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BASIS AND DIMENSION OF EXPONENTIAL VECTOR SPACE

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Abstract. Exponential vector space [shortly *evs*] is an algebraic order extension of a vector space in the sense that every evs contains a vector space and, conversely, every vector space can be embedded into such a structure. This evs structure consists of a semigroup structure, a scalar multiplication and a partial order. In this paper, we have developed the concepts of a basis and dimension of an evs by introducing the ideas of an orderly independent set and generating set with the help of partial order and algebraic operations. We have found that like a vector space, an evs does not contain basis always. We have established a necessary and sufficient condition for an evs to have a basis. It was shown that the equality of dimension is an evs property, but the converse is not true. We have studied the dimension of a subevs and found that every evs contains a subevs with all possible lower dimensions. Lastly, we have computed the basis and dimension of some evs which help us to explore the theory of basis by creating counter-examples in different aspects.

1. Introduction

Exponential vector space is an algebraic ordered extension of a vector space. The word 'extension' is used because of the fact that every exponential vector space contains a vector space and, conversely, every vector space can be embedded into such a structure. This structure comprises a semigroup structure, a scalar multiplication and a compatible partial order. We now start with the definition of evs.

Definition 1.1 ([7])**.** Let $(X, \leq)$ be a partially ordered set, '+' be a binary operation on X [called *addition*] and '$\cdot$': $K \times X \longrightarrow X$ be another composition [called *scalar multiplication*, K being a field]. If the operations and the partial order satisfy the axioms below, then $(X, +, \cdot, \leq)$ is called an *exponential vector space* (in short *evs*) over K [This structure was initiated under the name *quasi-vector space* or *qvs* by S. Ganguly et al. in [1]].

$A_1 : (X, +)$ is a commutative semigroup with identity θ

$.A_2 : x \leq y\ (x, y \in X) \Rightarrow x + z \leq y + z$ and $\alpha \cdot x \leq \alpha \cdot y,\ \forall z \in X, \forall \alpha \in K$

$.A_3$: (i) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$;

(ii) $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$;

(iii) $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$;

(iv) $1 \cdot x = x$, where '1' is the multiplicative identity in K,

$\forall\, x, y \in X,\ \forall\, \alpha, \beta \in K$;

$A_4 : \alpha \cdot x = \theta$ iff $\alpha = 0$ or $x = \theta$;

$A_5 : x + (-1) \cdot x = \theta$ iff $x \in X_0 := \big\{z \in X :\ y \not\leq z,\ \forall\, y \in X \smallsetminus \{z\}\big\}$;

A_6 : For each $x \in X$, $\exists\, p \in X_0$ such that $p \leq x$.

In the above definition, axiom A_3 (iii) indicates a rapid growth of the elements of X due to the fact that $x + x \geq 2x$ and axiom A_6 gives some positive sense of each element. These two facts express the exponential behaviour of the elements of an evs.

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In axiom A_5, we can notice that X_0 is precisely the set of all minimal elements of the evs X with respect to the partial order on X and forms the maximal vector space (within X) over the same field as that of X ([1]). We call this vector space X_0 as the '*primitive space*' or '*zero space*' of X and the elements of X_0 as '*primitive elements*'.

Also, given any vector space V over some field K, an evs X can be constructed (as is shown below) such that V is isomorphic to X_0. In this sense, an "exponential vector space" can be considered as an algebraic ordered extension of a vector space.

Example 1.2 ([7])**.** Let $X := \{(r,a) \in \mathbb{R} \times V : r \geq 0, a \in V\}$, where V is a vector space over some field K. Define operations and partial order on X as follows: for $(r,a),(s,b) \in X$ and $\alpha \in K$,
(i) $(r,a)+(s,b) := (r+s, a+b)$;
(ii) $\alpha(r,a) := (r, \alpha a)$, if $\alpha \neq 0$ and $0(r,a) := (0,\theta)$, θ being the identity in V;
(iii) $(r,a) \leq (s,b)$, iff $r \leq s$ and $a = b$.
Then X becomes an exponential vector space over K with the primitive space $\{0\} \times V$ which is evidently isomorphic to V.

Initially, the idea of this structure was given by S. Ganguly et al. under the name "quasi-vector space" in [1] and the following example of the hyperspace was the main motivation behind this new structure.

Example 1.3 ([1])**.** Let $\mathscr{C}(\mathcal{X})$ be the topological hyperspace consisting of all non-empty compact subsets of a Hausdörff topological vector space $\mathcal{X}$ over the field $\mathbb{K}$ of real or complex numbers. Then $\mathscr{C}(\mathcal{X})$ becomes an evs with respect to the operations and partial order defined as follows. For $A, B \in \mathscr{C}(\mathcal{X})$ and $\alpha \in \mathbb{K}$,
(i) $A + B := \{a + b : a \in A, b \in B\}$;
(ii) $\alpha A := \{\alpha a : a \in A\}$;
(iii) the usual set-inclusion as the partial order.

We now topologise an exponential vector space. For this we need the following concept.

Definition 1.4 ([5])**.** Let '$\leq$' be a preorder in a topological space Z; the preorder is said to be *closed* if its graph $G_{\leq}(Z) := \{(x,y) \in Z \times Z : x \leq y\}$ is closed in $Z \times Z$ (endowed with the product topology).

Theorem 1.5 ([5])**.** *A partial order '$\leq$' in a topological space Z will be a closed order iff for any $x, y \in Z$ with $x \nleq y$, $\exists$ open neighbourhoods U, V of x, y respectively in Z such that $(\uparrow U) \cap (\downarrow V) = \emptyset$, where $\uparrow U := \{z \in Z : z \geq u$ for some $u \in U\}$ and $\downarrow V := \{z \in Z : z \leq v$ for some $v \in V\}$.*

Definition 1.6 ([7])**.** An exponential vector space X over the field $\mathbb{K}$ of real or complex numbers is said to be a *topological exponential vector space* if there exists a topology on X with respect to which the addition and the scalar multiplication are continuous and the partial order '$\leq$' is closed (here, $\mathbb{K}$ is equipped with the usual topology).

Remark 1.7. If X is a topological exponential vector space, then its primitive space X_0 becomes a topological vector space, since the restriction of a continuous function is continuous. Moreover, the closedness of the partial order '$\leq$' in a topological exponential vector space X readily implies (in view of Theorem 1.5) that X is Hausdörff and hence X_0 becomes a Hausdörff topological vector space.

Example 1.8 ([2])**.** Let $X := [0,\infty) \times V$, where V is a vector space over the field $\mathbb{K}$ of real or complex numbers. Define operations and partial order on X as follows: for $(r,a),(s,b) \in X$ and $\alpha \in \mathbb{K}$,
(i) $(r,a)+(s,b) := (r+s, a+b)$;
(ii) $\alpha(r,a) := (|\alpha|r, \alpha a)$;
(iii) $(r,a) \leq (s,b)$ iff $r \leq s$ and $a = b$.
Then $[0,\infty) \times V$ becomes an exponential vector space with the primitive space $\{0\} \times V$ which is clearly isomorphic to V.

In this example, if we consider V as a Hausdörff topological vector space, then $[0,\infty) \times V$ becomes a topological exponential vector space with respect to the product topology, where $[0,\infty)$ is equipped with the subspace topology inherited from the real line $\mathbb{R}$.

If instead of V we take the trivial vector space $\{\theta\}$ in this example, then the resulting topological evs is $[0,\infty) \times \{\theta\}$ which can be clearly identified with the half-ray $[0,\infty)$ of the real line.

In this paper, we have developed the concept of the basis and dimension of an evs. We know that the basis of a vector space is a minimal part of it which generates the entire space. But in an evs it is impossible to express every element as a linear combination of some particular elements due to the exponential behaviour of its elements. In this paper, with the help of partial order we have developed the ideas of generating sets, orderly independent sets which allow us to define the basis. It has been shown that the basis of an evs is identified by a minimal generating set, whereas a maximal orderly independent set fails to form a basis [shown by a counter example], though every basis is a maximal orderly independent set. The main difference between a vector space and an evs in this respect is that an evs may not have a basis always (like a vector space). But for a topological evs, we have shown that if it has a basis, then it contains uncountably many bases. We have found out a property of every element of a basis which helped us to give a necessary and sufficient condition for an evs to have a basis. After that we have introduced the concept of dimension of an evs and shown that equality of dimension is an evs property, though two non-order-isomorphic evs may have the same dimension.

Lastly, we have studied the dimension of subevs and shown that every evs contains subevs(s) with all possible lower dimensions. In the last section of this paper, computations of the basis and dimension of some evs are given.

2. Prerequisites

In this section, we have discussed some definitions, results and examples of an exponential vector space which are very important in developing the main context. We now start with the definition of a subevs.

Definition 2.1 ([4])**.** A subset Y of an exponential vector space X is said to be a *sub- exponential vector space* (*subevs* in short) if Y itself is an exponential vector space with respect to the compositions of X being restricted to Y.

Note 2.2 ([4])**.** A subset Y of an exponential vector space X over a field K is a sub exponential vector space, iff Y satisfies the following:
(i) $\alpha x + y \in Y,\ \forall\, \alpha \in K, \forall\, x, y \in Y$;
(ii) $Y_0 \subseteq X_0 \bigcap Y$, where $Y_0 := \big\{z \in Y : y \nleq z, \forall\, y \in Y \smallsetminus \{z\}\big\}$;
(iii) for any $y \in Y$, $\exists\, p \in Y_0$ such that $p \leq y$.

If Y is a subevs of X, then actually $Y_0 = X_0 \cap Y$, since for any $Y \subseteq X$, we have $X_0 \cap Y \subseteq Y_0$. $[0, \infty) \times \{\theta\}$ is, clearly, a subevs of the evs $[0, \infty) \times V$.

We have used the following result to form a non-topological exponential vector space.

Result 2.3 ([3])**.** *In a topological evs X, if $a = a + x$ for some $a, x \in X$, then $x = \theta$.*

To talk about an evs property of this space we have to know the idea of an order-morphism.

Definition 2.4 ([2])**.** A mapping $f : X \longrightarrow Y$ (X, Y being two exponential vector spaces over the field K) is called an *order-morphism* if
(i) $f(x + y) = f(x) + f(y), \forall\, x, y \in X$;
(ii) $f(\alpha x) = \alpha f(x), \forall\, \alpha \in K, \forall\, x \in X$;
(iii) $x \leq y\ (x, y \in X) \Rightarrow f(x) \leq f(y)$;
(iv) $p \leq q\ \big(p, q \in f(X)\big) \Rightarrow f^{-1}(p) \subseteq\downarrow f^{-1}(q)$ and $f^{-1}(q) \subseteq\uparrow f^{-1}(p)$.

A bijective (injective, surjective) order-morphism is called an *order-isomorphism* (*order-monomorphism*, *order-epimorphism*, respectively).

If X, Y are two topological evs over $\mathbb{K}$, then an order-isomorphism $f : X \longrightarrow Y$ is said to be a *topological order-isomorphism* if f is a homeomorphism.

Definition 2.5. A property of an evs is called an *evs property* if it remains invariant under an order-isomorphism.

