9},$$
$$\Gamma_{lj}^{(0)}(\mathbf{x}) = -\frac{1}{8\pi}\left(\frac{\gamma_0}{k_0} + \frac{\gamma_1}{k_1}\right)\frac{\delta_{lj}}{|\mathbf{x}|} + \frac{1}{8\pi}\left(\frac{\gamma_0}{k_0} - \frac{\gamma_1}{k_1}\right)\frac{x_l x_j}{|\mathbf{x}|^3},$$
$$\Gamma_{l;j+3}^{(0)}(\mathbf{x}) = \Gamma_{l+3;j}^{(0)}(\mathbf{x}) = \frac{1}{8\pi}\left(\frac{\beta_0}{k_0} + \frac{\beta_1}{k_1}\right)\frac{\delta_{lj}}{|\mathbf{x}|} - \frac{1}{8\pi}\left(\frac{\beta_0}{k_0} - \frac{\beta_1}{k_1}\right)\frac{x_l x_j}{|\mathbf{x}|^3},$$
$$\Gamma_{l+3;j+3}^{(0)}(\mathbf{x}) = -\frac{1}{8\pi}\left(\frac{\alpha_0'}{k_0} + \frac{\alpha_1'}{k_1}\right)\frac{\delta_{lj}}{|\mathbf{x}|} + \frac{1}{8\pi}\left(\frac{\alpha_0'}{k_0} - \frac{\alpha_1'}{k_1}\right)\frac{x_l x_j}{|\mathbf{x}|^3},$$
$$\Gamma_{77}^{(0)}(\mathbf{x}) = -\frac{\alpha^{(2)}}{4\pi\alpha_0}\frac{1}{|\mathbf{x}|}, \quad \Gamma_{78}^{(0)}(\mathbf{x}) = \Gamma_{87}^{(0)}(\mathbf{x}) = \frac{\alpha^{(3)}}{4\pi\alpha_0}\frac{1}{|\mathbf{x}|}, \quad \Gamma_{88}^{(0)}(\mathbf{x}) = -\frac{\alpha^{(1)}}{4\pi\alpha_0}\frac{1}{|\mathbf{x}|},$$
$$\Gamma_{99}^{(0)}(\mathbf{x}) = -\frac{1}{4\pi k}\frac{1}{|\mathbf{x}|}, \quad \Gamma_{mr}^{(0)}(\mathbf{D_x}) = \Gamma_{rm}^{(0)}(\mathbf{D_x}) = \Gamma_{e9}^{(0)}(\mathbf{D_x}) = \Gamma_{9e}^{(0)}(\mathbf{D_x}) = 0,$$

where $l, j = 1, 2, 3$, $m = 1, 2, \dots, 6$, $e = 7, 8$ and $r = 7, 8, 9$.

Theorem 1 leads directly to the following basic properties of the matrix $\mathbf{\Gamma}^{(0)}(\mathbf{x})$.

Theorem 4. *The fundamental solution of the equation*

$$\mathbf{A}^{(0)}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$$

is the matrix $\mathbf{\Gamma}^{(0)}(\mathbf{x})$, *and the following relations:*

$$\Gamma_{lj}^{(0)}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right), \quad \Gamma_{mr}^{(0)}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right), \quad \Gamma_{99}^{(0)}(\mathbf{x}) = O\left(|\mathbf{x}|^{-1}\right),$$
$$l, j = 1, 2, \dots, 6, \quad m, r = 7, 8$$

hold in the neighborhood of the origin.

Theorem 5. *The relations*

$$\Gamma_{lj}(\mathbf{x}) - \Gamma_{lj}^{(0)}(\mathbf{x}) = \text{const} + O(|\mathbf{x}|), \quad l, j = 1, 2, \dots, 9 \tag{15}$$

hold in the neighborhood of the origin.

Thus, on the basis of Theorem 5 the matrix $\mathbf{\Gamma}^{(0)}(\mathbf{x})$ is the singular part of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$ in the neighborhood of the origin.

4. Basic Boundary Value Problems

Let S be the smooth closed surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $S \in C^{2,\nu'}$, $0 < \nu' \leq 1$; $\overline{\Omega^+} = \Omega^+ \cup S$, $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$, $\overline{\Omega^-} = \Omega^- \cup S$. We denote by $\mathbf{n}(\mathbf{z})$ the external (with respect to the Ω^+) unit vector, normal to S at $\mathbf{z}$.

Definition 2. A vector function $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta) = (U_1, U_2, \dots, U_9)$ is called *regular* in Ω^- (or Ω^+) if:

1)
$$U_j \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-}) \quad (\text{or } U_j \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})),$$

2)
$$U_j(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad U_{j,l}(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \quad \text{for} \quad |\mathbf{x}| \gg 1, \tag{16}$$

where $j = 1, 2, \dots, 9$ and $l = 1, 2, 3$.

In the sequel, we use the matrix differential operators

1)
$$\begin{gathered}
\mathbf{R}(\mathbf{D_x}, \mathbf{n}) = (R_{lj}(\mathbf{D_x}, \mathbf{n}))_{9\times 9},\\
R_{lj}(\mathbf{D_x}, \mathbf{n}) = \alpha_1' \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \alpha_2' n_l \frac{\partial}{\partial x_j} + \epsilon_1 \mathcal{M}_{lj}(\mathbf{D_x}, \mathbf{n}),\\
R_{l;j+3}(\mathbf{D_x}, \mathbf{n}) = R_{l+3;j}(\mathbf{D_x}, \mathbf{n}) = \beta_1 \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \beta_2 n_l \frac{\partial}{\partial x_j} + \epsilon_2 \mathcal{M}_{lj}(\mathbf{D_x}, \mathbf{n}),\\
R_{l+3;j+3}(\mathbf{D_x}, \mathbf{n}) = \gamma_1 \delta_{lj} \frac{\partial}{\partial \mathbf{n}} + \gamma_2 n_l \frac{\partial}{\partial x_j} + \epsilon_3 \mathcal{M}_{lj}(\mathbf{D_x}, \mathbf{n}),\\
R_{l7}(\mathbf{D_x}, \mathbf{n}) = \left(m^{(1)} + l^{(1)}\right) n_l, \quad R_{l8}(\mathbf{D_x}, \mathbf{n}) = \left(m^{(2)} + l^{(2)}\right) n_l,\\
R_{l9}(\mathbf{D_x}, \mathbf{n}) = -(\beta^{(1)} + \beta^{(2)})\, n_l, \quad R_{l+3;7}(\mathbf{D_x}, \mathbf{n}) = l^{(1)} n_l, \quad R_{l+3;8}(\mathbf{D_x}, \mathbf{n}) = l^{(2)} n_l,\\
R_{l+3;9}(\mathbf{D_x}, \mathbf{n}) = -\beta^{(2)}\, n_l, \quad R_{7l}(\mathbf{D_x}, \mathbf{n}) = -R_{7;l+3}(\mathbf{D_x}, \mathbf{n}) = b\, n_l,\\
R_{8l}(\mathbf{D_x}, \mathbf{n}) = -R_{8;l+3}(\mathbf{D_x}, \mathbf{n}) = c_0\, n_l, \quad R_{77}(\mathbf{D_x}, \mathbf{n}) = \alpha^{(1)} \frac{\partial}{\partial \mathbf{n}},\\
R_{78}(\mathbf{D_x}, \mathbf{n}) = R_{87}(\mathbf{D_x}, \mathbf{n}) = \alpha^{(3)} \frac{\partial}{\partial \mathbf{n}}, \quad R_{88}(\mathbf{D_x}, \mathbf{n}) = \alpha^{(2)} \frac{\partial}{\partial \mathbf{n}},\\
R_{9l}(\mathbf{D_x}, \mathbf{n}) = -R_{9;l+3}(\mathbf{D_x}, \mathbf{n}) = -i\omega f^*\, n_l, \quad R_{99}(\mathbf{D_x}, \mathbf{n}) = k \frac{\partial}{\partial \mathbf{n}},\\
R_{m9}(\mathbf{D_x}, \mathbf{n}) = -R_{9m}(\mathbf{D_x}, \mathbf{n}) = 0, \quad l, j = 1, 2, 3, \quad m = 7, 8,
\end{gathered} \tag{17}$$

