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ESTIMATION OF f-DIVERGENCE AND SHANNON ENTROPY BY LEVINSON TYPE INEQUALITIES VIA LIDSTONE INTERPOLATING POLYNOMIAL

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Abstract. Using Lidstone interpolating polynomial, some new generalizations of Levinson-type inequalities for $2p$-convex functions are obtained. In seek of applications to information theory, based on $\mathfrak{f}$-divergence, the estimates for new generalizations are also given. Moreover, inequalities for Shannon entropies are deduced.

1. Introduction and Preliminaries

The theory of convex functions has encountered a fast advancement. This can be attributed to a few causes: firstly, applications of convex functions are directly involved in the modern analysis, secondly, many important inequalities are applications of convex functions which are closely related to inequalities (see [24]).

Levinson generalized Ky Fan's inequality for 3-convex functions in [17] (see also [20, p.32, Theorem 1]) in the form of the following

Theorem 1.1. *Let $f : \mathbb{I} = (0, 2\lambda) \to \mathbb{R}$ be such that f is 3-convex. Also, let $0 < x_\rho < \lambda$ and $p_\rho > 0$. Then*

$$\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho\right) \leq \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(2\lambda - x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho (2\lambda - x_\rho)\right). \tag{1}$$

The difference of the right– and left-hand sides of (1) is the linear functional $J_1(f(\cdot))$, which can be written as follows:

$$J_1(f(\cdot)) = \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(2\lambda - x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho (2\lambda - x_\rho)\right) - \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(x_\rho) + f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho\right). \tag{2}$$

In [25], Popoviciu noticed that Levinson's inequality (1) is substantial on $(0, 2\lambda)$ for 3-convex functions, while in [9], (see additionally [20, p.32, Theorem 2]) Bullen gave distinctive confirmation of Popoviciu's result and furthermore the converse of (1).

Theorem 1.2. (a) *Let $f : \mathbb{I} = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a 3-convex function and $x_k, y_k \in [\zeta_1, \zeta_2]$ for $k = 1, 2, \ldots, \rho$ such that*

$$\max\{x_1 \ldots x_n\} \leq \min\{y_1 \ldots y_n\}, \ x_1 + y_1 = \cdots = x_n + y_n \tag{3}$$

and $p_\rho > 0$, then

$$\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(x_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho\right) \leq \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(y_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho y_\rho\right). \tag{4}$$

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(b) *If* $p_\rho > 0$, *inequality* (4) *is valid for all* x_k, y_k *satisfying condition* (3) *and the function* f *is continuous, then* f *is* 3*-convex.*

The difference of the right– and left-hand sides of (4) is the linear functional $J_2(f(\cdot))$, which can be written as follows:

$$J_2(f(\cdot)) = \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(y_\rho) - f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho y_\rho\right) - \frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho f(x_\rho) + f\left(\frac{1}{P_n}\sum_{\rho=1}^{n} p_\rho x_\rho\right). \tag{5}$$

Remark 1.1. It is essential to take note of the fact that under the suppositions of Theorem 1.1 and Theorem 1.2, if the function f is 3-convex, then $J_k(f(\cdot)) \geq 0$ for $k = 1, 2$, and $J_k(f(\cdot)) = 0$ for $f(x) = x$ or $f(x) = x^2$ or f is a constant function.

In the following result, Pečarić [21] (see also [20, p.32, Theorem 4]), proved inequality (4) by weakening condition (3).

Theorem 1.3. *Let* $f : \mathbb{I} = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be a* 3*-convex function,* $p_\rho > 0$*, and let* $x_\rho, y_\rho \in [\zeta_1, \zeta_2]$ *such that* $x_\rho + y_\rho = 2\check{c}$*, for* $\rho = 1, \ldots, n$ $x_\rho + x_{n-\rho+1} \leq 2\check{c}$ *and* $\frac{p_\rho x_\rho + p_{n-\rho+1}x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \leq \check{c}$*. Then inequality* (4) *holds.*

In [19], Mercer replaced the symmetry by the equality of the variances of points and proved in the following result that inequality (4) still holds.

