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RESTRICTED TESTING FOR THE HARDY–LITTLEWOOD MAXIMAL FUNCTION ON ORLICZ SPACES

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Abstract. In this short note, we formulate and prove a restricted testing condition for the Hardy–Littlewood maximal function acting between the weighted Orlicz spaces.

1. Introduction

Our interest in this note is for the two weight inequalities for the Hardy–Littlewood maximal function acting between the weighted Orlicz spaces of $\mathbb{R}^d$.

Recall that a weight ω on $\mathbb{R}^d$ is any positive locally integrable function. The Hardy–Littlewood maximal function is defined by

$$\mathcal{M}f(x) := \sup_{x\in Q} \frac{1_Q(x)}{|Q|} \int\limits_Q |f(y)|dy \tag{1}$$

with Q a cube whose sides are parallel to the coordinate axes, and $|Q|$ is the Lebesgue measure of Q.

In 1982, E. T. Sawyer (see [8]) obtained the following two weight characterizations for the Hardy–Littlewood maximal function.

Theorem 1.1. *For $1 < p < \infty$ and for any pair of weights (ω, σ), we have the inequality*

$$\|\mathcal{M}(\sigma f)\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\sigma)} \tag{2}$$

if and only if

$$\sup_{Q,\, \sigma(Q)>0} \sigma(Q)^{-1/p} \|1_Q \mathcal{M}(\sigma 1_Q)\|_{L^p(\omega)} < \infty. \tag{3}$$

Sawyer's result summarizes as follows: for (2) to hold for any $f \in L^p(\sigma)$, it suffices for it to hold on characteristic functions of cubes.

Pretty recently, it has been observed that the supremum in (3) doesn't need to be taken on all cubes provided the so-called A_p condition for the pair of weights (ω, σ) holds ([2, 3]). To be more precise, this type of new characterizations was first considered in [5, 7] for various operators. In particular, in [7], the authors introduced the restricted testing to doubling cubes in the two weight inequalities for the maximal operator $\mathcal{M}$. In [3], W. Chen and M. T. Lacey obtained similar conditions providing also a short proof. More recently, in [2], the authors exploited these new ideas to obtain corresponding results for the multilinear maximal operator.

Recall that the pair of weights (ω, σ) is said to satisfy the A_p condition if

$$[\omega, \sigma]_p := \sup_Q \langle \omega \rangle_Q^{1/p} \langle \sigma \rangle_Q^{1/p'} < \infty$$

where $\langle \omega \rangle_Q = |Q|^{-1} \int_Q \omega dx$, and $\frac{1}{p} + \frac{1}{p'} = 1$.

We recall the following definition.

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Definition 1.2. Let $1 < \rho, p, D < \infty$. We say the pair of weights (ω, σ) satisfies a (ρ, p, D) parent doubling testing condition if there is a positive finite constant $\mathcal{P} = \mathcal{P}_{\rho,D} = \mathcal{P}(\omega, \sigma, d, p, \rho, D)$, so that we have

$$\sigma(Q)^{-1/p}\|1_Q\mathcal{M}(\sigma 1_Q)\|_{L^p(\omega)} \le \mathcal{P}$$

for every cube Q for which there is another cube $R \supset Q$, with $\ell R \ge \rho \ell Q$, and $\sigma(R) \le D\sigma(Q)$.

Above and all over the test, $\ell Q = |Q|^{1/d}$ is the side length of the cube Q.

The result obtained by W. Chen and M. T. Lacey in [3] is the following

Theorem 1.3. *Let $1 < p, \rho < \infty$. Then there exists a constant $D = D_{d,p,\rho}$ such that for any pair of weights (ω, σ), we have*

$$\|\mathcal{M}(\sigma\cdot)\|_{L^p(\sigma)\to L^p(\omega)} \approx [\omega, \sigma]_p + \mathcal{P}_{\rho,D}.$$

Our aim here is to formulate and prove an analogue of Theorem 1.3 when the Lebesgue spaces are replaced by the Orlicz spaces.