where $\mathbf{n} = (n_1, n_2, n_3)$, $\frac{\partial}{\partial \mathbf{n}}$ is the derivative along the vector $\mathbf{n}$ and

$$\mathcal{M}_{lj}(\mathbf{D_x}, \mathbf{n}) = n_j \frac{\partial}{\partial x_l} - n_l \frac{\partial}{\partial x_j}, \quad \epsilon_1 = \mu - i\omega\mu^* + 2\gamma + 2\zeta, \quad \epsilon_2 = 2\kappa + \zeta, \quad \epsilon_3 = 2\gamma.$$

The basic internal and external BVPs of steady vibrations in the linear theory of thermoviscoelastic binary porous mixtures are formulated as follows.

Find a regular (classical) solution to (9) for $\mathbf{x} \in \Omega^\pm$ satisfying the boundary condition

$$\lim_{\Omega^\pm \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^\pm = \mathbf{f}(\mathbf{z})$$

in *Problem* $(I)^\pm_{\mathbf{F},\mathbf{f}}$, and

$$\lim_{\Omega^\pm \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^\pm = \mathbf{f}(\mathbf{z})$$

in *Problem* $(II)^\pm_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are the prescribed nine-component vector functions and $\operatorname{supp} \mathbf{F}$ is a finite domain in Ω^-.

5. Green's Identities

In this section, Green's identities in the linear theory of thermoviscoelasticity for binary porous mixtures are established.

Let $u_l', w_l', \varphi', \psi', \theta'$ $(l = 1,2,3)$ be complex functions, $\mathbf{u}' = (u_1', u_2', u_3')$, $\mathbf{w}' = (w_1', w_2', w_3')$, $\mathbf{U}' = (\mathbf{u}', \mathbf{w}', \varphi', \psi', \theta')$. We introduce the notation

$$\begin{aligned}
W^{(0)}(\mathbf{u},\mathbf{u}') &= \frac{1}{4}\sum_{l,j=1;\, l\neq j}^{3}(u_{j,l}+u_{l,j})\left(\overline{u'_{j,l}}+\overline{u'_{l,j}}\right)+\frac{1}{6}\sum_{l,j=1}^{3}\left(\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\right)\left(\frac{\overline{\partial u_l'}}{\partial x_l}-\frac{\overline{\partial u_j'}}{\partial x_j}\right),\\
W^{(1)}(\mathbf{u},\mathbf{u}') &= \frac{1}{3}(\alpha_1'+3\alpha_2'-2\epsilon_1)\operatorname{div}\mathbf{u}\operatorname{div}\overline{\mathbf{u}'}+\frac{1}{2}(\alpha_1'-\epsilon_1)\operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{u}'\\
&\quad+(\alpha_1'+\epsilon_1)W^{(0)}(\mathbf{u},\mathbf{u}')-\eta_1'\mathbf{u}\cdot\mathbf{u}',\\
W^{(2)}(\mathbf{u},\mathbf{w}') &= \frac{1}{3}(\beta_1+3\beta_2-2\epsilon_2)\operatorname{div}\mathbf{u}\operatorname{div}\overline{\mathbf{w}'}+\frac{1}{2}(\beta_1-\epsilon_2)\operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{w}'\\
&\quad+(\beta_1+\epsilon_2)W^{(0)}(\mathbf{u},\mathbf{w}')-\xi'\mathbf{u}\cdot\mathbf{w}',\\
W^{(3)}(\mathbf{w},\mathbf{w}') &= \frac{1}{3}(\gamma_1+3\gamma_2-2\epsilon_3)\operatorname{div}\mathbf{w}\operatorname{div}\overline{\mathbf{w}'}+\frac{1}{2}(\gamma_1-\epsilon_3)\operatorname{curl}\mathbf{w}\cdot\operatorname{curl}\mathbf{w}'\\
&\quad+(\gamma_1+\epsilon_3)W^{(0)}(\mathbf{w},\mathbf{w}')-\eta_2'\mathbf{w}\cdot\mathbf{w}'.
\end{aligned}\tag{18}$$

Using Green's first identity of the classical theory of elasticity (see e.g., Kupradze et al. [18]), it is a simple matter to verify that

$$\begin{aligned}
\int_{\Omega^+}\left[A_{lj}(\mathbf{D_x})\,u_j\overline{u_l'}+W^{(1)}(\mathbf{u},\mathbf{u}')\right]d\mathbf{x} &= \int_S R_{lj}(\mathbf{D_z},\mathbf{n})u_j(\mathbf{z})\overline{u_l'(\mathbf{z})}\,d_{\mathbf{z}}S,\\
\int_{\Omega^+}\left[A_{l;j+3}(\mathbf{D_x})\,w_j\overline{u_l'}+W^{(2)}(\mathbf{w},\mathbf{u}')\right]d\mathbf{x} &= \int_S R_{l;j+3}(\mathbf{D_z},\mathbf{n})w_j(\mathbf{z})\overline{u_l'(\mathbf{z})}\,d_{\mathbf{z}}S,\\
\int_{\Omega^+}\left[A_{l+3;j}(\mathbf{D_x})\,u_j\overline{w_l'}+W^{(2)}(\mathbf{u},\mathbf{w}')\right]d\mathbf{x} &= \int_S R_{l+3;j}(\mathbf{D_z},\mathbf{n})u_j(\mathbf{z})\overline{w_l'(\mathbf{z})}\,d_{\mathbf{z}}S,\\
\int_{\Omega^+}\left[A_{l+3;j+3}(\mathbf{D_x})\,w_j\overline{w_l'}+W^{(3)}(\mathbf{w},\mathbf{w}')\right]d\mathbf{x} &= \int_S R_{l+3;j+3}(\mathbf{D_z},\mathbf{n})w_j(\mathbf{z})\overline{w_l'(\mathbf{z})}\,d_{\mathbf{z}}S.
\end{aligned}\tag{19}$$