The concept of order-isomorphism is competent enough to extract the structural beauty of an evs by judging the invariance of its various properties. Since the composition of two order-isomorphisms, the inverse of an order-isomorphism and the identity map are again order-isomorphisms, the concept thereby produces a partition on the collection of all evs over some common field; this helps one to distinguish two evs belonging to two different classes under this partition.

Definition 2.6 ([4])**.** In an evs X, the *primitive* of $x \in X$ is defined as the set
$$P_x := \{p \in X_\circ : p \leq x\}.$$
Axiom A_6 in Definition 1.1 ensures that the primitive of each element of an evs is nonempty.

Definition 2.7 ([4])**.** An evs X is said to be a *single primitive evs* if P_x is a singleton set for each $x \in X$. Also, in a single primitive evs X, $P_{x+y} = P_x + P_y$ and $P_{\alpha x} = \alpha P_x$, $\forall\, x, y \in X$ and for all scalar α.

Single primitivity is an evs property [4].

Definition 2.8 ([4])**.** An evs X is said to be a *comparable evs* if $\forall x, y \in X$, $P_x = P_y \Rightarrow x$ and y are comparable with respect to the partial order of X.

This is also an evs property [4].

We now give some examples of an exponential vector space to build up some counter-examples of the main section.

Example 2.9 ([3])**. (Arbitrary product of exponential vector spaces)** Let $\{X_i : i \in \Lambda\}$ be an arbitrary family of exponential vector spaces over a common field K and $X := \prod\limits_{i\in\Lambda} X_i$ be the Cartesian product. Then X becomes an exponential vector space over K with respect to the following operations and partial order:

For $x = (x_i)_i, y = (y_i)_i \in X$ and $\alpha \in K$ we define (i) $x + y := (x_i + y_i)_i$, (ii) $\alpha x := (\alpha x_i)_i$, (iii) $x \ll y$ if $x_i \leq y_i$, $\forall\, i \in \Lambda$.
Here the notation $x = (x_i)_i \in X$ means that the point $x \in X$ is the map $x : i \mapsto x_i$ $(i \in \Lambda)$, where $x_i \in X_i$, $\forall\, i \in \Lambda$. The additive identity of X is given by $\theta = (\theta_i)_i$, θ_i being the additive identity in X_i. Also, the primitive space of X is given by $X_0 = \prod\limits_{i\in\Lambda}[X_i]_0$.

This product space X becomes a topological exponential vector space over the field $\mathbb{K}$ whenever each factor space X_i is a topological evs over $\mathbb{K}$ and X is endowed with the product topology, which is the weakest topology on X so that each projection map $p_i : X \longrightarrow X_i$ given by $p_i : x \longmapsto x_i$ is continuous.

Thus for any cardinal number β, $[0, \infty)^\beta$ becomes a topological evs.

Example 2.10. Let X be an evs over the field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$) and V be a vector space over the same field $\mathbb{K}$. We now give operations on $X \times V$ like $[0, \infty) \times V$, i.e.,
for $(x_1, e_1), (x_2, e_2), (x, e) \in X \times V$ and $\alpha \in \mathbb{K}$:
(i) $(x_1, e_1) + (x_2, e_2) := (x_1 + x_2, e_1 + e_2)$.
(ii) $\alpha(x, e) := (\alpha x, \alpha e)$.
The partial order '$\leq$' is defined as: $(x_1, e_1) \leq (x_2, e_2)$, iff $x_1 \leq x_2$ and $e_1 = e_2$. Then $X \times V$ becomes an evs over the field $\mathbb{K}$. Justification of this is straightforward.

Example 2.11. Let $\mathscr{C}_\theta(\mathcal{X})$ be the collection of all compact subsets of a Hausdörff topological vector space $\mathcal{X}$ containing θ (the identity in $\mathcal{X}$). So, $\mathscr{C}_\theta(\mathcal{X}) \subseteq \mathscr{C}(\mathcal{X})$. If we take any two members $A, B \in \mathscr{C}_\theta(\mathcal{X})$ and any $\alpha \in \mathbb{K}$, then $\alpha A + B \in \mathscr{C}_\theta(\mathcal{X})$. Again, $[\mathscr{C}_\theta(\mathcal{X})]_0 = \big\{\{\theta\}\big\} = [\mathscr{C}(\mathcal{X})]_0 \cap \mathscr{C}_\theta(\mathcal{X})$. For any $A \in \mathscr{C}_\theta(\mathcal{X})$, $\{\theta\} \subseteq A$ which shows that $\mathscr{C}_\theta(\mathcal{X})$ is a subevs of $\mathscr{C}(\mathcal{X})$ [by note 2.2].

Example 2.12 ([4])**.** Let $\mathcal{X}$ be a vector space over the field $\mathbb{K}$ of real or complex numbers. Let $\mathscr{L}(\mathcal{X})$ be the set of all linear subspaces of $\mathcal{X}$. We now define $+, \cdot, \leq$ on $\mathscr{L}(\mathcal{X})$ as follows: For $\mathcal{X}_1, \mathcal{X}_2 \in \mathscr{L}(\mathcal{X})$ and $\alpha \in \mathbb{K}$ we define
(i) $\mathcal{X}_1 + \mathcal{X}_2 := \operatorname{span}(\mathcal{X}_1 \cup \mathcal{X}_2)$, (ii) $\alpha \cdot \mathcal{X}_1 := \mathcal{X}_1$, if $\alpha \neq 0$ and $\alpha \cdot \mathcal{X}_1 := \{\theta\}$, if $\alpha = 0$ (θ being the additive identity of $\mathcal{X}$), (iii) $\mathcal{X}_1 \leq \mathcal{X}_2$ iff $\mathcal{X}_1 \subseteq \mathcal{X}_2$.

Then $\big(\mathscr{L}(\mathcal{X}), +, \cdot, \leq\big)$ is an exponential vector space over $\mathbb{K}$.

Since every element of $\mathscr{L}(\mathcal{X})$ is an idempotent $\big[\because \mathcal{X}_1 + \mathcal{X}_1 = \mathcal{X}_1$, for all $\mathcal{X}_1 \in \mathscr{L}(\mathcal{X})\big]$, we can say that there is no topology with respect to which $\mathscr{L}(\mathcal{X})$ can be a topological evs $\big[$since a topological evs cannot contain any idempotent element, as follows from the Result 2.3$\big]$. $\square$

Example 2.13 ([6])**.** Let us consider $\mathscr{D}^2([0,\infty)) := [0,\infty) \times [0,\infty)$. We define $+, \cdot, \leq$ on $\mathscr{D}^2([0,\infty))$ as follows:
For $(x_1,y_1),(x_2,y_2) \in \mathscr{D}^2([0,\infty))$ and $\alpha \in \mathbb{C}$, we define
(i) $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$;
(ii) $\alpha \cdot (x_1,y_1) = (|\alpha|x_1, |\alpha|y_1)$;
(iii) $(x_1,y_1) \leq (x_2,y_2) \Longleftrightarrow$ either $x_1 < x_2$ or if $x_1 = x_2$, then $y_1 \leq y_2$ [*dictionary order*].
Then $\big(\mathscr{D}^2([0,\infty)), +, \cdot, \leq\ \big)$ becomes a non-topological exponential vector space over the complex field $\mathbb{C}$.

Note 2.14 ([6])**.** For a ***well-ordered*** set I and an evs X, if we consider $\mathscr{D}(X:I) := X^I$, then also, like the above example, it forms a non-topological evs with dictionary order. If $I=\{1,2,\ldots,n\}$, we usually denote the evs $\mathscr{D}(X:I)$ as $\mathscr{D}^n(X)$. We can also generalise this by taking different evs i.e., $\mathscr{D}(X_\alpha : \alpha \in I) := \prod\limits_{\alpha\in I} X_\alpha$, which also becomes a non-topological evs with dictionary order.

3. Basis and Dimension: General Discussion

In this section, we have introduced the concepts of a basis and dimension of an exponential vector space. These concepts are different from those already given in a vector space. Like a vector space, it is not true that every evs contains a basis, rather it behaves like a module in this respect. We have found a necessary as well as sufficient condition for an evs to have a basis. Finally, we have computed a basis and dimension of some particular evs.

Definition 3.1. Let X be an evs over the field K and $x \in X \smallsetminus X_0$. Define

$$L(x) := \{z \in X : z \geq \alpha x + p, \alpha \in K^*, p \in X_0\}, \quad \text{where } K^* \equiv K \smallsetminus \{0\}.$$

We name these sets $L(x)$ for different x's in $X \smallsetminus X_0$ as *testing sets* of X.

We discuss below some properties of $L(x)$. First of all note that $L(x) = \uparrow(K^*x + X_0)$.

Proposition 3.2. (i) $\forall\, x \in X \smallsetminus X_0$*,* $x \in L(x)$ *and* $\uparrow L(x) = L(x)$*.*
(ii) $x \leq y$ $(x,y \in X \smallsetminus X_0) \Rightarrow L(x) \supseteq L(y)$*.*
(iii) *If* $x = \alpha y + p$ *for some* $\alpha \in K^*$*,* $p \in X_0$ *and* $y \in X \smallsetminus X_0$*, then* $L(x) = L(y)$*.*
(iv) $L(x) \cap X_0 = \emptyset$*.*
(v) *If* $a \in L(b)$*, then* $L(a) \subseteq L(b)$*.*
(vi) *For any* $x, y \in X \smallsetminus X_0$*,* $L(x) \cap L(y) \neq \emptyset$*.*

Proof. **(i)** Immediate from definition.

(ii) Let $z \in L(y) \Rightarrow \exists\, \alpha \in K^*$ and $p \in X_0$ such that $\alpha y + p \leq z$. Now, $x \leq y \Rightarrow \alpha x + p \leq \alpha y + p \leq z \Rightarrow z \in L(x)$.

(iii) As $y \in X \smallsetminus X_0$, so $x \in X \smallsetminus X_0$. Therefore we can talk about $L(x)$. Let $z \in L(y) \Rightarrow \exists\, \alpha_z \in K^*$ and $p_z \in X_0$ such that $\alpha_z y + p_z \leq z \Rightarrow \alpha_z\alpha^{-1}(x-p) + p_z \leq z \Rightarrow \alpha_z\alpha^{-1}x + (p_z - \alpha_z\alpha^{-1}p) \leq z \Rightarrow z \in L(x)$. Therefore $L(y) \subseteq L(x)$. Again, $x = \alpha y + p \Rightarrow y = \alpha^{-1}(x-p)$. So, by the above argument, we also have $L(x) \subseteq L(y)$. Thus $L(x) = L(y)$.

(iv) Let $y \in L(x) \Rightarrow \exists\, \alpha \in K^*$ and $p \in X_0$ such that $\alpha x + p \leq y$. If $y \in X_0$, then $\alpha x + p \in X_0 \Rightarrow x \in X_0$. This contradiction proves that $L(x) \cap X_0 = \emptyset$.