By a growth function we will mean a continuous and nondecreasing function Φ from $[0, \infty)$ onto itself. We note that this implies, in particular, that $\Phi(0) = 0$.

The growth function Φ is said to satisfy the Δ_2-condition if there exists a constant $K > 1$ such that for any $t \ge 0$,

$$\Phi(2t) \le K\Phi(t). \tag{4}$$

Given a convex growth function Φ satisfying the Δ_2-condition and a weight ω, the weighted Orlicz space $L^\Phi_\omega(\mathbb{R}^d)$ is defined to be the space of all functions f on $\mathbb{R}^d$ such that

$$\|f\|_{\Phi,\omega} := \int_{\mathbb{R}^d} \Phi(|f(t)|)\omega(t)dt < \infty.$$

Let us note that when $\Phi(t) = t^p$, $1 \le p < \infty$, the above space is just the usual weighted Lebesgue $L^p_\omega(\mathbb{R}^d)$.

Recall also that the complementary function Ψ of the convex growth function Φ is the function defined from $\mathbb{R}_+$ onto itself by

$$\Psi(s) = \sup_{t\in\mathbb{R}_+}\{ts - \Phi(t)\}. \tag{5}$$

A growth function Φ is said to satisfy the ∇_2-condition whenever both Φ and its complementary function satisfy the Δ_2-conditon.

Given a convex growth function Φ, we define $\phi(t) = \frac{\Phi(t)}{t}$ and observe that ϕ is nondecreasing. We then say a pair of weights (ω, σ) satisfies the A_Φ condition whenever

$$[\omega, \sigma]_\Phi := \sup_Q \langle\omega\rangle_Q \phi(\langle\sigma\rangle_Q) < \infty.$$

Let us now introduce the following

Definition 1.4. Let Φ be a convex growth function and $1 < \rho, D < \infty$. We say the pair of weights (ω, σ) satisfies a (ρ, Φ, D) parent doubling testing condition if there is a positive finite constant $\mathcal{P} = \mathcal{P}_{\rho,D} = \mathcal{P}(\omega, \sigma, d, \Phi, \rho, D)$ so that we have

$$\int_Q \Phi\left(\mathcal{M}(\sigma 1_Q)\right)\omega dx \le \mathcal{P}\sigma(Q) \tag{6}$$

for every cube Q for which there is another cube $R \supset Q$ with $\ell R \ge \rho\ell Q$, and $\sigma(R) \le D\sigma(Q)$.

Let us denote by $\mathscr{C}$ the set of all convex growth functions. We then define $\mathscr{C}'$ as the set of all $\Phi \in \mathscr{C} \cap \mathcal{C}^1$ such that $\Phi'(t) \approx \frac{\Phi(t)}{t}$.

We say a growth function Φ satisfies the Δ'-condition if there exists a constant $C_1 > 0$ such that for any $0 < s, t < \infty$,

$$\Phi(st) \le C_1\Phi(s)\Phi(t). \tag{7}$$

Obviously, power functions satisfy (7). As a nontrivial example of a growth function satisfying (7), we have the function $t \mapsto t^q \log^\alpha(C + t)$, where $q \ge 1$, $\alpha > 0$ and the constant $C > 0$ is large enough.

It is not difficult to prove the following extension of Theorem 1.1 (see also [9]).

Theorem 1.5. *Let $\Phi \in \mathcal{C}'$ be a growth function satisfying both the Δ'-condition and the ∇_2 condition. Then*

$$\sup_{0\neq f\in L^{\Phi}(\sigma)} \frac{\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}(\sigma f)(x)\right)\omega(x)dx}{\int_{\mathbb{R}^d} \Phi(|f(x)|)\sigma(x)dx} < \infty \tag{8}$$

holds if and only if

$$[\omega,\sigma]_{S_\Phi} := \sup_{Q,\,\sigma(Q)>0} \sigma(Q)^{-1} \int_Q \Phi\left(\mathcal{M}(\sigma 1_Q)\right)\omega dx < \infty. \tag{9}$$

Moreover,

$$\sup_{0\neq f\in L^{\Phi}(\sigma)} \frac{\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}(\sigma f)(x)\right)\omega(x)dx}{\int_{\mathbb{R}^d} \Phi(|f(x)|)\sigma(x)dx} \approx [\omega,\sigma]_{S_\Phi}. \tag{10}$$

It is obvious that (8) implies (9). The converse can be proved as in the power functions case, using Theorem 2.4 in the next section.