Theorem 1.4. *Let* f *be a* 3*-convex function on* $[\zeta_1, \zeta_2]$*, and let* p_ρ *be positive such that* $\sum_{\rho=1}^{n} p_\rho = 1$*. Also, let* x_ρ*,* y_ϱ *satisfy* $\max\{x_1 \ldots x_n\} \leq \min\{y_1 \ldots y_n\}$ *and*

$$\sum_{\rho=1}^{n} p_\rho\left(x_\rho - \sum_{\rho=1}^{n} p_\rho x_\rho\right)^2 = \sum_{\rho=1}^{n} p_\rho\left(y_\rho - \sum_{\rho=1}^{n} p_\rho y_\rho\right)^2, \tag{6}$$

then (4) *holds.*

In [22], Pečarić *et al.* gave probabilistic version of inequality (1) under condition (6). In [23] the operator version of probabilistic Levinsons inequality is discussed.
The following Lemma is given in [28].

Lemma 1.1. *If* $f \in C^\infty[0, 1]$*, then*

$$f(t) = \sum_{l=0}^{p-1}\left[f^{(2l)}(0)\Theta_l(1-t) + f^{(2l)}(0)\Theta_l(t)\right] + \int_0^1 G_p(t, s) f^{(2p)}(t)dt,$$

where Θ_l *is a polynomial of degree* $2l + 1$ *defined by the relations*

$$\Theta_0(t) = t, \ \Theta_p^{''}(t) = \Theta_{p-1}(t), \ \Theta_p(0) = \Theta_p(1) = 0, \ p \geq 1,$$

and

$$G_1(t, s) = G(t, s) = \begin{cases} (t-1)s, & s \leq t; \\ (s-1)t, & t \leq s, \end{cases} \tag{7}$$

is homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ *on* $[0, 1]$*, and with the successive iterates of* $G(t, s)$*,*

$$G_p(t, s) = \int_0^1 G_1(t, k)G_{p-1}(k, s)dk, \ p \geq 2. \tag{8}$$

The Lidstone polynomial can be expressed in terms of $G_p(t,s)$ as

$$\Theta_p(t) = \int\limits_0^1 G_p(t,s)sds. \tag{9}$$

Lidstone series representation of $f \in C^{2p}[\zeta_1, \zeta_2]$ given in [7] as follows:

$$\begin{aligned} f(x) = &\sum_{l=0}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_1)\Theta_l\left(\frac{\zeta_2 - x}{\zeta_2-\zeta_1}\right) + \sum_{l=0}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_2)\Theta_l\left(\frac{x-\zeta_1}{\zeta_2-\zeta_1}\right) \\ &+ (\zeta_2-\zeta_1)^{2p-1}\int\limits_{\zeta_1}^{\zeta_2} G_p\left(\frac{x-\zeta_1}{\zeta_2-\zeta_1}, \frac{t-\zeta_1}{\zeta_2-\zeta_1}\right) f^{(2p)}(t)dt \end{aligned} \tag{10}$$

In [8], Gazić *et al.* considered the class of $2p$-convex functions and generalized Jensen's inequality and converses of Jensen's inequality by using Lidstone's interpolating polynomials. Some other, new and thought provoking results and their applications for various divergences, can be found in the literature (see, for example, [1–6]). All generalizations existing in literature are only for one type of data points. But in this paper and motivated by the above discussion, Levinson type inequalities are generalized via the Lidstone interpolating polynomial involving two types of data points for higher order convex functions. Moreover, a new functional is introduced based on f-divergence and then some estimates for new functional are obtained. Some inequalities for Shannon entropies are also deduced.

2. Main Results

Motivated by functional (5), we generalize the following results with the help of the Lidstone interpolating polynomial given by (10).