(v) $a \in L(b) \Rightarrow a \in X \smallsetminus X_0$ and $\exists\, \alpha \in K^*, p \in X_0$ such that $\alpha b + p \leq a \Rightarrow L(a) \subseteq L(\alpha b + p) = L(b)$ [by (ii) and (iii) above].

(vi) For any $p \in X_0$ with $p \leq y$, $x + p \leq x + y \Rightarrow x + y \in L(x)$. Similarly, we can say that $x + y \in L(y)$. So, $x + y \in L(x) \cap L(y) \Rightarrow L(x) \cap L(y) \neq \emptyset$. □

Definition 3.3. A subset B of $X \smallsetminus X_0$ is said to be a *generator* of $X \smallsetminus X_0$ if

$$X \smallsetminus X_0 = \bigcup_{x \in B} L(x).$$

Note 3.4. The set $X \smallsetminus X_0$ always generates $X \smallsetminus X_0$. So, a generator always exists for $X \smallsetminus X_0$. It is clear that any superset of a generator of $X \smallsetminus X_0$ is also a generator of $X \smallsetminus X_0$.

Definition 3.5. Two elements $x, y \in X \smallsetminus X_0$ are said to be *orderly dependent* if either $x \in L(y)$ or $y \in L(x)$.

Definition 3.6. Two elements $x, y \in X \smallsetminus X_0$ are said to be *orderly independent* if they are not orderly dependent, i.e., neither $x \in L(y)$, nor $y \in L(x)$.

A subset B of $X \smallsetminus X_0$ is said to be *orderly independent* if any two members $x, y \in B$ are orderly independent.

Remark 3.7. Let Y be a subevs of an evs X. Then any two orderly dependent elements of $Y \smallsetminus Y_0$ are also orderly dependent in $X \smallsetminus X_0$ because of the fact that $Y_0 \subseteq X_0$. In other words, any two elements of $Y \smallsetminus Y_0$ which are orderly independent in $X \smallsetminus X_0$ are also orderly independent in $Y \smallsetminus Y_0$. But the converse is not true in general, i.e., two orderly independent elements in $Y \smallsetminus Y_0$ may not be orderly independent in $X \smallsetminus X_0$ [in contrast to the case of linear independence in vector space]. For example, $\{0, 2, 5\}$ and $\{0, -2, 3\}$ are orderly independent in $\mathscr{C}_\theta(\mathbb{R})$ [see Example 2.11], since $\nexists$ any $\alpha \in \mathbb{R}^*$ such that $\alpha\{0, 2, 5\} \subseteq \{0, -2, 3\}$ or $\alpha\{0, -2, 3\} \subseteq \{0, 2, 5\}$. $\big[$Here, $[\mathscr{C}_\theta(\mathbb{R})]_0 = \big\{\{0\}\big\}\ \big]$. But these two elements are not orderly independent in $\mathscr{C}(\mathbb{R})$, as we can write $\{0, -2, 3\} = \{0, 2, 5\} + \{-2\}$, where $\{-2\} \in [\mathscr{C}(\mathbb{R})]_0$.

In the above context, it should thus be noted that while discussing the orderly independence of two elements of a subevs Y of an evs X, there are two types of orderly independence — one with respect to Y, and the other with respect to X; while considering orderly independence with respect to Y, the testing sets should be of the form

$$L_Y(y) := \{z \in Y : z \geq \alpha y + p, \alpha \in K^*, p \in Y_0\} \text{ for any } y \in Y \smallsetminus Y_0,$$

and when considering orderly independence with respect to X, the testing sets should be of the form

$$L_X(y) := \{z \in X : z \geq \alpha y + p, \alpha \in K^*, p \in X_0\} \text{ for any } y \in Y \smallsetminus Y_0.$$

Since $Y_0 \subseteq X_0$, we have $L_Y(y) \subseteq L_X(y)$, for any $y \in Y \smallsetminus Y_0$. Thus it follows that an orderly independent set in $Y \smallsetminus Y_0$ need not be orderly independent in $X \smallsetminus X_0$. However, a set $B\,(\subseteq Y \smallsetminus Y_0)$ which is orderly independent in $X \smallsetminus X_0$ must be such in $Y \smallsetminus Y_0$.

Definition 3.8. A subset B of $X \smallsetminus X_0$ is said to be a *basis* of $X \smallsetminus X_0$ if B is orderly independent and generates $X \smallsetminus X_0$.

Note 3.9. For each $x \in X$, either $x \in X_0$, or $x \in X \smallsetminus X_0$. If $x \in X_0$, then it can be expressed as a finite linear combination of some basic vectors of some basis of X_0 [as a vector space]. If $x \in X \smallsetminus X_0$, then there exists a member of some basis [if exists] of $X \smallsetminus X_0$ which generates x. So, we can say that to represent an evs X it is necessary to consider a basis of $X \smallsetminus X_0$ together with a basis of X_0 [in the sense of a vector space]. Thus a basis of an evs X should be composed of two components, one for X_0 and the other for $X \smallsetminus X_0$. To express this fact in an easiest way we shall represent a basis of an evs X as $[B : B_0]$, where B is a basis of $X \smallsetminus X_0$ and B_0 is a basis of X_0 [as a vector space]. If for an evs X, $X_0 = \{\theta\}$ then we shall consider $B_0 = \{\theta\}$, since in that case X_0 has no basis.

Theorem 3.10. *For a topological evs X, $X \smallsetminus X_0$ either has no basis or has uncountably many bases.*

Proof. Let B be a basis of $X \smallsetminus X_0$. Then $G_\alpha := \{\alpha x : x \in B\}$ and $H_p := \{x + p : x \in B\}$ are also bases of $X \smallsetminus X_0$, for any $\alpha \in \mathbb{K}^*$ and any $p \in X_0$. This holds because of the result $L(x) = L(\alpha x + p)$ [Proposition 3.2]. If $G_\alpha = G_\beta$ for any $\alpha, \beta \in \mathbb{K}^*$, then $\alpha x = \beta x$, $\forall\, x \in B$ [$\because \alpha x \neq \beta y$ for any $x, y \in B$ as B is orderly independent]. If we choose $\alpha, \beta \in \mathbb{K}^*$ such that $|\alpha| < |\beta|$, then using the continuity of the scalar multiplication of the topological evs X, we must have $x = \theta$ [$\because \alpha x = \beta x \Rightarrow (\alpha\beta^{-1})^n x = x$, $\forall\, n \in \mathbb{N}$ which implies by taking limit $n \to \infty$ that $x = \theta$, as $|\alpha\beta^{-1}| < 1$] is a contradiction. Thus, it follows that for any $\alpha, \beta \in \mathbb{K}^*$ with $|\alpha| < |\beta|$ we must have $G_\alpha \neq G_\beta$. This immediately justifies that there are uncountably many bases of $X \smallsetminus X_0$.

If X_0 contains more than one element, then for $p, q \in X_0$ we may consider H_p, H_q. If $H_p = H_q$, then B, being orderly independent, we must have $x + p = x + q$, $\forall\, x \in B$. Then by Result 2.3, it follows that $p = q$. Since X is a topological evs, X_0 is a Hausdörff topological vector space. So, if $X_0 \neq \{\theta\}$, then it should be uncountable and hence ensure the existence of uncountably many bases of $X \smallsetminus X_0$. $\square$

For a non-topological evs it may happen that $G_\alpha = B$ for every $\alpha \in K^*$ [this will be discussed in the next section]. However, an evs (topological or not) need not have a basis. We show in the next section that the evs $\mathscr{D}\big([0,\infty) : \mathbb{N}\big)$ discussed in Note 2.14 cannot have a basis. The following result shows that having a basis is an evs property.

Result 3.11. *Let $\phi : X \longrightarrow Y$ be an order-isomorphism. Then*
(1) *for any generator B of $X \smallsetminus X_0$, $\phi(B)$ is a generator of $Y \smallsetminus Y_0$;*
(2) *for any orderly independent subset B of $X \smallsetminus X_0$, $\phi(B)$ is also an orderly independent subset of $Y \smallsetminus Y_0$.*

Thus, for a basis B of $X \smallsetminus X_0$, $\phi(B)$ becomes a basis of $Y \smallsetminus Y_0$.

Proof. (1) $B \subseteq X \smallsetminus X_0 \Rightarrow \phi(B) \subseteq Y \smallsetminus Y_0$ [As $\phi(X_0) = Y_0$]. Let $y \in Y \smallsetminus Y_0 \Rightarrow \phi^{-1}(y) \in X \smallsetminus X_0$ $\Rightarrow \exists\, b \in B$ and $\alpha \in K^*$, $p \in X_0$ such that $\phi^{-1}(y) \geq \alpha b + p \Rightarrow y \geq \alpha\phi(b) + \phi(p) \Rightarrow y \in L(\phi(b)) \subseteq \bigcup\limits_{b\in B} L(\phi(b))$. Therefore $Y \smallsetminus Y_0 \subseteq \bigcup\limits_{b\in B} L(\phi(b))$. Again, by Proposition 3.2, $L(\phi(b)) \cap Y_0 = \emptyset$, $\forall\, b \in B$. So $Y \smallsetminus Y_0 = \bigcup\limits_{b\in B} L(\phi(b))$. Thus $\phi(B)$ is a generator of $Y \smallsetminus Y_0$.

(2) We first show that for any two orderly dependent members y_1, y_2 of $Y \smallsetminus Y_0$, $\phi^{-1}(y_1), \phi^{-1}(y_2)$ are orderly dependent in $X \smallsetminus X_0$. As y_1, y_2 are orderly dependent, so without loss of generality, we can take $y_1 \in L(y_2) \Rightarrow \exists\, \alpha \in K^*$ and $p \in Y_0$ such that $\alpha y_2 + p \leq y_1$. Then ϕ^{-1}, also being an order-isomorphism, we have $\phi^{-1}(\alpha y_2 + p) \leq \phi^{-1}(y_1) \Rightarrow \alpha\phi^{-1}(y_2) + \phi^{-1}(p) \leq \phi^{-1}(y_1) \Rightarrow \phi^{-1}(y_1) \in L(\phi^{-1}(y_2))$ [as $\phi^{-1}(p) \in X_0$]. This justifies our assertion. Then contra-positively, the result follows. □

The next theorem characterises a basis (if exists) of $X \smallsetminus X_0$, for any evs X.

Theorem 3.12. *A subset of $X \smallsetminus X_0$ is a basis of $X \smallsetminus X_0$, iff it is a minimal generating subset of $X \smallsetminus X_0$.* [Here, a minimal generating subset B of $X \smallsetminus X_0$ means that there does not exist any proper subset of B which can generate $X \smallsetminus X_0$.]

Proof. Suppose B is a basis of $X \smallsetminus X_0$. Then B generates $X \smallsetminus X_0$. Now B, being an orderly independent subset of $X \smallsetminus X_0$, if we take an element $x \in B$ then $\forall\, y \in B \smallsetminus \{x\}$, x and y are orderly independent. Therefore $x \notin L(y)$, $\forall\, y \in B \smallsetminus \{x\}$. This shows that $B \smallsetminus \{x\}$ cannot generate $X \smallsetminus X_0$ and this holds for any $x \in B$. Therefore B is a minimal generator of $X \smallsetminus X_0$.