Our result here is about restricting the global testing condition (9). It is given as follows.

Theorem 1.6. *Let $\Phi \in \mathcal{C}'$ be a growth function satisfying both the Δ'-condition and the ∇_2 condition, and let $1<\rho<\infty$. Then there exists a constant $D = D_{d,\Phi,\rho}$ such that for any pair of weights (ω,σ), we have*

$$\sup_{0\neq f\in L^{\Phi}(\sigma)} \frac{\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}(\sigma f)(x)\right)\omega(x)dx}{\int_{\mathbb{R}^d} \Phi(|f(x)|)\sigma(x)dx} \approx [\omega,\sigma]_{\Phi} + \mathcal{P}_{\rho,D}. \tag{11}$$

2. Preliminaries

2.1. **Indices of a Growth function.** We recall that for Φ a $\mathcal{C}^1$ growth function, the lower index of Φ is defined by

$$a = a_\Phi := \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

Following [4, Lemma 2.6], we find that if a convex growth function Φ satisfies the ∇_2-condition, then $1 < a_\Phi < \infty$.

It is easy to see that if Φ is a $\mathcal{C}^1$ growth function, then the function $\frac{\Phi(t)}{t^{a_\Phi}}$ is increasing.

2.2. **Dyadic grids and sparse families.** The standard dyadic grid $\mathcal{D}$ in $\mathbb{R}^d$ is the collection of all cubes of the form

$$2^{-k}\left([0,1)^d + m\right), \quad k\in\mathbb{Z}, m\in\mathbb{Z}^d.$$

Definition 2.1. A (general) dyadic grid $\mathcal{D}^\beta$ in $\mathbb{R}^d$ is any collection of cubes such that:

(i) the sidelength ℓQ of any cube $Q\in\mathcal{D}^\beta$ is 2^k for some $k\in\mathbb{Z}$;
(ii) for $Q, Q'\in\mathcal{D}^\beta$, $Q\cap Q'\in\{Q,Q',\emptyset\}$;
(iii) for each $k\in\mathbb{Z}$, the family $\mathcal{D}^\beta_k := \{Q\in\mathcal{D}^\beta : \ell Q = 2^k\}$ forms a partition of $\mathbb{R}^d$.

We say a collection of dyadic cubes $\mathcal{S}^\beta = \{Q_{j,k}\}_{j,k\in\mathbb{Z}} \subset \mathcal{D}^\beta$ is a sparse family if

(i) for each fixed k, the family $\{Q_{j,k}\}_{j\in\mathbb{Z}}$ is pairwise disjoint;
(ii) if $A_k = \cup_{j\in\mathbb{Z}} Q_{j,k}$, then $A_{k+1}\subset A_k$;
(iii) $|A_{k+1}\cap Q_{j.k}| \le \frac{1}{2}|Q_{j,k}|$.

In particular, given a sparse family $\mathcal{S}^\beta = \{Q_{j,k}\}_{j,k\in\mathbb{Z}} \subset \mathcal{D}^\beta$, if we define for $Q_{j,k}\in\mathcal{S}^\beta$, the set $E_{Q_{j,k}} := Q_{j,k}\setminus A_{k+1}$, then we find that the family $\{E_Q\}_{Q\in\mathcal{S}^\beta}$ is pairwise disjoint.

We refer to [6] for the following

Lemma 2.2. *There are 2^d dyadic grids $\mathcal{D}^\beta$ such that for any cube $Q\in\mathbb{R}^d$, there exists a cube $R\in\mathcal{D}^\beta$ for some β such that $Q\subset R$ and $\ell R\le 6\ell Q$.*

2.3. **Extended Carleson embedding lemma.** Recall that for σ a weight, the weighted (dyadic) Hardy–Littlewood maximal function is defined by

$$\mathcal{M}_\sigma^{\mathcal{D}^\beta} f(x) := \sup_{Q\in\mathcal{D}^\beta} \frac{1_Q(x)}{\sigma(Q)} \int_Q |f(s)|\sigma(s)ds.$$

We have the following easy fact.