2.1. Generalization of Bullen type inequalities for $2p$-convex functions.

First, we define the following functional:

$\mathcal{F}$**:** Let $f : \mathbb{I}_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ be a function, $x_1, \ldots, x_n$ and $y_1, \ldots, y_m \in \mathbb{I}_1$ such that

$$\max\{x_1 \ldots x_n\} \leq \min\{y_1 \ldots y_m\}. \tag{11}$$

Also, let $(p_1, \ldots, p_n) \in \mathbb{R}^n$ and $(q_1, \ldots, q_m) \in \mathbb{R}^m$ be such that $\sum_{\rho=1}^{n} p_\rho = 1$, $\sum_{\varrho=1}^{m} q_\varrho = 1$ and x_ρ, y_ϱ, $\sum_{\rho=1}^{n} p_\rho x_\rho$, $\sum_{\varrho=1}^{m} q_\varrho y_\varrho \in \mathbb{I}_1$. Then

$$\breve{J}(f(\cdot)) = \sum_{\varrho=1}^{m} q_\varrho f(y_\varrho) - f\left(\sum_{\varrho=1}^{m} q_\varrho y_\varrho\right) - \sum_{\rho=1}^{n} p_\rho f(x_\rho) + f\left(\sum_{\rho=1}^{n} p_\rho x_\rho\right). \tag{12}$$

Theorem 2.1. *Assume $\mathcal{F}$ with $f \in C^{2p}[\zeta_1, \zeta_2]$ $(p > 2)$ and let $\Theta_p(t)$ be the same as defined in Lemma 1.1. Then*

$$\begin{aligned} \breve{J}(f(\cdot)) = &\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_1)\breve{J}\Big(\Theta_l(.)\Big) + \sum_{l=1}^{p-1}(\zeta_2-\zeta_2)^{2l} f^{(2l)}(\zeta_2)\breve{J}\Big(\ddot{\Theta}_l(\cdot)\Big) \\ &+ (\zeta_2-\zeta_1)^{2p-1}\int\limits_{\zeta_1}^{\zeta_2} \breve{J}\Big(G_p(t,\cdot)\Big) f^{(2p)}(t)dt, \end{aligned} \tag{13}$$

where

$$\breve{J}\Big(\Theta_l(\cdot)\Big) = \sum_{\varrho=1}^{m} q_\varrho \Theta_l\left(\frac{\zeta_2 - y_\varrho}{\zeta_2 - \zeta_1}\right) - \Theta_l\left(\frac{\zeta_2 - \sum_{\varrho=1}^{m} q_\varrho y_\varrho}{\zeta_2 - \zeta_1}\right)$$
$$- \sum_{\rho=1}^{n} p_\rho \Theta_l\left(\frac{\zeta_2 - x_\rho}{\zeta_2 - \zeta_1}\right) + \Theta_l\left(\frac{\zeta_2 - \sum_{\rho=1}^{n} p_\rho x_\rho}{\zeta_2 - \zeta_1}\right), \tag{14}$$

$$\breve{J}\Big(\ddot{\Theta}_l(\cdot)\Big) = \sum_{\varrho=1}^{m} q_\varrho \Theta_l\left(\frac{y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}\right) - \Theta_l\left(\frac{\sum_{\varrho=1}^{m} q_\varrho y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}\right)$$
$$- \sum_{\rho=1}^{n} p_\rho \Theta_l\left(\frac{x_\rho - \zeta_1}{\zeta_2 - \zeta_1}\right) + \Theta_l\left(\frac{\sum_{\rho=1}^{n} p_\rho x_\rho - \zeta_1}{\zeta_2 - \zeta_1}\right) \tag{15}$$

and

$$\breve{J}\Big(G_p(t,\cdot)\Big) = \sum_{\varrho=1}^{m} q_\varrho G_p\left(\frac{y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) - G_p\left(\frac{\sum_{\varrho=1}^{m} q_\varrho y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right)$$
$$- \sum_{\rho=1}^{n} p_\rho G_p\left(\frac{x_\rho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) + G_p\left(\frac{\sum_{\rho=1}^{n} p_\rho x_\rho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right). \tag{16}$$