On the basis of (18) and identity

$$\int_{\Omega^+}\left[\nabla\varphi(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x})+\varphi(\mathbf{x})\operatorname{div}\overline{\mathbf{u}'(\mathbf{x})}\right]d\mathbf{x}=\int_S\varphi(\mathbf{z})\mathbf{n}(\mathbf{z})\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S,\tag{20}$$

from (19), it follows that

$$\begin{aligned}
&\int_{\Omega^+}\left[(A_{lj}u_j+A_{l;j+3}w_j+A_{l7}\varphi+A_{l8}\psi+A_{l9}\theta)\,\overline{u_l'}+W_1(\mathbf{U},\mathbf{u}')\right]d\mathbf{x}\\
&\qquad=\int_S[R_{lj}u_j+R_{l;j+3}w_j+R_{l7}\varphi+R_{l8}\psi+R_{l9}\theta]\,\overline{u_l'}d_{\mathbf{z}}S,\\
&\int_{\Omega^+}\left[(A_{l+3;j}u_j+A_{l+3;j+3}w_j+A_{l+3;7}\varphi+A_{l+3;8}\psi+A_{l+3;9}\theta)\,\overline{w_l'}+W_2(\mathbf{U},\mathbf{w}')\right]d\mathbf{x}\\
&\qquad=\int_S(R_{l+3;j}u_j+R_{l+3;j+3}w_j+R_{l+3;7}\varphi+R_{l+3;8}\psi+R_{l+3;9}\theta)\,\overline{w_l'}d_{\mathbf{z}}S,\quad l=1,2,3,
\end{aligned}\tag{21}$$

where

$$\begin{aligned} W_1(\mathbf{U},\mathbf{u}') &= W^{(1)}(\mathbf{u},\mathbf{u}') + W^{(2)}(\mathbf{w},\mathbf{u}') \\ &+ \left[\left(m^{(1)}+l^{(1)}\right)\varphi + \left(m^{(2)}+l^{(2)}\right)\psi - (\beta^{(1)}+\beta^{(2)})\theta\right] \operatorname{div}\overline{\mathbf{u}'} \\ &+\nabla(b\varphi + c_0\psi + b^*\theta)\cdot\mathbf{u}', \\ W_2(\mathbf{U},\mathbf{w}') &= W^{(2)}(\mathbf{u},\mathbf{w}') + W^{(3)}(\mathbf{w},\mathbf{w}') + \left(l^{(1)}\varphi + l^{(2)}\psi - \beta^{(2)}\theta\right)\operatorname{div}\overline{\mathbf{w}'} \\ &-\nabla(b\varphi + c_0\psi + b^*\theta)\cdot\mathbf{w}' \end{aligned} \tag{22}$$

and $W^{(1)}(\mathbf{u},\mathbf{u}'), W^{(2)}(\mathbf{w},\mathbf{u}')$ and $W^{(3)}(\mathbf{w},\mathbf{w}')$ are defined by (18).

Now, taking into account the identities (20) and

$$\int_{\Omega^+}\left[\Delta\varphi(\mathbf{x})\,\overline{\psi'(\mathbf{x})} + \nabla\varphi(\mathbf{x})\cdot\nabla\psi'(\mathbf{x})\right]d\mathbf{x} = \int_S \frac{\partial\varphi(\mathbf{z})}{\partial\mathbf{n}(\mathbf{z})}\overline{\psi'(\mathbf{z})}\,d_{\mathbf{z}},S$$

we deduce that

$$\begin{aligned} &\int_{\Omega^+}\left[(A_{7j}u_j + A_{7;j+3}w_j + A_{77}\varphi + A_{78}\psi + A_{79}\theta)\,\overline{\varphi'} + W_3(\mathbf{U},\varphi')\right]d\mathbf{x} \\ &\qquad= \int_S (R_{7j}u_j + R_{7;j+3}w_j + R_{77}\varphi + R_{78}\psi)\,\overline{\varphi'}d_{\mathbf{z}}S, \\ &\int_{\Omega^+}\left[(A_{8j}u_j + A_{8;j+3}w_j + A_{87}\varphi + A_{88}\psi + A_{89}\theta)\,\overline{\psi'} + W_4(\mathbf{U},\psi')\right]d\mathbf{x} \\ &\qquad= \int_S (R_{8j}u_j + R_{8;j+3}w_j + R_{87}\varphi + R_{88}\psi)\,\overline{\psi'}d_{\mathbf{z}}S, \\ &\int_{\Omega^+}\left[(A_{9j}u_j + A_{9;j+3}w_j + A_{97}\varphi + A_{98}\psi + A_{99}\theta)\,\overline{\theta'} + W_5(\mathbf{U},\theta')\right]d\mathbf{x} \\ &\qquad= \int_S (R_{9j}u_j + R_{9;j+3}w_j + R_{99}\theta)\,\overline{\theta'}d_{\mathbf{z}}S, \end{aligned} \tag{23}$$

where

$$\begin{aligned} W_3(\mathbf{U},\varphi') &= \left[\nabla(\alpha^{(1)}\varphi + \alpha^{(3)}\psi) + b(\mathbf{u}-\mathbf{w})\right]\cdot\nabla\varphi' \\ &+ \left[(m^{(1)}+l^{(1)})\operatorname{div}\mathbf{u} + l^{(1)}\operatorname{div}\mathbf{w} - \eta_1\varphi + \zeta^{(3)}\psi - b^{(1)}\theta\right]\overline{\varphi'}, \\ W_4(\mathbf{U},\psi') &= \left[\nabla(\alpha^{(3)}\varphi + \alpha^{(2)}\psi) + c_0(\mathbf{u}-\mathbf{w})\right]\cdot\nabla\psi' \\ &+ \left[(m^{(2)}+l^{(2)})\operatorname{div}\mathbf{u} + l^{(2)}\operatorname{div}\mathbf{w} + \zeta^{(3)}\varphi - \eta_2\psi - b^{(2)}\theta\right]\overline{\psi'}, \\ W_5(\mathbf{U},\theta') &= k\nabla\theta\cdot\nabla\theta' - a'\theta\overline{\theta'} \\ &- i\omega T_0\left[(\beta^{(1)}+\beta^{(2)})\operatorname{div}\mathbf{u} + \beta^{(2)}\operatorname{div}\mathbf{w} + b^{(1)}\varphi + b^{(2)}\psi\right]\overline{\theta'} - i\omega f^*(\mathbf{u}-\mathbf{w})\cdot\nabla\theta'. \end{aligned} \tag{24}$$

Finally, we combine the relations (21) and (23) to deduce the identity

$$\int_{\Omega^+}\left[\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{U}\cdot\mathbf{U}' + W(\mathbf{U},\mathbf{U}')\right]d\mathbf{x} = \int_S \mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S, \tag{25}$$

where

$$W(\mathbf{U},\mathbf{U}') = W_1(\mathbf{U},\mathbf{u}') + W_2(\mathbf{U},\mathbf{w}') + W_3(\mathbf{U},\varphi') + W_4(\mathbf{U},\psi') + W_5(\mathbf{U},\theta'). \tag{26}$$

Hence, the following theorem is proved.

Theorem 6. *If* $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta)$ *and* $\mathbf{U}' = (\mathbf{u}', \mathbf{w}', \varphi', \psi', \theta')$ *are regular vectors in* Ω^+*, then the identity* (25) *is valid, where* $\mathbf{R}(\mathbf{D_z}, \mathbf{n})$ *and* $W(\mathbf{U}, \mathbf{U}')$ *are defined by* (17) *and* (26)*, respectively.*

Quite similarly as in Theorem 6, on the basis of (16), we obtain the following

Theorem 7. *If* $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta)$ *and* $\mathbf{U}' = (\mathbf{u}', \mathbf{w}', \varphi', \psi', \theta')$ *are regular vectors in* Ω^-*, then*

$$\int\limits_{\Omega^-} [\mathbf{A}(\mathbf{D_x})\, \mathbf{U} \cdot \mathbf{U}' + W(\mathbf{U}, \mathbf{U}')]\, d\mathbf{x} = -\int\limits_S \mathbf{R}(\mathbf{D_z}, \mathbf{n}) \mathbf{U}(\mathbf{z}) \cdot \mathbf{U}'(\mathbf{z})\, d_{\mathbf{z}} S. \tag{27}$$

Formulas (25) and (27) are Green's first identity in the linear theory of thermoviscoelasticity of binary porous mixtures for domains Ω^+ and Ω^-, respectively.