Conversely, suppose B is a minimal generator of $X \smallsetminus X_0$. For any two members $x, y \in B$ if $x \in L(y)$ then by Proposition 3.2, $L(x) \subseteq L(y) \Longrightarrow B \smallsetminus \{x\}$ also generates $X \smallsetminus X_0$, which contradicts that B is a minimal generator of $X \smallsetminus X_0$. Again, if $y \in L(x)$, we get a similar contradiction. So, neither $x \in L(y)$, nor $y \in L(x) \Longrightarrow x, y$ are orderly independent. Arbitrariness of x, y shows that B is an orderly independent subset of $X \smallsetminus X_0$. Consequently, B is a basis of $X \smallsetminus X_0$. □

Result 3.13. *Every basis of $X \smallsetminus X_0$ is a maximal orderly independent subset of $X \smallsetminus X_0$.* [Here, a maximal orderly independent subset B of $X \smallsetminus X_0$ means that there does not exist any orderly independent subset of $X \smallsetminus X_0$ containing B.]

Proof. Let B be a basis of $X \smallsetminus X_0$. Then for any $x \in X \smallsetminus (B \cup X_0)$, $\exists\, b \in B$ such that $x \in L(b)$. This shows that $B \cup \{x\}$ is not orderly independent. Thus B is maximal orderly independent in $X \smallsetminus X_0$. □

The converse of the above result is not true in general, i.e., a maximal orderly independent subset of $X \smallsetminus X_0$ may not be a basis of $X \smallsetminus X_0$. For example, in the evs $\mathscr{C}_\theta(\mathcal{X})$ [discussed in 2.11], let us consider the collection

$$\mathscr{G} := \big\{A \in \mathscr{C}_\theta(\mathcal{X}) : A \text{ consists of three distinct elements of } \mathcal{X}\big\}.$$

Then $\mathscr{G} \subset \mathscr{C}_\theta(\mathcal{X}) \smallsetminus \big\{\{\theta\}\big\}$. Now we define a relation '$\sim$' on $\mathscr{G}$ by "$A \sim B$, iff $A = \alpha B$ for some $\alpha \in \mathbb{K}^*$". Then this relation becomes an equivalence relation on $\mathscr{G}$. Let us consider the subcollection $\mathscr{H}$ of $\mathscr{G}$ taking exactly one member from each equivalence class produced by the equivalence relation '$\sim$'. Then $\mathscr{H}$ becomes an orderly independent subset of $\mathscr{C}_\theta(\mathcal{X}) \smallsetminus \big\{\{\theta\}\big\}$, because any two elements $A, B \in \mathscr{G}$ are orderly dependent, iff $A = \alpha B$ for some $\alpha \in \mathbb{K}^*$ and hence belong to the same equivalence class. For any member $C \in \mathscr{C}_\theta(\mathcal{X}) \setminus (\mathscr{H} \cup [\mathscr{C}_\theta(\mathcal{X})]_0)$ if $\operatorname{card}(C) \geq 3$, then there exists a member $A_C \in \mathscr{H}$ and $\alpha \in \mathbb{K}^*$ such that $\alpha A_C \subseteq C$. If $\operatorname{card}(C) = 2$, then also there exists $\beta \in \mathbb{K}^*$ and $A_C \in \mathscr{H}$ such that $C \subseteq \beta A_C$. This shows that $\mathscr{H} \cup \{C\}$ is orderly dependent. So, we can say that $\mathscr{H}$ forms a maximal orderly independent set in $\mathscr{C}_\theta(\mathcal{X}) \smallsetminus \big\{\{\theta\}\big\}$. But it does not generate $\mathscr{C}_\theta(\mathcal{X}) \smallsetminus \big\{\{\theta\}\big\}$. In fact, for any $D \in \mathscr{C}_\theta(\mathcal{X})$ with $\operatorname{card}(D) = 2$ there does not exist any $A \in \mathscr{H}$ such that $D \in L(A)$, since each member of $L(A)$ contains three or more elements of $\mathcal{X}$. Hence $\mathscr{H}$ cannot be a basis of $\mathscr{C}_\theta(\mathcal{X}) \smallsetminus [\mathscr{C}_\theta(\mathcal{X})]_0$, although it is maximal orderly independent $\big[$note here that $[\mathscr{C}_\theta(\mathcal{X})]_0 = \big\{\{\theta\}\big\}\big]$.

Remark 3.14. If A is an orderly independent set in $X \smallsetminus X_0$, then for any $a_1, a_2 \in A$ with $a_1 \neq a_2$ neither $a_1 \in L(a_2)$, nor $a_2 \in L(a_1)$. In other words, if $a_1 \in L(a_2)$ for some $a_1, a_2 \in A$, then $a_1 = a_2$. Moreover, any two elements of an orderly independent set A must be incomparable with respect to the partial order '$\leq$' of the evs X; in fact, $x \leq y \Rightarrow y \in L(x)$.

Lemma 3.15. *Let A and B be two bases of $X \smallsetminus X_0$. Then for any $a \in A$, there exists one and only one $b_a \in B$ such that $L(a) = L(b_a)$.*

Proof. As B is a basis of $X \smallsetminus X_0$, so for the member $a \in A$, there must exist some $b \in B$ such that $a \in L(b)$. Let us suppose, $\exists\, b_1, b_2 \in B$ such that $a \in L(b_1) \cap L(b_2) \Rightarrow L(a) \subseteq L(b_1) \cap L(b_2)$ [by Proposition 3.2] — $(*)$. Again, $a \in L(b_1) \Rightarrow \exists\, \alpha \in K^*$ and $p \in X_0$ such that $\alpha b_1 + p \leq a$ — $(**)$. Now, since A is a basis, so for $b_1, b_2 \in B$, $\exists\, a_1, a_2 \in A$ such that $b_1 \in L(a_1)$ and $b_2 \in L(a_2) \Rightarrow L(b_1) \subseteq L(a_1)$ and $L(b_2) \subseteq L(a_2)$ [by Proposition 3.2]. By $(*)$, $L(a) \subseteq L(a_1)$ and $L(a) \subseteq L(a_2) \Rightarrow a \in L(a_1)$ and $a \in L(a_2)$. As a, a_1, a_2 are the members of the basis A, so we can say that $a_2 = a = a_1$ [by the above Remark 3.14]. Therefore $b_1, b_2 \in L(a) \Rightarrow \exists\, \alpha_1, \alpha_2 \in K^*$ and $p_1, p_2 \in X_0$ such that $\alpha_1 a + p_1 \leq b_1$ and $\alpha_2 a + p_2 \leq b_2$ — $(***)$. From $(**)$ and $(***)$, we get $b_2 \geq \alpha_2 a + p_2 \geq \alpha_2(\alpha b_1 + p) + p_2 = \alpha_2\alpha b_1 + (\alpha_2 p + p_2)$. $\therefore b_2 \in L(b_1)$ [as $\alpha_2\alpha \in K^*$ and $\alpha_2 p + p_2 \in X_0$]. Since b_1, b_2 are the members of the basis B so $b_2 = b_1$ [by the above Remark 3.14]. Thus there exists one and only one member (say) $b_a \in B$ such that $a \in L(b_a)$ and also, $b_a \in L(a) \Rightarrow L(a) \subseteq L(b_a)$ and $L(b_a) \subseteq L(a) \Rightarrow L(a) = L(b_a)$. □

Theorem 3.16. *If A and B are two bases of $X \smallsetminus X_0$, then card(A) = card(B).*

Proof. From the proof of the above Lemma 3.15, we can say that for each $a \in A$, $\exists$ unique $b \in B$ such that $L(b) = L(a)$. This property creates a one-to-one correspondence between A and B. Hence the theorem is complete. □

This theorem motivates us to introduce the concept of dimension of an evs.

Definition 3.17. For an evs X we define *dimension* of $X \smallsetminus X_0$ as

$$\dim(X \smallsetminus X_0) := \operatorname{card}(B), \text{ where } B \text{ is a basis of } X \smallsetminus X_0.$$

Then we represent *dimension* of the evs X as $\dim X := [\dim(X \smallsetminus X_0) : \dim X_0]$. If $X_0 = \{\theta\}$, dimension of X_0 will be taken as 0, since then X_0 has no basis [as a vector space].

Note 3.18. Theorem 3.16 makes the above definition well-defined. From Result 3.11 we can say that if X and Y are order-isomorphic evs, then $\dim X = \dim Y$. Here, by "$\dim X = \dim Y$" we mean that $\dim(X \smallsetminus X_0) = \dim(Y \smallsetminus Y_0)$, as well as $\dim X_0 = \dim Y_0$. However, the converse of this is not true in general, i.e., there are the evs X, Y such that $\dim X = \dim Y$, but X, Y are not order-isomorphic. This will be clear in the next section, in computing the dimension of some particular evs.

Result 3.19. *Let X be an evs and B be a basis of $X \smallsetminus X_0$. Then $\downarrow x \smallsetminus X_0 \subseteq L(x)$, for each $x \in B$.*

Proof. Let $x \in B$ and $y \in \downarrow x \smallsetminus X_0$. Since B is a basis of $X \smallsetminus X_0$, $\exists\, x_1 \in B$ such that $y \in L(x_1) \Rightarrow \exists\, \alpha_1 \in K^*$ and $p_1 \in X_0$ such that $\alpha_1 x_1 + p_1 \leq y \Rightarrow \alpha_1 x_1 + p_1 \leq x$ [$\because y \leq x$] $\Rightarrow x \in L(x_1)$. Since B is orderly independent and both $x, x_1 \in B$, we can say by Remark 3.14 that $x = x_1$. Therefore $y \in L(x)$. Thus $\downarrow x \smallsetminus X_0 \subseteq L(x)$, for each $x \in B$. □

This result reveals an important property of each member of a basis of $X \smallsetminus X_0$ which helps us to set up a precise domain of basic elements of $X \smallsetminus X_0$. The collection of all x in $X \smallsetminus X_0$ satisfying the property stated in the above Result 3.19 makes our task of finding a basis of $X \smallsetminus X_0$ easier. To make this assertion precise, let us consider the following.

For an evs X, let

$$Q(X) := \big\{x \in X \smallsetminus X_0 : (\downarrow x \smallsetminus X_0) \subseteq L(x)\big\}.$$

From Result 3.19, we can say that $B \subseteq Q(X)$, for any basis B of $X \smallsetminus X_0$. It is thus enough to find any basis of $X \smallsetminus X_0$ within $Q(X)$. We call this set $Q(X)$ as a *feasible set* of X. At this point it is important to note that $Q(X)$ may be empty; in fact, if for an evs X, $Q(X) = \emptyset$, then such evs X cannot have any basis (as we have claimed earlier). We shall encounter such evs later. If for an evs X, $Q(X) \neq \emptyset$, then also X may not have a basis. In fact, we shall prove a theorem shortly in terms of $Q(X)$ which identifies an evs when it has a basis or not. We now prove a lemma that will be useful in the sequel.