Proof. Using (10) in (12), we have

$$\breve{J}(f(\cdot)) = \sum_{\varrho=1}^{m} q_\varrho \Bigg[\sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_1) \Theta_l\left(\frac{\zeta_2 - y_\varrho}{\zeta_2 - \zeta_1}\right) + \sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_2)$$
$$\times \Theta_l\left(\frac{y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}\right) + (\zeta_2 - \zeta_1)^{2p-1} \int_{\zeta_1}^{\zeta_2} G_p\left(\frac{y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) f^{(2p)}(t)dt\Bigg]$$
$$- \Bigg[\sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_1) \Theta_l\left(\frac{\zeta_2 - \sum_{\varrho=1}^{m} q_\varrho y_\varrho}{\zeta_2 - \zeta_1}\right)$$
$$+ \sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_2) \Theta_l\left(\frac{\sum_{\varrho=1}^{m} q_\varrho y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}\right)$$
$$+ (\zeta_2 - \zeta_1)^{2p-1} \int_{\zeta_1}^{\zeta_2} G_p\left(\frac{\sum_{\varrho=1}^{m} q_\varrho y_\varrho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) f^{(2p)}(t)dt\Bigg]$$
$$- \sum_{\rho=1}^{n} p_\rho \Bigg[\sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_1) \Theta_l\left(\frac{\zeta_2 - x_\rho}{\zeta_2 - \zeta_1}\right) + \sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_2)$$
$$\times \Theta_l\left(\frac{x_\rho - \zeta_1}{\zeta_2 - \zeta_1}\right) + (\zeta_2 - \zeta_1)^{2p-1} \int_{\zeta_1}^{\zeta_2} G_p\left(\frac{x_\rho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) f^{(2p)}(t)dt\Bigg]$$
$$+ \Bigg[\sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_1) \Theta_l\left(\frac{\zeta_2 - \sum_{\rho=1}^{n} p_\rho x_\rho}{\zeta_2 - \zeta_1}\right) +$$
$$+ \sum_{l=0}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_2) \Theta_l\left(\frac{\sum_{\rho=1}^{n} p_\rho x_\rho - \zeta_2}{\zeta_2 - \zeta_1}\right)$$
$$+ (\zeta_2 - \zeta_1)^{2p-1} \int_{\zeta_1}^{\zeta_2} G_p\left(\frac{\sum_{\rho=1}^{n} p_\rho x_\rho - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1}\right) f^{(2p)}(t)dt\Bigg]$$

$$+(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\left(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)f^{(2p)}(t)dt\Bigg].$$

After some simple calculations, we have

$$\begin{aligned}
\breve{J}(f(\cdot)) =& \sum_{\varrho=1}^{m} q_\varrho\left[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)\Theta_l\left(\frac{\zeta_2-y_\varrho}{\zeta_2-\zeta_1}\right)\right]-\left[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)\Theta_l\right.\\
&\left.\left(\frac{\zeta_2-\sum_{\varrho=1}^{m}q_\varrho y_\varrho}{\zeta_2-\zeta_1}\right)\right]-\left[\sum_{\rho=1}^{n}p_\rho\left(\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)\Theta_l\left(\frac{\zeta_2-x_\rho}{\zeta_2-\zeta_1}\right)\right)\right]\\
&+\left[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)\Theta_l\left(\frac{\zeta_2-\sum_{\rho=1}^{n}p_\rho x_\rho}{\zeta_2-\zeta_1}\right)\right]\\
&+\left[\sum_{\varrho=1}^{m} q_\varrho\left[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\Theta_l\left(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\right)\right]-\left[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\Theta_l\right.\right.\\
&\left.\left(\frac{\sum_{\varrho=1}^{m}q_\varrho y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\right)\right]-\left[\sum_{\rho=1}^{n}p_\rho\left(\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\Theta_l\left(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1}\right)\right)\right]\\
&\left.+\left[\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\Theta_l\left(\frac{\sum_{\rho=1}^{n}p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1}\right)\right]\right]\\
&+\sum_{\varrho=1}^{m} q_\varrho\left[(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\left(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)f^{(2p)}(t)dt\right]\\
&-\left[(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\left(\frac{\sum_{\varrho=1}^{m}q_\varrho y_\varrho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)f^{(2p)}(t)dt\right]\\
&-\sum_{\rho=1}^{n} p_\rho\left[(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\left(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)f^{(2p)}(t)dt\right]\\
&+\left[(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2} G_p\left(\frac{\sum_{\rho=1}^{n}p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)f^{(2p)}(t)dt\right].
\end{aligned}$$