We introduce the matrix differential operator $\tilde{\mathbf{A}}(\mathbf{D_x})$, where $\tilde{\mathbf{A}}(\mathbf{D_x}) = \mathbf{A}^\top(-\mathbf{D_x})$ and $\mathbf{A}^\top$ is the transpose of the matrix $\mathbf{A}$. Obviously, the fundamental matrix of the operator $\tilde{\mathbf{A}}(\mathbf{D_x})$ is $\tilde{\mathbf{\Gamma}}(\mathbf{x})$, where

$$\tilde{\mathbf{\Gamma}}(\mathbf{x}) = \mathbf{\Gamma}^\top(-\mathbf{x}). \tag{28}$$

Let $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \varphi, \psi, \theta)$ and the vector $\tilde{\mathbf{U}}_j$ be the j-th column of the matrix $\tilde{\mathbf{U}} = (\tilde{U}_{lj})_{9\times 9}$. By a direct calculation we obtain the following results.

Theorem 8. *If* U *and* $\tilde{U}_j$ $(j = 1, 2, \dots, 9)$ *are regular vectors in* Ω^+*, then*

$$\begin{aligned}
&\int\limits_{\Omega^+} \Big\{ [\tilde{\mathbf{A}}(\mathbf{D_y}) \tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{A}(\mathbf{D_y}) \mathbf{U}(\mathbf{y}) \Big\}\, d\mathbf{y} \\
&= \int\limits_S \Big\{ [\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n}) \tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{R}(\mathbf{D_z}, \mathbf{n}) \mathbf{U}(\mathbf{z}) \Big\}\, d_{\mathbf{z}} S,
\end{aligned} \tag{29}$$

where the operator $\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n})$ *is defined by*

$$\begin{gathered}
\tilde{\mathbf{R}}(\mathbf{D_x}, \mathbf{n}) = (\tilde{R}_{lj}(\mathbf{D_x}, \mathbf{n}))_{9\times 9}, \quad \tilde{R}_{lj}(\mathbf{D_x}, \mathbf{n}) = R_{lj}(\mathbf{D_x}, \mathbf{n}), \\
\tilde{R}_{r9}(\mathbf{D_x}, \mathbf{n}) = -i\omega T_0 \left(\beta^{(1)} + \beta^{(1)}\right) n_r, \quad \tilde{R}_{r+3;9}(\mathbf{D_x}, \mathbf{n}) = -i\omega T_0 \beta^{(2)} n_r, \\
\tilde{R}_{9r}(\mathbf{D_x}, \mathbf{n}) = -\tilde{R}_{9;r+3}(\mathbf{D_x}, \mathbf{n}) = b^* n_r, \quad \tilde{R}_{99}(\mathbf{D_x}, \mathbf{n}) = k \frac{\partial}{\partial \mathbf{n}}, \\
l, j = 1, 2, \dots, 8, \quad r = 1, 2, 3.
\end{gathered} \tag{30}$$

Theorem 9. *If* U *and* $\tilde{U}_j$ $(j = 1, 2, \dots, 9)$ *are regular vectors in* Ω^-*, then*

$$\begin{aligned}
&\int\limits_{\Omega^-} \Big\{ [\tilde{\mathbf{A}}(\mathbf{D_y}) \tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{A}(\mathbf{D_y}) \mathbf{U}(\mathbf{y}) \Big\}\, d\mathbf{y} \\
&= -\int\limits_S \Big\{ [\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n}) \tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{R}(\mathbf{D_z}, \mathbf{n}) \mathbf{U}(\mathbf{z}) \Big\}\, d_{\mathbf{z}} S.
\end{aligned} \tag{31}$$

Formulas (30) and (31) are Green's second identities in the linear theory of thermoviscoelasticity of binary porous mixtures for the domains Ω^+ and Ω^-, respectively.

With the help of the relations (28), (29) and (31) we can derive the following useful consequences.

Theorem 10. *If* U *is a regular vector in* Ω^+*, then*

$$\begin{aligned}
\mathbf{U}(\mathbf{x}) = &\int\limits_S \Big\{ [\tilde{\mathbf{R}}(\mathbf{D_z}, \mathbf{n}) \mathbf{\Gamma}^\top(\mathbf{x} - \mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x} - \mathbf{z})\, \mathbf{R}(\mathbf{D_z}, \mathbf{n}) \mathbf{U}(\mathbf{z}) \Big\}\, d_{\mathbf{z}} S \\
&+ \int\limits_{\Omega^+} \mathbf{\Gamma}(\mathbf{x} - \mathbf{y})\, \mathbf{A}(\mathbf{D_y}) \mathbf{U}(\mathbf{y}) d\mathbf{y} \quad \text{for} \quad \mathbf{x} \in \Omega^+.
\end{aligned} \tag{32}$$

Theorem 11. *If U is a regular vector in Ω^-, then*

$$\mathbf{U}(\mathbf{x}) = -\int\limits_S \left\{[\tilde{\mathbf{R}}(\mathbf{D_z},\mathbf{n})\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - \mathbf{\Gamma}(\mathbf{x}-\mathbf{z})\,\mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\right\} d_\mathbf{z}S$$
$$+\int\limits_{\Omega^-} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\,\boldsymbol{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y})d\mathbf{y} \quad \textit{for} \quad \mathbf{x}\in\Omega^-. \tag{33}$$

Formulas (32) and (33) are integral representations of the regular vector (Green's third identity) in the linear theory of thermoviscoelasticity of binary porous mixtures for the domains Ω^+ and Ω^-, respectively.

6. Uniqueness Theorems

In this section, on the basis of Green's first identity we prove the uniqueness of regular (classical) solutions of BVPs $(K)^+_{\mathbf{F},\mathbf{f}}$ and $(K)^-_{\mathbf{F},\mathbf{f}}$, where $K = I, II$. The scalar product of two vectors $\mathbf{U} = (U_1, U_2, \ldots, U_l)$ and $\mathbf{V} = (V_1, V_2, \ldots, V_l)$ is denoted by $\mathbf{U}\cdot\mathbf{V} = \sum\limits_{j=1}^{l} U_j\overline{V_j}$, where $\overline{V_j}$ is the complex conjugate of V_j.

We have the following

Theorem 12. *If the conditions*

$$\mu^* > 0, \qquad 3\lambda^* + 2\mu^* > 0, \qquad \xi^* > 0,$$
$$4k\xi^* T_0 > (b^* T_0 + f^*)^2, \qquad \sigma_1\tau_2 - \sigma_2\tau_1 \neq 0 \tag{34}$$

are satisfied, then the internal BVP $(I)^+_{\mathbf{F},\mathbf{f}}$ admits at most one regular solution.