Lemma 3.20. *For an evs X, if $x \in Q(X)$ then for each $y \in \downarrow x \smallsetminus X_0$, $L(x) = L(y)$.*

Proof. $y \in \downarrow x \smallsetminus X_0 \Rightarrow y \leq x$. So, by Proposition 3.2, we have $L(x) \subseteq L(y)$. Again, by the construction of $Q(X)$, $x \in Q(X) \Rightarrow \downarrow x \smallsetminus X_0 \subseteq L(x) \Rightarrow y \in L(x) \Rightarrow L(y) \subseteq L(x)$ [by Proposition 3.2]. Thus $L(x) = L(y)$. □

The following theorem may be compared with the so-called '*Replacement theorem*' in the context of the basis of a vector space.

Theorem 3.21. *For an evs X, let B be a basis of $X \smallsetminus X_0$ and $x \in B$. Then for any $y \in \downarrow x \smallsetminus X_0$, $\big(B \smallsetminus \{x\}\big) \cup \{y\}$ is also a basis of $X \smallsetminus X_0$.*

Proof. Let $A = \big(B \smallsetminus \{x\}\big) \cup \{y\}$. As $y \in \downarrow x \smallsetminus X_0$ and $x \in B \subseteq Q(X)$ so, by Lemma 3.20, $L(x) = L(y)$. Therefore $X \smallsetminus X_0 = \bigcup_{z \in B} L(z) = \bigcup_{z \in A} L(z) \Rightarrow A$ generates $X \smallsetminus X_0$. To show that A is orderly independent it is sufficient to show that for any $z \in B \smallsetminus \{x\}$, z, y are orderly independent. If not, then for some $z_1 \in B \smallsetminus \{x\}$ either $y \in L(z_1)$ or $z_1 \in L(y)$. Now, if $y \in L(z_1)$, then by Proposition 3.2, $L(y) \subseteq L(z_1) \Rightarrow x \in L(x) = L(y) \subseteq L(z_1)$ which contradicts that x, z_1 are two members of the basis B. Again, if $z_1 \in L(y)$, then $z_1 \in L(y) = L(x)$ which again contradicts that x, z_1 are orderly independent. Thus it follows that A is orderly independent. □

The above theorem makes it convenient to construct a new basis from the old one. The following theorem is the key to ensure the existence of a basis of an evs.

Theorem 3.22. *An evs X has a basis, iff $Q(X)$ is a generator of $X \smallsetminus X_0$.*

Proof. Suppose that X has a basis $[B : B_0]$, where B and B_0 are bases of $X \smallsetminus X_0$ and X_0 respectively. Then by Result 3.19, $B \subseteq Q(X)$. As B is a generator of $X \smallsetminus X_0$, so $Q(X)$ is also a generator of $X \smallsetminus X_0$.

Conversely, suppose $Q(X)$ is a generator of $X \smallsetminus X_0 \Rightarrow Q(X) \neq \emptyset$. We now give a relation $\sim$ in $Q(X)$ as follows: For $x, y \in Q(X)$, we say $x \sim y \Leftrightarrow L(x) = L(y)$. Then, obviously, this becomes an equivalence relation on $Q(X)$. Let us consider a collection taking exactly one representative from each equivalence class and denote this collection as B. Then $B \subseteq Q(X) \subseteq X \smallsetminus X_0$. Also, $x, y \in B$ with $x \neq y \Leftrightarrow x, y \in Q(X)$ and $L(x) \neq L(y)$. We claim that B is a basis of $X \smallsetminus X_0$. Let $z \in X \smallsetminus X_0 \Rightarrow \exists\ x_z \in Q(X)$ [$\because Q(X)$ is a generator] such that $z \in L(x_z) \Rightarrow \exists$ an element $x'_z \in B$ such that $L(x_z) = L(x'_z)$ and hence $z \in L(x'_z) \Rightarrow B$ generates $X \smallsetminus X_0$. Now, we have to show that B is orderly independent. Suppose not, then $\exists$ two distinct elements $x_1, x_2 \in B$ such that they are orderly dependent. So, without loss of generality, we can think that $x_1 \in L(x_2)$. Now, $x_1 \in L(x_2) \Rightarrow \exists\ \alpha \in \mathbb{K}^*$ and $p \in X_0$ such that $\alpha x_2 + p \leq x_1$. Since $x_1 \in Q(X)$ [as $B \subseteq Q(X)$] and $\alpha x_2 + p \in \downarrow x_1 \smallsetminus X_0$, using Lemma 3.20 we can say that $L(\alpha x_2 + p) = L(x_1)$ and then by Proposition 3.2, we have $L(x_2) = L(\alpha x_2 + p) = L(x_1)$ which contradicts that x_1, x_2 are two distinct elements of B. Thus B becomes a basis of $X \smallsetminus X_0$. Let us take any basis B_0 of X_0. Then $[B : B_0]$ becomes a basis of X. □

From the proof of the above theorem it is clear that the hypothesis of $Q(X)$, being a generator of $X \smallsetminus X_0$, is used only to justify that B, constructed by using the equivalence relation $\sim$ within $Q(X)$, is a generator of $X \smallsetminus X_0$; to ensure the orderly independence of B, the structure of $Q(X)$ is enough. Thus we can conclude that for any evs X, $Q(X)$ (if nonempty) always contains an orderly independent set like B (as constructed in the proof of the above Theorem 3.22). This orderly independent set is also a maximal orderly independent set in $Q(X)$. In fact, if D is another orderly independent set in

$Q(X)$ such that $B \subset D$, then for any $x \in D$, $\exists\, z \in B$ such that $L(x) = L(z) \Rightarrow x = z$ [$\because x, z \in D$ and D is orderly independent] and hence $x \in B$. Thus $B = D$. Summarising all these facts, we get the following

Theorem 3.23. *For an evs X, if $Q(X) \neq \emptyset$, then it contains a maximal orderly independent set.*

The next theorem is useful in finding a basis of $X \smallsetminus X_0$, for any evs X.

Theorem 3.24. *For an evs X, every maximal orderly independent set of $Q(X)$ is a basis of $X \smallsetminus X_0$, provided $Q(X)$ generates $X \smallsetminus X_0$.*

Proof. Let B be a maximal orderly independent set in $Q(X)$. Since $Q(X)$ generates $X \smallsetminus X_0$, for any $x \in X \smallsetminus X_0$, $\exists\, z \in Q(X)$ such that $x \in L(z)$. If $z \in B$, we are done. If $z \notin B$, then B, being a maximal orderly independent set in $Q(X)$, $B \cup \{z\}$ is orderly dependent. So, $\exists\, b \in B$ such that either $z \in L(b)$ or $b \in L(z)$. If $z \in L(b)$ then by Proposition 3.2, $x \in L(z) \subseteq L(b)$. If $b \in L(z)$, $\exists\, \alpha \in \mathbb{K}^*$ and $p \in X_0$ such that $b \geq \alpha z + p$, then $b \in B \subseteq Q(X) \Rightarrow L(z) = L(b)$ [by Lemma 3.20] $\Rightarrow x \in L(b)$. Thus B generates $X \smallsetminus X_0$. Consequently, B is a basis of $X \smallsetminus X_0$. ☐

The above Theorem 3.24 shows the converse of Result 3.13 to some extent; as we have explained, just after Result 3.13, through the example of $\mathscr{C}_\theta(\mathcal{X})$ that every maximal orderly independent subset of $X \smallsetminus X_0$ need not be a basis of $X \smallsetminus X_0$, the above Theorem 3.24 shows that every maximal orderly independent subset of $Q(X)$ [but *not only* of $X \smallsetminus X_0$] becomes a basis of $X \smallsetminus X_0$, provided of course, $Q(X)$ generates $X \smallsetminus X_0$ [note that the necessity of $Q(X)$ being a generator of $X \smallsetminus X_0$ is the principal key for an evs X to have a basis]. From Remark 3.14, we may recall one more point that while finding a basis of $X \smallsetminus X_0$, we have to gather only suitable incomparable elements from $Q(X)$. In this context it should also be noted that any two elements of $Q(X)$ need not be orderly independent. In fact, for any $x \in Q(X)$, if $y \in \downarrow x \smallsetminus X_0$, then $y \in Q(X)$, as well [see Result 3.26(ii)]. Clearly, these x, y are orderly dependent, since $L(x) = L(y)$.

Result 3.25. *If X and Y are order-isomorphic, then $Q(X)$ and $Q(Y)$ are in a one-to-one correspondence.*

Proof. Let $\phi : X \longrightarrow Y$ be an order-isomorphism. We now show that $\phi(Q(X)) = Q(Y)$. Let $x_0 \in Q(X) \Rightarrow \downarrow x_0 \smallsetminus X_0 \subseteq L(x_0)$. Also let $y \in \downarrow \phi(x_0) \smallsetminus Y_0 \Rightarrow y \leq \phi(x_0)$ and $y \notin Y_0 \Rightarrow \phi^{-1}(y) \leq x_0$ and $\phi^{-1}(y) \notin X_0 \Rightarrow \phi^{-1}(y) \in \downarrow x_0 \smallsetminus X_0 \subseteq L(x_0) \Rightarrow \exists\, \alpha \in \mathbb{K}^*$ and $p \in X_0$ such that $\alpha x_0 + p \leq \phi^{-1}(y)$ $\Rightarrow \alpha\phi(x_0) + \phi(p) \leq y \Rightarrow y \in L(\phi(x_0))$ [$\because \phi(p) \in Y_0$]. Therefore $\downarrow \phi(x_0) \smallsetminus Y_0 \subseteq L(\phi(x_0)) \Rightarrow$ $\phi(Q(X)) \subseteq Q(Y)$. Similarly, we can say that $\phi^{-1}(Q(Y)) \subseteq Q(X)$ [$\because \phi^{-1}$ is an order-isomorphism from Y onto X] $\Rightarrow Q(Y) \subseteq \phi(Q(X))$.
$\therefore \phi(Q(X)) = Q(Y)$. Thus $Q(X)$ and $Q(Y)$ are in a one-to-one correspondence. ☐

Result 3.26. (i) *If $x \in Q(X)$, then for any $\alpha \in \mathbb{K}^*$ and $p \in X_0$, $\alpha x + p \in Q(X)$, i.e., $Q(X)$ is closed under dilation and translation by primitive elements.*
(ii) *If $x \in Q(X)$, then $\downarrow x \smallsetminus X_0 \subseteq Q(X)$, i.e., $\downarrow Q(X) \smallsetminus X_0 \subseteq Q(X)$.*

Proof. (i) $x \in Q(X) \Rightarrow \downarrow x \smallsetminus X_0 \subseteq L(x)$. We now show that $\downarrow (\alpha x + p) \smallsetminus X_0 \subseteq L(\alpha x + p)$. Let $y \in \downarrow (\alpha x + p) \smallsetminus X_0 \Rightarrow y \leq \alpha x + p$ and $y \notin X_0 \Rightarrow \alpha^{-1}(y - p) \leq x$ and $y \notin X_0 \Rightarrow \alpha^{-1}(y - p) \in \downarrow x \smallsetminus X_0$ $\Rightarrow \alpha^{-1}(y - p) \in L(x) \Rightarrow \exists\, \beta \in \mathbb{K}^*$ and $q \in X_0$ such that $\beta x + q \leq \alpha^{-1}(y - p) \Rightarrow \alpha(\beta x + q) + p \leq y \Rightarrow$ $\alpha\beta x + \alpha q + p \leq y \Rightarrow y \in L(x)$ [$\because \alpha q + p \in X_0$] $\Rightarrow y \in L(\alpha x + p)$ [$\because L(\alpha x + p) = L(x)$, by Proposition 3.2]. Therefore $\alpha x + p \in Q(X)$.