Theorem 2.3. *Let Φ be a convex growth function in $\mathcal{C}'$ satisfying the ∇_2-condition, and let σ be a weight in $\mathbb{R}^d$. Then there exists a constant $C = C_\Phi > 0$ such that for any $f \in L_\sigma^\Phi(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}_\sigma^{\mathcal{D}^\beta} f(x)\right)\sigma(x)dx \le C \int_{\mathbb{R}^d} \Phi\left(|f(x)|\right)\sigma(x)dx. \tag{12}$$

The following Carleson embeddding result can be proved as in the power functions case (see [1,6]).

Theorem 2.4. *Let Φ be a growth function in $\mathcal{C}'$ satisfying the ∇_2-condition. Let σ be a weight on $\mathbb{R}^d$ and let $\{\lambda_Q\}_{Q\in\mathcal{D}^\beta}$ be a sequence of positive numbers indexed over the set of dyadic cubes $\mathcal{D}^\beta$ in $\mathbb{R}^d$. Then the following assertions are equivalent.*

(a) *$\{\lambda_Q\}_{Q\in\mathcal{D}^\beta}$ is a σ-Carleson sequence, i.e., there is a constant $A > 0$ such that*

$$\sum_{Q\subseteq R, Q\in\mathcal{D}^\beta} \lambda_Q \le A\sigma(R).$$

(b) *There exists a constant $C > 0$ such that for any function f,*

$$\sum_{Q\in\mathcal{D}^\beta} \lambda_Q \Phi\left(\frac{1}{\sigma(Q)}\int_Q |f(x)|\sigma(x)dx\right) \le CA \int_{\mathbb{R}^d} \Phi(|f(x)|)\sigma(x)dx. \tag{13}$$

3. Proof of Theorem 1.6

First, the fact that the condition (8) implies the A_Φ condition for the pair (ω, σ) is obvious. From Theorem 1.5, we have that (8) implies (9). Hence the heart of the matter is to prove the existence of a sufficiently large (doubling) constant D such that for any pair of weights (ω, σ) and for any $f \in L^\Phi(\sigma)$,

$$\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}(\sigma f)(x)\right)\omega(x)dx \lesssim \left([\omega,\sigma]_\Phi + \mathcal{P}_{\rho,D}\right) \int_{\mathbb{R}^d} \Phi(|f(x)|)\sigma(x)dx. \tag{14}$$

As in the power function case, we restrict our proof to $1 < \rho \le 2$ and use dyadic grids introduced above as for $r = 3, 4, \dots$, and choices of $r - 1 < \rho \le r$, the proof proceeds by replacing dyadic grids by r–ary grids.

Indeed, following Lemma 2.2, it is enough to prove (14) with $\mathcal{M}^{\mathcal{D}^\beta}$ in place of $\mathcal{M}$. We only consider the standard grid $\mathcal{D}$ as the proof doesn't depend on the choice of the dyadic grid.

We set $D = 2^{d\frac{a+1}{a-1}}$, where a is the lower index of Φ. We note that if $\mathcal{S} = \{Q_{j,k}\}_{j,k\in\mathbb{Z}} \subset \mathcal{D}$ is the set of all maximal dyadic cubes $Q_{j,k} \in \mathcal{D}$ (with respect to the inclusion) such that

$$\frac{1}{|Q_{j,k}|}\int_{Q_{j,k}} |f(y)|\sigma(y)dy > 2^k,$$

then $\mathcal{S}$ is a sparse family. Moreover,

$$A_k = \bigcup_{j\in\mathbb{Z}} Q_{j,k} = \{x \in \mathbb{R}^d : \mathcal{M}^{\mathcal{D}} f(x) > 2^k\}.$$