Proof. Suppose that there are two regular solutions of the problem $(I)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ corresponds to the zero data ($\mathbf{F} = \mathbf{f} = \mathbf{0}$), i.e., $\mathbf{U}$ is a regular solution of the homogeneous equation

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{35}$$

for $\mathbf{x}\in\Omega^+$ and satisfies the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0}. \tag{36}$$

Then, employing the conditions (35) and (36), we can derive from (21) and (23)

$$\int\limits_{\Omega^+} W_1(\mathbf{U},\mathbf{u})d\mathbf{x} = 0, \quad \int\limits_{\Omega^+} W_2(\mathbf{U},\mathbf{w})d\mathbf{x} = 0, \quad \int\limits_{\Omega^+} W_3(\mathbf{U},\varphi)d\mathbf{x} = 0,$$
$$\int\limits_{\Omega^+} W_4(\mathbf{U},\psi)d\mathbf{x} = 0, \quad \int\limits_{\Omega^+} W_5(\mathbf{U},\theta)d\mathbf{x} = 0, \tag{37}$$

where $W_1, W_2, \ldots, W_5$ are defined by (22) and (24).

In view of the relations (6) and (8), we can write

$$\alpha_1' + 3\alpha_2' - 2\epsilon_1 = \alpha_1 + 3\alpha_2 - 2(\mu + 2\zeta + 2\gamma) - i\omega(3\lambda^* + 2\mu^*),$$
$$\alpha_1' + \epsilon_1 = \alpha_1 + \mu + 2\zeta + 2\gamma - 2i\omega\mu^*,$$
$$\eta_1'\,|\mathbf{u}|^2 + 2\xi'\mathrm{Re}\,[\mathbf{u}\cdot\mathbf{w})] + \eta_2'\,|\mathbf{w}|^2 = (\rho_1\omega^2 - \xi)\,|\mathbf{u}|^2 + 2\xi\mathrm{Re}\,[\mathbf{u}\cdot\mathbf{w})]$$
$$+(\rho_2\omega^2 - \xi)\,|\mathbf{w}|^2 + i\omega\xi^*\,|\mathbf{u}-\mathbf{w}|^2. \tag{38}$$

Obviously, on the basis of (22), (24) and (38), it follows that

$$\mathrm{Im}\,[W_1(\mathbf{U},\mathbf{u}) + W_2(\mathbf{U},\mathbf{w}) + W_3(\mathbf{U},\varphi) + W_4(\mathbf{U},\psi)]$$
$$= -\frac{\omega}{3}(3\lambda^* + 2\mu^*)\,|\mathrm{div}\,\mathbf{u}|^2 - 2\omega W^{(0)}(\mathbf{u},\mathbf{u}) - \omega\xi^*|\mathbf{u}-\mathbf{w}|^2$$
$$-\mathrm{Im}\left\{\left[(\beta^{(1)} + \beta^{(2)})\mathrm{div}\,\overline{\mathbf{u}} + \beta^{(2)}\mathrm{div}\,\overline{\mathbf{w}} + b^{(1)}\overline{\varphi} + b^{(2)}\overline{\psi}\right]\theta - b^*\nabla\theta\cdot(\mathbf{u}-\mathbf{w})\right\}, \tag{39}$$

$$\frac{1}{\omega T_0}\text{Re}W_5(\mathbf{U},\theta)=\frac{k}{\omega T_0}|\nabla\theta|^2+\text{Im}\left[(\beta^{(1)}+\beta^{(2)})\text{div}\,\mathbf{u}+\beta^{(2)}\text{div}\,\mathbf{w}+b^{(1)}\varphi+b^{(2)}\psi\right]\overline{\theta}$$
$$+\frac{1}{T_0}f^*\text{Im}\left[(\mathbf{u}-\mathbf{w})\cdot\nabla\theta\right].$$

Clearly, from (39), we get

$$\begin{aligned}\frac{1}{\omega T_0}\text{Re}W_5(\mathbf{U},\theta)&-\text{Im}\left[W_1(\mathbf{U},\mathbf{u})+W_2(\mathbf{U},\mathbf{w})+W_3(\mathbf{U},\varphi)+W_4(\mathbf{U},\psi)\right]\\ &=\frac{\omega}{3}(3\lambda^*+2\mu^*)\,|\text{div}\,\mathbf{u}|^2+2\omega W^{(0)}(\mathbf{u},\mathbf{u})+\omega\xi^*|\mathbf{u}-\mathbf{w}|^2\\ &\quad+\frac{k}{\omega T_0}|\nabla\theta|^2-\left(b^*+\frac{f^*}{T_0}\right)\text{Im}\left[(\mathbf{u}-\mathbf{w})\cdot\nabla\theta\right].\end{aligned}\tag{40}$$

In view of (37) and (40), we have

$$\begin{aligned}&\frac{\omega}{3}(3\lambda^*+2\mu^*)\,|\text{div}\,\mathbf{u}|^2+2\omega W^{(0)}(\mathbf{u},\mathbf{u})+\omega\xi^*|\mathbf{u}-\mathbf{w}|^2\\ &\quad+\frac{k}{\omega T_0}|\nabla\theta|^2-\left(b^*+\frac{f^*}{T_0}\right)\text{Im}\left[(\mathbf{u}-\mathbf{w})\cdot\nabla\theta\right]=0.\end{aligned}$$

By virtue of (34), the last equation leads to the following relations:

$$\begin{gathered}\text{div}\,\mathbf{u}(\mathbf{x})=0,\quad W^{(0)}(\mathbf{u},\mathbf{u})=0,\quad \mathbf{u}(\mathbf{x})=\mathbf{w}(\mathbf{x}),\\ \nabla\theta(\mathbf{x})=0,\quad \mathbf{x}\in\Omega^+.\end{gathered}\tag{41}$$

Then, employing (18), (22) and (41), we can derive from (37)

$$\begin{aligned}W_1(\mathbf{u},\mathbf{u})+W_2(\mathbf{u},\mathbf{u})&=W^{(1)}(\mathbf{u},\mathbf{u})+2W^{(2)}(\mathbf{u},\mathbf{u})+W^{(3)}(\mathbf{u},\mathbf{u})\\ =\frac{1}{2}\left[\alpha_1'-\epsilon_1+2(\beta_1-\epsilon_2)+\gamma_1-\epsilon_3\right]|\text{curl}\,\mathbf{u}|^2&-(\eta_1'+2\xi'+\eta_2')|\mathbf{u}|^2=-\rho\omega^2|\mathbf{u}|^2=0\end{aligned}\tag{42}$$

and consequently, from (42), we have

$$\mathbf{u}(\mathbf{x})\equiv\mathbf{0}\quad\text{for}\quad\mathbf{x}\in\Omega^+.\tag{43}$$

Now, taking into account (41) and (43), from (35), we deduce the system

$$\sigma_1\nabla\varphi+\sigma_2\nabla\psi=0,\quad \tau_1\nabla\varphi+\tau_2\nabla\psi=0.\tag{44}$$

By virtue of the last relation of (34), from (44), we obtain $\nabla\varphi=\nabla\psi=0$. Combining this relation with (41) and (43), we may further conclude that

$$\mathbf{u}(\mathbf{x})=\mathbf{w}(\mathbf{x})\equiv\mathbf{0},\quad\varphi(\mathbf{x})=c_1,\quad\psi(\mathbf{x})=c_2,\quad\theta(\mathbf{x})=c_3\quad\text{for}\quad\mathbf{x}\in\Omega^+,\tag{45}$$

where c_1,c_2 and c_3 are arbitrary complex numbers. Finally, in view of the homogeneous boundary condition (36), from (45), we get $c_1=c_2=c_3=0$. Thus, $\mathbf{U}(\mathbf{x})\equiv\mathbf{0}$ for $\mathbf{x}\in\Omega^+$, and we have the desired result. □