(ii) Let $y \in \downarrow x \smallsetminus X_0$. Then by Lemma 3.20, $L(x) = L(y)$. Now, for each $z \in \downarrow y \smallsetminus X_0$, we have $z \leq y \leq x$ with $z \notin X_0 \Rightarrow L(z) = L(x)$ [by Lemma 3.20] $\Rightarrow z \in L(x) = L(y)$. Thus $\downarrow y \smallsetminus X_0 \subseteq L(y)$. Consequently, $y \in Q(X)$ and hence $\downarrow x \smallsetminus X_0 \subseteq Q(X)$. ☐

As we have explained in Remark 3.7 regarding orderly independence in a subevs of an evs, the theory of basis of a subevs does not behave nicely like the theory of basis of a subspace of a vector space. However, we have the following theorems and examples which reveal some technical aspects of the dimension theory of evs.

Theorem 3.27. *Every evs contains a subevs of dimension* $[1:0]$.

Proof. Let X be an evs over $\mathbb{K}$ and $B(x) := \left\{ \sum_{i=1}^{n} \alpha_i x : \alpha_i \in \mathbb{K}, n \in \mathbb{N} \right\}$, where $x \in \uparrow \theta \smallsetminus \{\theta\}$. Then for any $\alpha, \beta \in \mathbb{K}$ and any $\sum_{i=1}^{n} \alpha_i x, \sum_{j=1}^{m} \beta_j x \in B(x)$, we have $\alpha \sum_{i=1}^{n} \alpha_i x + \beta \sum_{j=1}^{m} \beta_j x = \sum_{i=1}^{n} \alpha\alpha_i x + \sum_{j=1}^{m} \beta\beta_j x \in B(x)$. Also, $[B(x)]_0 = \{\theta\} = B(x) \cap X_0$ and for any $y \in B(x)$, $\theta \leq y$ [$\because \theta \leq x$]. So, $B(x)$ forms a subevs of X for any $x \in \uparrow \theta \smallsetminus \{\theta\}$. In this case, $\{x\}$ forms a basis of $B(x) \smallsetminus [B(x)]_0$. In fact, for any $\sum_{i=1}^{n} \alpha_i x \in B(x)$, $\alpha_j x + \theta \leq \sum_{i=1}^{n} \alpha_i x$ for some $j \in \{1, 2, \ldots, n\}$ for which $\alpha_j \neq 0$. Again, any singleton set consisting of non-zero elements is always orderly independent. So, we can say that $\dim B(x) = [1 : 0]$. ☐

The following example shows that corresponding to any cardinal α, there exists an evs of dimension $[\alpha : 0]$.

Example 3.28. For any cardinal number α, let us consider the evs $[0, \infty)^\alpha$, discussed in Example 2.9. Let us take a set I such that $\text{card}(I) = \alpha$. We now show that $B := \{e_i : i \in I\}$ is a basis of $[0, \infty)^\alpha$, where $e_i = (\delta_j^i)_{j \in I}$ and $\delta_j^i = \begin{cases} 1, & \text{when } i = j, \\ 0, & \text{when } i \neq j. \end{cases}$

For any $x \in [0, \infty)^\alpha \smallsetminus \big[[0, \infty)^\alpha\big]_0$ $\Big[$here $\big[[0, \infty)^\alpha\big]_0 = \{\theta\}$ and $\theta = (z_j)_{j \in I}$, where $z_j = 0, \forall\, j \in I\Big]$ with representation $x = (x_j)_{j \in I}$, $\exists\, p \in I$ such that $x_p \neq 0 \Rightarrow x_p e_p \leq x$ [$\because x_j \geq 0, \forall\, j \in I$] $\Rightarrow x_p e_p + \theta \leq x$ $\Rightarrow x \in L(e_p) \Rightarrow B$ generates $[0, \infty)^\alpha \smallsetminus \big[[0, \infty)^\alpha\big]_0$. Now clearly, any two members of B are orderly independent in $[0, \infty)^\alpha \smallsetminus \big[[0, \infty)^\alpha\big]_0$. This shows that B is a basis of $[0, \infty)^\alpha \smallsetminus \big[[0, \infty)^\alpha\big]_0$. Therefore $\dim[0, \infty)^\alpha = [\alpha : 0]$, since $\text{card}(B) = \text{card}(I) = \alpha$ and $\dim \big[[0, \infty)^\alpha\big]_0 = \dim\{\theta\} = 0$.

Thus any two cardinal numbers α, β with $\alpha \neq \beta$, $[0, \infty)^\alpha$ and $[0, \infty)^\beta$ cannot be order-isomorphic, since they are of different dimension. We now show that for any two cardinal numbers α, β, there exists an evs of dimension $[\alpha : \beta]$. Toward this end, we need first the following

Theorem 3.29. *For an evs X and a vector space V, both being over the common field $\mathbb{K}$, the evs $Y := X \times V$ has a basis, iff the evs X has a basis.* [The evs $X \times V$ is discussed in Example 2.10]. *Also,* $\dim(X \times V) = [\dim(X \smallsetminus X_0) : \dim X_0 + \dim V]$.

Proof. Let X has a basis. We first show that $A := \{(b, \theta_V) : b \in B\}$ is a basis of $Y \smallsetminus Y_0$, where B is a basis of $X \smallsetminus X_0$ and θ_V is the identity of V. As B is orderly independent in $X \smallsetminus X_0$, we can say that any two members of A are orderly independent $\Rightarrow A$ is an orderly independent set in $Y \smallsetminus Y_0$. Let $(x, v) \in Y \smallsetminus Y_0 \Rightarrow x \in X \smallsetminus X_0$ [$\because Y_0 = X_0 \times V$]. Since B generates $X \smallsetminus X_0$, for this x, $\exists\, b \in B$ such that $x \in L(b) \Rightarrow \alpha b + p \leq x$ for some $\alpha \in \mathbb{K}^*$ and $p \in X_0 \Rightarrow \alpha(b, \theta_V) + (p, v) = (\alpha b + p, v) \leq (x, v)$ and $(p, v) \in [X \times V]_0 \Rightarrow (x, v) \in L\big((b, \theta_V)\big) \Rightarrow A$ is a generator of $Y \smallsetminus Y_0$. So A becomes a basis of $Y \smallsetminus Y_0$. Consequently, Y has a basis. Now $\dim(Y \smallsetminus Y_0) = \text{card}(A) = \text{card}(B) = \dim(X \smallsetminus X_0)$ and $\dim[X \times V]_0 = \dim(X_0 \times V) = \dim X_0 + \dim V$. Therefore $\dim(X \times V) = [\dim(X \smallsetminus X_0) : \dim X_0 + \dim V]$.

Conversely, suppose $Y := X \times V$ has a basis. Let B be a basis of $Y \smallsetminus Y_0$. Now consider $B' := \{x : (x, v_x) \in B$ for some $v_x \in V\}$. Then $x \in B' \Rightarrow x \notin X_0$. Therefore $B' \subseteq X \smallsetminus X_0$. We now show that B' forms a basis of $X \smallsetminus X_0$. For any $z \in X \smallsetminus X_0$, $(z, \theta_V) \in Y \smallsetminus Y_0$. As B is a basis of $Y \smallsetminus Y_0$, $\exists$ $(x, v_x) \in B$ such that $\alpha(x, v_x) + (p, v) \leq (z, \theta_V)$ for some $\alpha \in \mathbb{K}^*$ and $(p, v) \in [X \times V]_0 = X_0 \times V \Rightarrow$ $(\alpha x + p, \alpha v_x + v) \leq (z, \theta_V) \Rightarrow \alpha x + p \leq z \Rightarrow z \in L(x)$. So, B' generates $X \smallsetminus X_0$. If two members of B', say x', z', are orderly dependent, then without loss of generality, we can take $x' \in L(z') \Rightarrow \exists\, \alpha \in \mathbb{K}^*$ and $p \in X_0$ such that $\alpha z' + p \leq x' \Rightarrow \alpha(z', v_{z'}) + (p, v_{x'} - \alpha v_{z'}) \leq (x', v_{x'}) \Rightarrow (x', v_{x'})$ and $(z', v_{z'})$ are orderly dependent in $Y \smallsetminus Y_0$. Therefore we can say that B' is orderly independent in $X \smallsetminus X_0$ as B is orderly independent in $Y \smallsetminus Y_0$. So B' becomes a basis of $X \smallsetminus X_0$. Consequently, X has a basis. ☐

Example 3.30. For any two cardinal numbers α, β there exists an evs X such that $\dim X = [\alpha : \beta]$. For example, if we consider the evs $X := Y \times E$, where Y is an evs whose dimension is $[\alpha : 0]$ (existence of such evs has been established in Example 3.28) and E is a vector space with dimension β, then by the above Theorem 3.29, $\dim X = [\alpha : \beta]$.

Theorem 3.31. *Let X be an evs whose dimension is $[\alpha : \beta]$. Also, let γ and δ be two cardinal numbers such that $\gamma \leq \alpha$ and $\delta \leq \beta$. Then $\exists$ a subevs Y of X such that $\dim Y = [\gamma : \delta]$.*

Proof. Let B be a basis of $X \smallsetminus X_0$. Then $\operatorname{card}(B) = \alpha$. Since $\gamma \leq \alpha$, there exists $C \subseteq B$ such that $\operatorname{card}(C) = \gamma$. For each $c \in C$, we choose one element $p_c \in P_c$ and fix it.
<u>**Case 1**</u>**:** If $\delta < \gamma$, then $\exists\ E \subsetneqq C$ such that $\operatorname{card}(E) = \delta$. Consider the set

$$D := E \cup \{c - p_c : c \in C \smallsetminus E\}.$$

Since C is orderly independent, it follows that $\operatorname{card}(D) = \operatorname{card}(C) = \gamma$. As $L(c-p_c) = L(c)$, it follows that D is an orderly independent set in $X \smallsetminus X_0$. Also, consider for any $d \in D$, $q_d = p_d$ if $d \in E$, otherwise $q_d = \theta$. Then there exists a subspace W of the vector space X_0 such that $q_d \in W$, $\forall\, d \in D$ and $\dim W = \delta$.
<u>**Case 2**</u>**:** If $\gamma \leq \delta$, then consider $D = C$ and $q_d = p_d$, $\forall\, d \in D$. Then also there exists a subspace W of X_0 such that $q_d \in W$, $\forall\, d \in D$ and $\dim W = \delta$.
Thus for both cases we get
(i) an orderly independent set D in $X \smallsetminus X_0$ whose cardinality is γ.
(ii) a subspace W of X_0 such that $q_d \in W$, where $q_d < d$,[1] $\forall\, d \in D$ and $\dim W = \delta$.
Now we consider the set

$$G(D) := \left\{ \sum_{i=1}^{n} \alpha_i d_i + p : \alpha_i \in \mathbb{K}, d_i \in D, p \in W, n \in \mathbb{N} \right\}.$$