As Φ satisfies the Δ'-condition, we obtain

$$\int_{\mathbb{R}^d} \Phi\left(\mathcal{M}^{\mathcal{D}}(\sigma f)(x)\right)\omega(x)dx \lesssim \sum_{Q\in\mathcal{D}} \lambda_Q \Phi\left(\frac{1}{\sigma(Q)}\int_Q |f(x)|\sigma(x)dx\right),$$

where

$$\lambda_Q = \begin{cases} \Phi\left(\frac{\sigma(Q)}{|Q|}\right)\omega(E_Q) & \text{if} \quad Q \in \mathcal{S} \\ 0 & \text{if} \quad Q \in \mathcal{D} \setminus \mathcal{S}. \end{cases}$$

By Theorem 2.4, (14) follows provided the sequence $\{\lambda_Q\}_{Q\in\mathcal{D}}$ satisfies

$$\sum_{Q\subset R,\, Q\in\mathcal{D}} \lambda_Q = \sum_{Q\in\mathcal{S}_R} \lambda_Q \lesssim ([\omega,\sigma]_\Phi + \mathcal{P}_{\rho,D})\,\sigma(R), \ \ \forall R\in\mathcal{D}, \tag{15}$$

where $\mathcal{S}_R := \{Q \in \mathcal{D} : Q \subset R\}$.

We partition $\mathcal{S}_R$ into the following four subcollections.

- (The Testing Collection). Let $\mathcal{T}$ be the subcollection of cubes in $\mathcal{S}_R$ such that the testing inequality (6) is satisfied.
- (The Top Cubes). Let $\mathcal{U} := \{Q \in \mathcal{S}_R \setminus \mathcal{T} :\, 2^k \ell Q \geq \ell R\}$, where k is chosen large enough so that $2^{dk}k^{-a} > 1$. One can observe that this collection has at most $2^{1+d(k+1)}$ cubes.
- (The Small A_Φ Cubes). Let $\mathcal{A}$ be the set of cubes in $Q \in \mathcal{S}_R \setminus (\mathcal{T}\cup\mathcal{U})$ such that

$$\langle\omega\rangle_Q \phi(\langle\sigma\rangle_Q) \leq \frac{[\omega,\sigma]_\Phi}{(\log_2 \ell R/\ell Q)^a}. \tag{16}$$

- (The Remaining Cubes). Let $\mathcal{L} := \mathcal{S}_R \setminus (\mathcal{T}\cup\mathcal{U}\cup\mathcal{A})$.

We now show that the estimate (15) holds when the sum is restricted to each of the above subcollections.

Starting with the Testing Collection, we easily obtain

$$\begin{aligned}\sum_{Q\in\mathcal{T}} \omega(E_Q)\Phi\left(\frac{\sigma(Q)}{|Q|}\right) &\leq \sum_{Q\in\mathcal{T}} \int_{E_Q} \Phi\left(\mathcal{M}^{\mathcal{D}}(\sigma 1_Q)\right)\omega(x)dx \\ &\leq \int_R \Phi\left(\mathcal{M}^{\mathcal{D}}(\sigma 1_Q)\right)\omega(x)dx \\ &\leq \mathcal{P}_{\rho,D}\sigma(R).\end{aligned}$$

Recalling that the Top Collection $\mathcal{U}$ has at most $2^{1+d(k+1)}$ cubes, that $\phi(t) = \frac{\Phi(t)}{t}$, and using our definition of A_Φ, we obtain

$$\begin{aligned}\sum_{Q\in\mathcal{U}} \omega(E_Q)\Phi\left(\frac{\sigma(Q)}{|Q|}\right) &\leq \sum_{Q\in\mathcal{U}} \sigma(Q)\frac{\omega(Q)}{|Q|}\phi\left(\frac{\sigma(Q)}{|Q|}\right) \\ &\lesssim_k [\omega,\sigma]_\Phi \sigma(R).\end{aligned}$$

Here, the notation $\lesssim_k$ means that the implied constant depends on the integer k.