Theorem 13. *If the conditions* (34) *and*

$$\det\begin{pmatrix}\eta_1 & -\zeta^{(3)} & b^{(1)}\\ -\zeta^{(3)} & \eta_2 & b^{(2)}\\ b^{(1)} & b^{(2)} & a\end{pmatrix}_{3\times3}\neq0\tag{46}$$

are satisfied, then the internal BVP $(II)^+_{\mathbf{F},\mathbf{f}}$ *admits at most one regular solution.*

Proof. Suppose that there are two regular solutions of problem $(II)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ corresponds to zero data ($\mathbf{F}=\mathbf{f}=\mathbf{0}$), i.e., $\mathbf{U}$ is a regular solution of problem $(II)^+_{\mathbf{0},\mathbf{0}}$. Consequently, $\mathbf{U}$ is a regular solution of the system of homogeneous equations (35) satisfying the homogeneous boundary condition

$$\{\mathbf{R}(\mathbf{D}_\mathbf{z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+=\mathbf{0}\quad\text{for}\quad\mathbf{z}\in S.$$

In a similar manner as in Theorem 12 we obtain the relations (45). We now combine (45) with (35) to deduce the system

$$\begin{aligned} \eta_1 c_1 - \zeta^{(3)} c_2 + b^{(1)} c_3 = 0, \\ -\zeta^{(3)} c_1 + \eta_2 c_2 + b^{(2)} c_3 = 0, \\ b^{(1)} c_1 + b^{(2)} c_2 + a c_3 = 0. \end{aligned} \tag{47}$$

By virtue of (46), from (47), we obtain $c_1 = c_2 = c_3 = 0$ and, therefore, we get the relation $\mathbf{U}(\mathbf{x}) \equiv \mathbf{0}$ for $\mathbf{x} \in \Omega^+$. Hence, the uniqueness of a regular solution to problem $(II)^+_{\mathbf{F},\mathbf{f}}$ follows. □

Quite similarly, on the basis of the condition (16) and the identity (27), we obtain the following

Theorem 14. *If condition* (34) *is satisfied, then the external BVP* $(K)^-_{\mathbf{F},\mathbf{f}}$ *admits at most one regular solution, where* $K = I, II$.

7. Surface and Volume Potentials

We introduce the following notation:

i) $\mathbf{Q}^{(1)}(\mathbf{x},\mathbf{g}) = \int\limits_S \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{g}(\mathbf{y}) d_{\mathbf{y}}S$ is the single-layer potential,

ii) $\mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}) = \int\limits_S [\tilde{\mathbf{R}}(\mathbf{D}_{\mathbf{y}},\mathbf{n}(\mathbf{y}))\mathbf{\Gamma}^\top(\mathbf{x}-\mathbf{y})]^\top \mathbf{g}(\mathbf{y}) d_{\mathbf{y}}S$ is the double-layer potential, and

iii) $\mathbf{Q}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^\pm) = \int\limits_{\Omega^\pm} \mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\boldsymbol{\phi}(\mathbf{y}) d\mathbf{y}$ is the volume potential,

where the matrices $\mathbf{\Gamma}(\mathbf{x})$ and $\tilde{\mathbf{R}}(\mathbf{D}_{\mathbf{x}})$ are given by (14) and (30), respectively; $\mathbf{g}$ and $\boldsymbol{\phi}$ are the nine-component vector functions.

Obviously, on the basis of Green's third identities (32) and (33), the regular vector $\mathbf{U}$ in Ω^+ is represented by the sum of the single-layer, double-layer and volume potentials as follows:

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(2)}(\mathbf{x},\mathbf{U}) - \mathbf{Q}^{(1)}(\mathbf{x},\mathbf{R}\mathbf{U}) + \mathbf{Q}^{(3)}(\mathbf{x},\mathbf{A}\mathbf{U},\Omega^+) \quad \text{for} \quad \mathbf{x} \in \Omega^+.$$

Similarly, the regular vector $\mathbf{U}$ in Ω^- is represented by the sum

$$\mathbf{U}(\mathbf{x}) = -\mathbf{Q}^{(2)}(\mathbf{x},\mathbf{U}) + \mathbf{Q}^{(1)}(\mathbf{x},\mathbf{R}\mathbf{U}) + \mathbf{Q}^{(3)}(\mathbf{x},\mathbf{A}\mathbf{U},\Omega^-) \quad \text{for} \quad \mathbf{x} \in \Omega^-.$$

On the basis of (14) and (15), we have the following results.

Theorem 15. *If* $S \in C^{m+1,\nu'}$, $\mathbf{g} \in C^{m,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, *and* m *is a non-negative integer, then:*

a)
$$\mathbf{Q}^{(1)}(\cdot,\mathbf{g}) \in C^{0,\nu''}(\mathbb{R}^3) \cap C^{m+1,\nu''}(\overline{\Omega^\pm}) \cap C^\infty(\Omega^\pm),$$

b)
$$\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{Q}^{(1)}(\mathbf{x},\mathbf{g}) = \mathbf{0},$$

c)
$$\{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g})\}^\pm = \mp\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}), \tag{48}$$

d)
$$\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g})$$

is a singular integral, where $\mathbf{z} \in S$, $\mathbf{x} \in \Omega^\pm$ *and*

$$\{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g})\}^\pm \equiv \lim_{\Omega^\pm \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D}_{\mathbf{x}},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(1)}(\mathbf{x},\mathbf{g}).$$

Theorem 16. *If* $S \in C^{m+1,\nu'}$, $\mathbf{g} \in C^{m,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, *then:*

a)
$$\mathbf{Q}^{(2)}(\cdot,\mathbf{g}) \in C^{m,\nu''}(\overline{\Omega^\pm}) \cap C^\infty(\Omega^\pm),$$

b)
$$\mathbf{A}(\mathbf{D}_{\mathbf{x}})\,\mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}) = \mathbf{0},$$

c)

$$\{\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{\pm} = \pm\,\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}) \tag{49}$$

for the non-negative integer m,

d) $\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})$ *is a singular integral, where* $\mathbf{z}\in S$,

e)

$$\{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{+} = \{\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{-}$$

for the natural number m, *where* $\mathbf{z}\in S$, $\mathbf{x}\in\Omega^{\pm}$ *and*

$$\{\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g})\}^{\pm} \equiv \lim_{\Omega^{\pm}\ni\mathbf{x}\to\ \mathbf{z}\in S} \mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}).$$

Theorem 17. *If* $S\in C^{1,\nu'}$, $\boldsymbol{\phi}\in C^{0,\nu''}(\Omega^+)$, $0<\nu''<\nu'\le 1$, *then:*

a)

$$\mathbf{Q}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^+)\in C^{1,\nu''}(\mathbb{R}^3)\cap C^2(\Omega^+)\cap C^{2,\nu''}\left(\overline{\Omega_0^+}\right),$$

b)

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^+) = \boldsymbol{\phi}(\mathbf{x}),$$

where $\mathbf{x}\in\Omega^+$, Ω_0^+ *is a domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^+}\subset\Omega^+$.

Theorem 18. *If* $S\in C^{1,\nu'}$, $\mathrm{supp}\boldsymbol{\phi}=\Omega\subset\Omega^-$, $\boldsymbol{\phi}\in C^{0,\nu''}(\Omega^-)$, $0<\nu''<\nu'\le 1$, *then:*

a)

$$\mathbf{Q}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^-)\in C^{1,\nu''}(\mathbb{R}^3)\cap C^2(\Omega^-)\cap C^{2,\nu''}(\overline{\Omega_0^-}),$$

b)

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{Q}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^-) = \boldsymbol{\phi}(\mathbf{x}),$$

where $\mathbf{x}\in\Omega^-$, Ω *is a bounded domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^-}\subset\Omega^-$.