<u>**Step 1**</u>**:** In this step we prove that $G(D)$ becomes a subevs of X with $D \subseteq G(D)$ and $[G(D)]_0 = W$. For any $d \in D$, $d = 1.d + \theta \in G(D) \Rightarrow D \subseteq G(D)$. Also, for any $p \in W$, $0.d + p \in G(D) \Rightarrow W \subseteq G(D)$. For any two elements $x = \sum_{i=1}^{m} \alpha_i d_i + p$, $y = \sum_{j=1}^{n} \beta_j d_j + q$ in $G(D)$ and any two scalars α, β, $\alpha x + \beta y = \sum_{i=1}^{m} \alpha\alpha_i d_i + \sum_{j=1}^{n} \beta\beta_j d_j + (\alpha p + \beta q) \in G(D)$ [as W is a subspace]. Let $y \in [G(D)]_0$. Then y is a minimal element of $G(D)$. As $y \in G(D)$, y can be written as $y = \sum_{i=1}^{n} \alpha_i d_i + p$. Our claim is that all $\alpha_i = 0$. If not, there exists $j \in \{1, 2, \ldots, n\}$ such that $\alpha_j \neq 0$. Then there exists $q_{d_j} \in W$ such that $q_{d_j} < d_j \Rightarrow \sum_{i=1}^{n} \alpha_i q_{d_i} + p < y$ which contradicts that $y \in [G(D)]_0$, as $\sum_{i=1}^{n} \alpha_i q_{d_i} + p \in W \subseteq G(D)$. So, all $\alpha_i = 0$. Therefore $y = p \in W \Rightarrow [G(D)]_0 \subseteq W \subseteq G(D) \cap X_0$, and hence $[G(D)]_0 = G(D) \cap X_0 = W$ [by Note 2.2]. Also, for any $x = \sum_{i=1}^{n} \alpha_i d_i + p \in G(D)$, $\sum_{i=1}^{n} \alpha_i q_{d_i} + p \in W = [G(D)]_0$ such that $x \geq \sum_{i=1}^{n} \alpha_i q_{d_i} + p$. Thus it follows that $G(D)$ is a subevs of X.
<u>**Step 2**</u>**:** In this step we show that D is a basis of $G(D) \smallsetminus [G(D)]_0$. Since D is an orderly independent subset of $X \smallsetminus X_0$ and $G(D)$ is a subevs of X containing D, by Remark 3.7, we can say that D is orderly independent in $G(D) \smallsetminus [G(D)]_0$. Now, let $y \in G(D) \smallsetminus [G(D)]_0$. Then y can be written as $y = \sum_{i=1}^{n} \alpha_i d_i + p$, where not all $\alpha_i = 0$. Let $\alpha_j \neq 0$. Then $\alpha_j d_j + \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i q_{d_i} + p \right) \leq y$. As $\left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i q_{d_i} + p \right) \in W = [G(D)]_0$, so $y \in L(d_j)$ in $G(D) \smallsetminus [G(D)]_0$. Thus D becomes a basis of

[1]Here the notation '$q_d < d$' is used to mean that $q_d \leq d$ but $q_d \neq d$.

$G(D) \smallsetminus [G(D)]_0$.
Therefore $\dim G(D) = [\operatorname{card}(D) : \dim W] = [\gamma : \delta]$. $\square$

4. Computation of Basis and Dimension of Some Exponential Vector Space

In this section, we discuss the existence of a basis of some particular evs and thereby compute their dimensions. We show that there are evs which do not have basis.

Theorem 4.1. *Let X be a single-primitive comparable topological evs. Then X has a basis and $\dim X = [1 : \dim X_0]$.*

Proof. Since X is single-primitive, for each $z \in X$ let us write $P_z = \{p_z\}$. Let $x \in\uparrow \theta$ with $x \neq \theta$. Then $P_x = \{p_x\} = \{\theta\}$. Now for $y \in X \smallsetminus X_0$, $y - p_y \in\uparrow \theta$. Then X, being comparable evs, x and $y - p_y$ are comparable as $P_x = P_{y-p_y} = \{\theta\}$. If $x \leq y - p_y$, then $x + p_y \leq y \Rightarrow y \in L(x)$. If $x > y - p_y$, our claim is that there exists $\alpha \in \mathbb{K}^*$ such that $\alpha x \leq y - p_y$ with $|\alpha| < 1$. For, otherwise we can choose a sequence $\{\alpha_n\}$ in $\mathbb{K}^*$ such that $y - p_y < \alpha_n x \ \forall n \in \mathbb{N}$ and $\alpha_n \to 0$ as $n \to \infty$. Since X is a topological evs, we then have $y - p_y \leq \theta$ [taking the limit $n \to \infty$], which is a contradiction. So, there must exist one $\alpha \in \mathbb{K}^*$ such that $\alpha x \leq y - p_y \Rightarrow y \in L(x)$. Thus $L(x) = X \smallsetminus X_0$. Clearly, $\{x\}$ is orderly independent. Therefore $\{x\}$ is a basis of $X \smallsetminus X_0$. Consequently, X has a basis and $\dim X = [1 : \dim X_0]$. $\square$

As the evs $[0, \infty) \times V$ is a single primitive comparable evs by the above theorem, we can say that $\dim([0, \infty) \times V) = [1 : \dim V]$, for any Hausdörff topological vector space V. So, in particular, if $V = \{\theta\}$, then the resulting evs is order-isomorphic to $[0, \infty)$ and hence $\dim[0, \infty) = [1 : 0]$. We have shown in the previous section that $\dim[0, \infty)^\alpha = [\alpha : 0]$, for any cardinal α. This can also be justified from the following more general example.

Example 4.2. Let $\{X_i : i \in I\}$ be an arbitrary collection of exponential vector spaces, over the common field $\mathbb{K}$, each having a basis. Let B_i be a basis of $X_i \smallsetminus [X_i]_0$, $\forall i \in I$. Consider the product evs $X := \prod_{i \in I} X_i$ [see Example 2.9]. Then $X_0 = \prod_{i \in I} [X_i]_0$. For any $j \in I$, consider the set $D_j := \prod_{i \in I} C_i$, where $C_i := \begin{cases} \{\theta_{X_i}\}, & \text{when } i \neq j, \\ B_j, & \text{when } i = j. \end{cases}$ Here, θ_{X_i} is the identity in X_i. Then $D_j \subseteq X \smallsetminus X_0$, $\forall j \in I$. Let $D := \bigcup_{j \in I} D_j$. Then $D \subseteq X \smallsetminus X_0$. Now, two different members in different D_i are orderly independent. As each B_i is a basis of X_i, so two different members of one D_i are orderly independent. Thus any two different members of D are orderly independent and hence D is orderly independent in $X \smallsetminus X_0$. We now show that D is a basis of $X \smallsetminus X_0$. For any $x = (x_i)_{i \in I} \in X \smallsetminus X_0$, $\exists$ some $k \in I$ such that $x_k \in X_k \smallsetminus [X_k]_0 \Rightarrow \exists\ b_k \in B_k$, $\alpha_k \in \mathbb{K}^*$ and $p_k \in [X_k]_0$ such that $\alpha_k b_k + p_k \leq x_k$. Now for $i \neq k$, $\exists$ $p_i \in [X_i]_0$ such that $p_i \leq x_i$. Let $b = (b_i)_{i \in I}$, where $b_i = \theta_{X_i}$ for $i \neq k$ and $p = (p_i)_{i \in I} \in X_0$. Then $\alpha_k b + p = (\alpha_k b_i + p_i)_{i \in I} \leq (x_i)_{i \in I} = x$ and $b \in D_k \subset D \Rightarrow x \in L(b)$. This shows that D generates $X \smallsetminus X_0$ and hence is a basis of $X \smallsetminus X_0$. Consequently, X has a basis and $\dim X = [\operatorname{card}(D) : \dim X_0]$.

If I is finite, then $\operatorname{card}(D) = \sum_{i \in I} \operatorname{card}(D_i) = \sum_{i \in I} \operatorname{card}(B_i) = \sum_{i \in I} \dim\big(X_i \smallsetminus [X_i]_0\big)$ and $\dim X_0 = \sum_{i \in I} \dim[X_i]_0$. For any four cardinal number $\alpha, \beta, \gamma, \delta$, if we use the notation $[\alpha + \gamma : \beta + \delta] = [\alpha : \beta] + [\gamma : \delta]$, then we can write

$$\dim \prod_{i \in I} X_i = \left[\sum_{i \in I} \dim\left(X_i \smallsetminus [X_i]_0\right) : \sum_{i \in I} \dim[X_i]_0\right] = \sum_{i \in I} \big[\dim\left(X_i \smallsetminus [X_i]_0\right) : \dim[X_i]_0\big].$$

If I is infinite, then also we get the similar expression as above, provided the sums (over I) are properly defined.

If all X_i's are the same, say $X_i = Y$, $\forall i \in I$ and $\operatorname{card}(I) = \alpha$, then we have

$$\dim(Y^\alpha) = [\alpha \cdot \dim(Y \smallsetminus Y_0) : \alpha \cdot \dim Y_0].$$

Thus it follows that for any cardinal α, $\dim[0, \infty)^\alpha = [\alpha : 0]$, since $\dim[0, \infty) = [1 : 0]$.