The Small A_Φ Cubes are handled by using the condition (16) defining them as follows:

$$\begin{aligned}\sum_{Q\in\mathcal{A}} \omega(E_Q)\Phi\left(\frac{\sigma(Q)}{|Q|}\right) &\leq \sum_{Q\in\mathcal{A}} \sigma(Q)\frac{\omega(Q)}{|Q|}\phi\left(\frac{\sigma(Q)}{|Q|}\right) \\ &\lesssim [\omega,\sigma]_\Phi \sum_{Q\in\mathcal{A}} \frac{\sigma(Q)}{(\log_2 \ell R/\ell Q)^a} \\ &= [\omega,\sigma]_\Phi \sum_{s>k} \sum_{Q\in\mathcal{A},\, \ell R = 2^s \ell Q} \frac{\sigma(Q)}{(\log_2 \ell R/\ell Q)^a} \\ &= [\omega,\sigma]_\Phi \sum_{s>k} \frac{1}{s^a} \sum_{Q\in\mathcal{A},\, \ell R=2^s\ell Q} \sigma(Q) \\ &\lesssim [\omega,\sigma]_\Phi \sigma(R).\end{aligned}$$

It now remains to deal with the last subcollection. We will prove that $\mathcal{L}$ is also empty in our case. Indeed, suppose that $\mathcal{L} \neq \emptyset$. Then there is a cube $Q \in \mathcal{S}_R$ such that $2^k\ell Q < \ell R$ and (16) fails and no ancestor of Q contained in R has a doubling parent in the sense of Definition 1.2.

Denote by $Q^{(1)}$ the D-parent of Q and let $Q^{(j+1)} = \left(Q^{(j)}\right)^{(1)}$. Let k_0 be the integer such that $R = Q^{(k_0)}$. Observe that for any $1 \leq j < k_0$, $\sigma(Q^{(j+1)}) > D\sigma(Q^{(j)})$. Hence $\sigma(R) \geq D^{k_0}\sigma(Q)$.

We recall that the function $\frac{\Phi(t)}{t^a} = \frac{\phi(t)}{t^{a-1}}$ is increasing. From this and the above observations, we obtain

$$\begin{aligned}
[\omega,\sigma]_\Phi &\geq \langle\omega\rangle_R \phi\left(\frac{\langle\sigma\rangle_R}{|R|}\right) \\
&\geq \frac{\omega(Q)}{|Q^{(k_0)}|}\phi\left(\frac{D^{k_0}\sigma(Q)}{|Q^{(k_0)}|}\right) \\
&= \frac{\omega(Q)}{2^{dk_0}|Q|}\phi\left(\left(\frac{D}{2^d}\right)^{k_0}\frac{\sigma(Q)}{|Q|}\right) \\
&\geq \frac{\omega(Q)}{2^{dk_0}|Q|}\left(\frac{D}{2^d}\right)^{k_0(a-1)}\phi\left(\frac{\sigma(Q)}{|Q|}\right) \\
&\geq 2^{-dk_0}\left(\frac{D}{2^d}\right)^{k_0(a-1)}\frac{[\omega,\sigma]_\Phi}{(\log_2 \ell R/\ell Q)^a} \\
&= [\omega,\sigma]_\Phi 2^{-dk_0}\left(\frac{D}{2^d}\right)^{k_0(a-1)} k_0^{-a} \\
&= 2^{dk_0}[\omega,\sigma]_\Phi k_0^{-a}.
\end{aligned}$$

The last line follows from our choice of D. We easily deduce that $k_0 < k$, which implies that the cube Q belongs to $\mathcal{U}$. This is a contraction. Thus $\mathcal{L} = \emptyset$ and the proof is complete.

Remark 3.1. Following the equivalence (10), one could have chosen to prove directly that under the conditions in Theorem 1.6,

$$[\omega,\sigma]_{S_\Phi} \approx [\omega,\sigma]_\Phi + \mathcal{P}_{\rho,D}.$$

This can be done combining the ideas in this text with those in [3]. Our choice of the method in this text is motivated only by the fact that as the proof of Theorem 1.5 is left to the reader, we wanted the reader to have an idea of how the extended Carleson embedding result can be used.

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