We here introduce the following notation:

$$\begin{gathered}
\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) \equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}), \qquad \mathcal{K}^{(2)}\mathbf{g}(\mathbf{z}) \equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}^{(3)}\mathbf{g}(\mathbf{z}) \equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}), \quad \mathcal{K}^{(4)}\mathbf{g}(\mathbf{z}) \equiv \frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{Q}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{K}_{\varsigma}\mathbf{g}(\mathbf{z}) \equiv -\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \varsigma\,\mathbf{Q}^{(2)}(\mathbf{z},\mathbf{g}) \quad \text{for} \quad \mathbf{z}\in S,
\end{gathered} \tag{50}$$

where ς is a complex parameter. On the basis of Theorems 15 and 16, $\mathcal{K}^{(j)}$ $(j=1,2,3,4)$ and $\mathcal{K}_{\varsigma}$ are singular integral operators.

We introduce the notation

$$\begin{gathered}
\mu_1' = \mu_1 - i\omega\mu^*, \quad \mu_1 = \mu+\kappa+\gamma+2\zeta, \quad \mu_2 = \kappa+\gamma,\\
\mu_3 = \kappa+\gamma+\zeta, \quad \mu_0' = \mu_1'+\mu_2+3\mu_3, \quad e_1' = \alpha_2'+\kappa-\gamma,\\
e_1 = \alpha_2+\kappa-\gamma, \quad e_2 = \gamma_2+\kappa-\gamma, \quad e_3 = \beta_2-\kappa+\gamma,\\
e_0' = e_1'+e_2+e_3, \quad b_1 = \alpha_1\gamma_1-\beta_1^2, \quad b_2' = \mu_1'\mu_2-\mu_3^2,\\
b_2 = \mu_1\mu_2-\mu_3^2, \quad b_3' = e_1'e_2-e_3^2, \quad b_3 = e_1e_2-e_3^2.
\end{gathered} \tag{51}$$

Let $\boldsymbol{\sigma}^{(j)} = (\sigma_{lm}^{(j)})_{9\times 9}$ be the symbol of the singular integral operator $\mathcal{K}^{(j)}$ $(j=1,2,3,4)$. Taking into account (50) and (51), by a long calculation for det $\boldsymbol{\sigma}^{(j)}$, we find that

$$\det\boldsymbol{\sigma}^{(j)} = -\frac{1}{512}\,\frac{k_2k_3}{k_0k_1}, \quad j=1,2,3,4, \tag{52}$$

where k_0 and k_1 are defined by (10) and

$$k_2 = (\alpha_0'+\alpha_1)(\gamma_0+\gamma_1) - (\beta_0+\beta_1)^2, \quad k_3 = k_1b_3' + (\kappa-\gamma)(\mu_0'b_3'+e_0'b_2'). \tag{53}$$

On the basis of (6), (8) and (51), from (10) and (53), we have

$$\mathrm{Im}k_0 = -\omega(\lambda^*+2\mu^*)\gamma_0, \quad \mathrm{Im}k_1 = -\omega\mu^*\gamma_1, \quad \mathrm{Im}k_2 = -\omega(\lambda^*+3\mu^*)(\gamma_0+\gamma_1),$$

$$\mathrm{Im}k_3 = -\omega(\lambda^* + \mu^*)b_1 e_2 - \omega\mu^* b_3\gamma_1 - \omega(\kappa - \gamma)\left[\lambda^*(\mu_0 e_2 + b_2) + \mu^*(\mu_2 e_0 + b_3)\right].$$

Clearly, if

$$\mathrm{Im}k_l \neq 0, \quad l = 0,1,2,3, \tag{54}$$

then from (52), it follows that

$$\det \boldsymbol{\sigma}^{(j)} \neq 0 \tag{55}$$

which proves that the singular integral operator $\mathcal{K}^{(j)}$ is of the normal type, where $j = 1,2,3,4$. Hence we have the following

Theorem 19. *If condition* (54) *is satisfied, then the singular integral operator $\mathcal{K}^{(j)}$ is of the normal type, where $j = 1,2,3,4$.*

Let $\boldsymbol{\sigma}_\varsigma$ and $\mathrm{ind}\,\mathcal{K}_\varsigma$ be the symbol and the index of the operator $\mathcal{K}_\varsigma$, respectively. It can be easily shown that $\det\boldsymbol{\sigma}_\varsigma$ vanishes only at four points ς_j $(j = 1,2,3,4)$ of the complex plane. By virtue of (55) and $\det\boldsymbol{\sigma}_1 = \det\boldsymbol{\sigma}^{(1)}$, we get $\varsigma_j \neq 1$ for $j = 1,2,3,4$, and we obtain

$$\mathrm{ind}\,\mathcal{K}^{(1)} = \mathrm{ind}\,\mathcal{K}_1 = 0.$$

In a quite similar manner, we have the relation $\mathrm{ind}\,\mathcal{K}^{(2)} = 0$. We can easily verify that the operators $\mathcal{K}^{(3)}$ and $\mathcal{K}^{(4)}$ are the adjoint operators for $\mathcal{K}^{(2)}$ and $\mathcal{K}^{(1)}$, respectively. Consequently, we have

$$\mathrm{ind}\,\mathcal{K}^{(3)} = -\mathrm{ind}\,\mathcal{K}^{(2)} = 0, \quad \mathrm{ind}\,\mathcal{K}^{(4)} = -\mathrm{ind}\,\mathcal{K}^{(1)} = 0.$$

Hence, the singular integral operator $\mathcal{K}^{(j)}$ $(j = 1,2,3,4)$ is of the normal type with an index equal to zero, i.e., Fredholm's theorems are valid for $\mathcal{K}^{(j)}$. Thus, we have proved the following

Theorem 20. *If condition* (54) *is satisfied, then Fredholm's theorems are valid for the singular integral operator $\mathcal{K}^{(j)}$, where $j = 1,2,3,4$.*

Remark 1. The definitions of a normal type singular integral operator, the symbol and the index of the operator are given in [18,20]. In addition, in these books, one can find the method for calculating the symbol of singular integral operator.

8. Existence Theorems

In what follows, we assume that the constitutive coefficients satisfy the conditions (34), (46) and (54). Obviously, by Theorems 17 and 18, the volume potential $\mathbf{Q}^{(3)}(\mathbf{x},\mathbf{F},\Omega^\pm)$ is a partial regular solution of the nonhomogeneous equation (9), where $\mathbf{F} \in C^{0,\nu'}(\Omega^\pm)$, $0 < \nu' \leq 1$ and $\mathrm{supp}\,\mathbf{F}$ is a finite domain in Ω^-. Therefore, in this section, we prove the existence theorems for classical solutions of the BVPs $(K)^+_{\mathbf{0},\mathbf{f}}$ and $(K)^-_{\mathbf{0},\mathbf{f}}$, where $K = I, II$.