Theorem 4.3. *For every Hausdörff topological vector space $\mathcal{X}$, $\mathscr{C}(\mathcal{X})$ [discussed in 1.3] has a basis.*

Proof. Let us consider the relation '$\sim$' on $\mathcal{X} \smallsetminus \{\theta\}$, defined as

$$x \sim y \Leftrightarrow \exists\ \alpha \in \mathbb{K}^* \text{ such that } x = \alpha y.$$

Then '$\sim$' becomes an equivalence relation on $\mathcal{X} \smallsetminus \{\theta\}$. Let us construct a set $\mathcal{X}'$ taking exactly one representative from each equivalence class, relative to '$\sim$', and consider the set

$$\mathscr{N} := \big\{\{\theta, x\} : x \in \mathcal{X}'\big\}.$$

We now show that $\mathscr{N}$ becomes a basis of $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(\mathcal{X})]_0$. If $A \in \mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(\mathcal{X})]_0$, then there must exist two elements x, y of $\mathcal{X}$ with $\{x, y\} \subseteq A$ and $x \neq y$. Then $\{\theta, x-y\} + \{y\} = \{x, y\} \subseteq A$. Now, $x - y \in \mathcal{X} \smallsetminus \{\theta\} \Rightarrow \exists\ z \in \mathcal{X}'$ and $\alpha \in \mathbb{K}^*$ such that $x - y = \alpha z$. So we can write $\alpha\{\theta, z\} + \{y\} \subseteq A$ $\Rightarrow A \in L(\{\theta, z\})$. Therefore $\mathscr{N}$ generates $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(\mathcal{X})]_0$. We now show that $\mathscr{N}$ is an orderly independent set in $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(X)]_0$. For any two elements $\{\theta, x\}$ and $\{\theta, y\}$ in $\mathscr{N}$, if $\{\theta, x\} \in L(\{\theta, y\})$, then $\exists\ \alpha \in K^*$ such that $\alpha\{\theta, y\} + \{z\} \subseteq \{\theta, x\}$ for some $z \in \mathcal{X} \Rightarrow \{z, \alpha y + z\} = \{\theta, x\}$ [$\because z \neq \alpha y + z$] $\Rightarrow$ either $z = \theta$ or $z = x$. If $z = \theta$, then $\alpha y = x$ which means that x, y belong to the same equivalence class, relative to '$\sim$', and hence $\{\theta, x\}, \{\theta, y\}$ cannot be two distinct elements of $\mathscr{N}$, which is not the case. If $z = x$, then $\alpha y + x = \theta \Rightarrow x = -\alpha y$ which again leads to the same contradiction. This proves that any two elements of $\mathscr{N}$ are orderly independent. Therefore $\mathscr{N}$ is orderly independent in $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(X)]_0$ and hence becomes a basis of $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(\mathcal{X})]_0$. Consequently, $\mathscr{C}(\mathcal{X})$ has a basis and $\dim \mathscr{C}(\mathcal{X}) = [\operatorname{card}(\mathscr{N}) : \dim \mathcal{X}]$. □

Remark 4.4. We have shown in the above Theorem 4.3 that $\mathscr{N}$ forms a basis of $\mathscr{C}(\mathcal{X}) \smallsetminus [\mathscr{C}(\mathcal{X})]_0$. We now show that this basis depends on a basis (as vector space) of $\mathcal{X}$.

(i) If $\mathcal{X}$ is a Hausdörff topological vector space of dimension 1, then any non-zero element of $\mathcal{X}$ is a scalar multiple of a single basic vector of $\mathcal{X}$ and hence $\mathscr{N}$ contains exactly one element. So, $\dim \mathscr{C}(\mathcal{X}) = [1 : 1]$. For that reason dimension of $\mathscr{C}(\mathbb{R})$ over $\mathbb{R}$ is $[1 : 1]$ and dimension of $\mathscr{C}(\mathbb{C})$ over $\mathbb{C}$ is $[1 : 1]$.

(ii) Let $\mathcal{X}$ be a Hausdörff topological vector space of dimension 2 and $B = \{a, b\}$ be a basis of $\mathcal{X}$. We first show that $\mathcal{X}' = \{a + \beta b : \beta \in \mathbb{K}\} \cup \{b\}$, where $\mathcal{X}'$ is as defined in the proof of Theorem 4.3. Any two distinct elements $a + \beta_1 b, a + \beta_2 b \in \mathcal{X} \smallsetminus \{\theta\}$ must lie in two different equivalence classes, relative to '$\sim$', since for any $\alpha \in \mathbb{K}^*$, if $\alpha(a + \beta_1 b) = a + \beta_2 b$, we have $\alpha = 1$ and hence $\beta_1 = \beta_2$ [as $\{a, b\}$ is a linearly independent subset of $\mathcal{X}$] — this contradicts that $a + \beta_1 b \neq a + \beta_2 b$. Also, the linear independence of a, b implies that $a + \beta b$ and b must lie in two different equivalence classes, relative to '$\sim$', for any $\beta \in \mathbb{K}$. Now, for any non-zero element $x \in \mathcal{X}$, $\exists\, \alpha, \beta \in \mathbb{K}$ (not both zero) such that $x = \alpha a + \beta b$ [since $\{a, b\}$ is a basis of $\mathcal{X}$]. If $\alpha \neq 0$, then $x = \alpha(a + \beta\alpha^{-1} b) \Rightarrow x$ lies in the class (relative to '$\sim$'), whose representative is $(a + \beta\alpha^{-1} b)$. If $\alpha = 0$, then x lies in the equivalence class (relative to '$\sim$'), whose representative is b. Therefore $\mathcal{X}' = \{a + \beta b : \beta \in \mathbb{K}\} \cup \{b\}$. Now, the map $\alpha \longmapsto a + \alpha b$ creates a bijection between $\mathbb{K}$ and $\mathcal{X}' \smallsetminus \{b\}$. So, we can say that the cardinality of $\mathcal{X}'$ and hence that of $\mathscr{N}$ is c, the cardinality of the set of real numbers $\mathbb{R}$. Therefore $\dim \mathscr{C}(\mathcal{X}) = [c : 2]$. For that reason dimension of $\mathscr{C}(\mathbb{C})$ over $\mathbb{R}$ is $[c : 2]$.

(iii) In a similar manner as above, we can show that for a *well-ordered* basis B of a Hausdörff topological vector space $\mathcal{X}$,

$$\mathcal{X}' = (e_1 + < B_1 >) \cup (e_2 + < B_2 >) \cup \cdots \cup (e_n + < B_n >) \cup \cdots,$$

where $B = \{e_1, e_2, \ldots, e_n, \ldots\}$, $B_1 = B \smallsetminus \{e_1\}$, $B_n = B_{n-1} \smallsetminus \{e_n\}$, $\forall\, n \geq 2$ and $< B_i >$ denotes the linear span of B_i in $\mathcal{X}$, $\forall\, i$.

Theorem 4.5. *For every vector space $\mathcal{X}$, the evs $\mathscr{L}(\mathcal{X})$ has a basis.* [The evs $\mathscr{L}(\mathcal{X})$ is discussed in Example 2.12]

Proof. Let $\mathscr{T}$ be the collection of all one-dimensional subspaces of $\mathcal{X}$. We now show that $\mathscr{T}$ forms a basis of $\mathscr{L}(\mathcal{X}) \smallsetminus [\mathscr{L}(\mathcal{X})]_0$. For any non-trivial subspace $\mathcal{Y}$ of $\mathcal{X}$, there exists a non-zero element $x \in \mathcal{Y}$ such that $< x > \subseteq \mathcal{Y}$ [here, $< x >$ denotes the linear span of x in $\mathcal{X}$]. So, $\mathcal{Y} \in L(< x >)$. Also, $< x > \in \mathscr{T}$. Thus $\mathscr{T}$ generates $\mathscr{L}(\mathcal{X}) \smallsetminus [\mathscr{L}(\mathcal{X})]_0$. For any two distinct elements $< x >, < y > \in \mathscr{T}$, if $\alpha < x > \subseteq < y >$ for some $\alpha \in \mathbb{K}^*$, then $< x > = \alpha < x > \subseteq < y > \Rightarrow < x > = < y >$ which contradicts that $< x >$ and $< y >$ are distinct. So, we can say that $\mathscr{T}$ is an orderly independent subset of $\mathscr{L}(\mathcal{X}) \smallsetminus [\mathscr{L}(\mathcal{X})]_0$. Therefore $\mathscr{T}$ forms a basis of $\mathscr{L}(\mathcal{X}) \smallsetminus [\mathscr{L}(\mathcal{X})]_0$. Consequently, $\mathscr{L}(\mathcal{X})$ has a basis and $\dim \mathscr{L}(\mathcal{X}) = [\text{card}(\mathscr{T}) : 0]$, since $[\mathscr{L}(\mathcal{X})]_0 = \big\{\{\theta\}\big\}$. $\square$

From the above theorem, we can immediately get the following result. Also, $\mathscr{T}$ is the *only* basis of $\mathscr{L}(\mathcal{X}) \smallsetminus [\mathscr{L}(\mathcal{X})]_0$.

Result 4.6. $\dim \mathscr{L}(\mathcal{X}) = [1 : 0]$*, when* $\dim \mathcal{X} = 1$ *and* $\dim \mathscr{L}(\mathcal{X}) = [c : 0]$*, when* $\dim \mathcal{X} = 2$*, c being the cardinality of the set of all reals* $\mathbb{R}$*.*

Note 4.7. From the previous result we can say that $\dim \mathscr{L}(\mathbb{R}) = [1 : 0]$ which is the same with the $\dim[0, \infty)$. But $\mathscr{L}(\mathbb{R})$ and $[0, \infty)$ are not order-isomorphic as the first one is non-topological evs, whereas the second one is a topological evs and, being topological, is an evs property. This example shows that a converse part of the statement that equality of dimension is an evs property which we have discussed in 3.18, is not true.

Theorem 4.8. *For any $n \in \mathbb{N}$, $\mathscr{D}^n[0, \infty)$ has a basis and* $\dim \mathscr{D}^n[0, \infty) = [1 : 0]$.

Proof. We first show that $(0, 0, \ldots, 0, 1)$ generates $\mathscr{D}^n[0, \infty) \smallsetminus [\mathscr{D}^n[0, \infty)]_0$. Let $x = (x_1, \ldots, x_n) \in \mathscr{D}^n[0, \infty) \smallsetminus [\mathscr{D}^n[0, \infty)]_0$. Since $[\mathscr{D}^n[0, \infty)]_0 = \big\{(0, \ldots, 0)\big\}$, there exists $i \in \{1, 2, \ldots, n\}$ such that $x_i \neq 0$ and $x_j = 0$, for all $j < i$. If $i < n$, then, obviously, $(0, 0, \ldots, 0, 1) \leq x$. If $i = n$, then $\frac{x_i}{2}(0, 0, \ldots, 0, 1) \leq x$. In any case, $x \in L\big((0, 0, \ldots, 1)\big)$. Since $\big\{(0, \ldots, 0, 1)\big\}$ is orderly independent, it follows that $\big\{(0, \ldots, 0, 1)\big\}$ is a basis of $\mathscr{D}^n[0, \infty) \smallsetminus [\mathscr{D}^n[0, \infty)]_0$ and hence $\dim \mathscr{D}^n[0, \infty) = [1 : 0]$. $\square$

The following example shows that there exists an evs which has no basis.

Theorem 4.9. $X := \mathscr{D}\big([0, \infty) : \mathbb{N}\big)$ *has no basis.*

Proof. Let $x = (x_i)_{i \in \mathbb{N}} \in X \smallsetminus X_0$. Since here $X_0 = \big\{(0, 0, \ldots)\big\}$, there must exist a least positive integer p such that $x_p \neq 0$. If we consider $y = (y_i)_{i \in \mathbb{N}}$, where $y_i = x_i$, $\forall\, i \neq p, p+1$ and $y_p = 0$, $y_{p+1} = 1$, then $y \leq x$ and $y \notin X_0$; but there does not exist any $\alpha \in \mathbb{K}^*$ such that $\alpha x \leq y$, which means that $y \notin L(x)$. This shows that $x \notin Q(X)$ and this holds for any non-zero element x of X. Therefore $Q(X) = \emptyset$. So, $\mathscr{D}\big([0, \infty) : \mathbb{N}\big)$ has no basis. $\square$

Looking at the proof of the above theorems, we can get the following generalised theorem.

Theorem 4.10. *For a well-ordered set I, $\mathscr{D}(X : I)$ has a basis, iff I has a maximum element.*

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