Problem $(I)^+_{\mathbf{0},\mathbf{f}}$. We are looking for a regular solution to this problem in the form of a double-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(2)}(\mathbf{x},\mathbf{g}) \quad \text{for} \quad \mathbf{x} \in \Omega^+, \tag{56}$$

where $\mathbf{g}$ is the required nine-component vector function. By Theorem 16, the vector function $\mathbf{U}$ is a solution of the homogeneous equation (35) for $\mathbf{x} \in \Omega^+$. Keeping in mind the boundary condition $\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$ and using (49), from (56), for determining the unknown vector $\mathbf{g}$, we obtain a singular integral equation

$$\mathcal{K}^{(1)}\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S. \tag{57}$$

By Theorem 20, Fredholm's theorems are valid for the operator $\mathcal{K}^{(1)}$. We prove that (57) is always solvable for an arbitrary vector $\mathbf{f}$. Let us consider the adjoint homogeneous equation

$$\mathcal{K}^{(4)}\mathbf{h}_0(\mathbf{z}) = \mathbf{0} \quad \text{for} \quad \mathbf{z} \in S, \tag{58}$$

where $\mathbf{h}_0$ is the required nine-component vector function.

Now we prove that (58) has only the trivial solution. Indeed, let $\mathbf{h}_0$ be a solution of the homogeneous equation (58). On the basis of Theorem 15 and equation (58), the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x},\mathbf{h}_0)$ is a regular solution of problem $(II)^-_{\mathbf{0},\mathbf{0}}$. Using Theorem 14, problem $(II)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^-. \tag{59}$$

On the other hand, by Theorem 15 and equation (59), we get $\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0}$ for $\mathbf{z} \in S$, i.e., the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of problem $(I)^+_{\mathbf{0},\mathbf{0}}$. Using Theorem 12, problem $(I)^+_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, i.e.,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^+. \tag{60}$$

By virtue of (59), (60) and identity (48), we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ \equiv \mathbf{0} \quad \text{for} \quad \mathbf{z} \in S.$$

Thus, the homogeneous equation (58) has only the trivial solution and, therefore, (57) is always solvable for an arbitrary vector $\mathbf{f}$.

We have thereby proved

Theorem 21. *If $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{1,\nu''}(S)$, $0 < \nu'' < \nu' \leq 1$, then a regular solution of problem $(I)^+_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by the double-layer potential* (56)*, where $\mathbf{g}$ is a solution of the singular integral equation* (57) *which is always solvable for an arbitrary vector $\mathbf{f}$.*

Problem $(II)^-_{\mathbf{0},\mathbf{f}}$**.** We are looking for a regular solution to this problem in the form of a single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{h}) \quad \text{for} \quad \mathbf{x} \in \Omega^-, \tag{61}$$

where $\mathbf{h}$ is the required nine-component vector function. Obviously, by Theorem 15, the vector function $\mathbf{U}$ is a solution of (35) for $\mathbf{x} \in \Omega^-$. Keeping in mind the boundary condition $\{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$ and using (48), from (61), for determining the unknown vector $\mathbf{h}$, we obtain a singular integral equation

$$\mathcal{K}^{(4)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S. \tag{62}$$

It has been proved above that the corresponding homogeneous equation (58) has only the trivial solution. Hence, it follows that (62) is always solvable.

We have thereby proved

Theorem 22. *If $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{0,\nu''}(S)$, $0 < \nu'' < \nu' \leq 1$, then a regular solution of problem $(II)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by a single-layer potential* (61)*, where $\mathbf{h}$ is a solution of the singular integral equation* (62) *which is always solvable for an arbitrary vector $\mathbf{f}$.*

Problem $(II)^+_{\mathbf{0},\mathbf{f}}$**.** We are looking for a regular solution to this problem in the form of a single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{h}) \quad \text{for} \quad \mathbf{x} \in \Omega^+, \tag{63}$$

where $\mathbf{h}$ is the required nine-component vector function. Obviously, by Theorem 15, the vector function $\mathbf{U}$ is a solution of (35) for $\mathbf{x} \in \Omega^+$. Keeping in mind the boundary condition $\{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z})$ and using (48), from (63), for determining the unknown vector $\mathbf{h}$, we obtain a singular integral equation

$$\mathcal{K}^{(2)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S. \tag{64}$$

We now prove that (64) is always solvable for an arbitrary vector $\mathbf{f}$. Let $\mathbf{h}_0$ be a solution of the homogeneous equation

$$\mathcal{K}^{(2)}\mathbf{h}(\mathbf{z}) = \mathbf{0} \quad \text{for} \quad \mathbf{z} \in S. \tag{65}$$

On the basis of Theorem 15 and equation (65), the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x}, \mathbf{h}_0)$ is a regular solution of problem $(II)^+_{\mathbf{0},\mathbf{0}}$. Using Theorem 13, problem $(II)^+_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^+. \tag{66}$$

On the other hand, by Theorem 15 and equation (66), we get $\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0}$ for $\mathbf{z} \in S$, i.e., the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of problem $(I)^-_{\mathbf{0},\mathbf{0}}$. Now, using Theorem 14, problem $(I)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^-. \tag{67}$$

By virtue of (66), (67) and identity (48), we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ \equiv \mathbf{0} \quad \text{for} \quad \mathbf{z} \in S.$$

Thus, the homogeneous equation (65) has only the trivial solution and, therefore, (64) is always solvable for an arbitrary vector $\mathbf{f}$.

We have thereby proved

Theorem 23. *If* $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{0,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, *then a regular solution of problem* $(II)^+_{\mathbf{0},\mathbf{f}}$ *exists, is unique and represented by a single-layer potential* (63)*, where* $\mathbf{h}$ *is a solution of the singular integral equation* (64) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Problem $(I)^-_{\mathbf{0},\mathbf{f}}$**.** Finally, we are looking for a regular solution to this problem in the form of a double-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{Q}^{(2)}(\mathbf{x}, \mathbf{g}) \quad \text{for} \quad \mathbf{x} \in \Omega^-, \tag{68}$$

where $\mathbf{g}$ is the required nine-component vector function. Obviously, by Theorem 16, the vector function $\mathbf{U}$ is a solution of (35) for $\mathbf{x} \in \Omega^-$. Keeping in mind the boundary condition $\{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z})$ and using (49), from (68), for determining the unknown vector $\mathbf{g}$, we obtain a singular integral equation

$$\mathcal{K}^{(3)}\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S. \tag{69}$$

It has been proved above that the adjoint homogeneous equation (65) has only the trivial solution. Hence, it follows that (69) is always solvable.

Thus, we have thereby proved

Theorem 24. *If* $S \in C^{2,\nu'}$, $\mathbf{f} \in C^{1,\nu''}(S)$, $0 < \nu'' < \nu' \le 1$, *then a regular solution of problem* $(I)^-_{\mathbf{0},\mathbf{f}}$ *exists, is unique and represented by a double-layer potential* (68)*, where* $\mathbf{g}$ *is a solution of the singular integral equation* (69) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

9. Concluding Remarks

In this paper, the linear theory of thermoviscoelasticity for binary porous mixtures is considered and the following results are obtained.

a) The fundamental solution of the system of equations of steady vibrations is constructed explicitly and its basic properties are established.

b) Green's identities are obtained.

c) The uniqueness theorems for classical solutions of the internal and external basic BVPs of steady vibrations are proved.

d) The surface and volume potentials are constructed and their basic properties are given.

e) The determinants of symbolic matrices are calculated explicitly.

f) The BVPs are reduced to the always solvable singular integral equations for which Fredholm's theorems are valid.

g) Finally, the existence theorem for classical solutions of the internal and external BVPs of steady vibrations are proved by means of the potential method and the theory of singular integral equations.

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