$$\begin{aligned}
\breve{J}(f(\cdot)) =& \sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)\left[\sum_{\varrho=1}^{m}q_\varrho\Theta_l\left(\frac{\zeta_2-y_\varrho}{\zeta_2-\zeta_1}\right)-\Theta_l\left(\frac{\zeta_2-\sum_{\varrho=1}^{m}q_\varrho y_\varrho}{\zeta_2-\zeta_1}\right)\right.\\
&\left.-\sum_{\rho=1}^{n}p_\rho\Theta_l\left(\frac{\zeta_2-x_\rho}{\zeta_2-\zeta_1}\right)+\Theta_l\left(\frac{\zeta_2-\sum_{\rho=1}^{n}p_\rho x_\rho}{\zeta_2-\zeta_1}\right)\right]\\
&+\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\left[\sum_{\varrho=1}^{m}q_\varrho\Theta_l\left(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\right)-\Theta_l\left(\frac{\sum_{\varrho=1}^{m}q_\varrho y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\right)\right.\\
&\left.-\sum_{\rho=1}^{n}p_\rho\Theta_l\left(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1}\right)+\Theta_l\left(\frac{\sum_{\rho=1}^{n}p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1}\right)\right]\\
&+(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2}\left[\sum_{\varrho=1}^{m}q_\varrho G_p\left(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)-\right.
\end{aligned}$$

$$G_p\left(\frac{\sum_{\varrho=1}^{m} q_\varrho y_\varrho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)-\sum_{\rho=1}^{n} p_\rho G_p\left(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)$$
$$+G_p\left(\frac{\sum_{\rho=1}^{n} p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)\Bigg]f^{(2p)}(t)dt.$$

Using definition of (14), (15) and (16), we get (13). □

As an application, we obtain a generalization of Bullen type inequality for $2p$-convex functions for $p>2$.

Theorem 2.2. *Assuming the conditions of Theorem* 2.1 *and*

$$\breve{J}\Big(G_p(t,\cdot)\Big)\geq 0. \tag{17}$$

If f is a $2p$-convex function, then

$$\breve{J}(f(\cdot))\geq\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\breve{J}\Big(\Theta_l(.)\Big)+\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_2)\breve{J}\Big(\ddot{\Theta}_l(\cdot)\Big). \tag{18}$$

Proof. As the function f is $2p$-convex and $2p$-times differentiable, so

$$f^{(2p)}(x)\geq 0\ \forall\ x\in[\zeta_1,\zeta_2],$$

then using (17) in (13), we get (18). □

Remark 2.1.
(i) In Theorem 2.2, the reverse inequality in (17) leads to the reverse inequality in (18).
(ii) Inequality in (18) is also reversed if f is a $2p$-concave function.

If we put $m=n$, $p_\rho=q_\varrho$ and by using positive weights in (12), then $\breve{J}(\cdot)$ converts to the functional $J_2(\cdot)$ defined in (5), and also in this case, (13), (14), (15), (16), (17) and (18) become

$$J_2(f(\cdot))=\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l}f^{(2l)}(\zeta_1)J_2\Big(\Theta_l(\cdot)\Big)+\sum_{l=1}^{p-1}(\zeta_2-\zeta_2)^{2l}f^{(2l)}(\zeta_2)J_2\Big(\ddot{\Theta}_l(\cdot)\Big)$$
$$+(\zeta_2-\zeta_1)^{2p-1}\int_{\zeta_1}^{\zeta_2}J_2\Big(G_p(t,\cdot)\Big)f^{(2p)}(t)dt, \tag{19}$$

$$J_2\Big(\Theta_l(\cdot)\Big)=\sum_{\rho=1}^{n}p_\rho\Theta_l\left(\frac{\zeta_2-y_\varrho}{\zeta_2-\zeta_1}\right)-\Theta_l\left(\frac{\zeta_2-\sum_{\rho=1}^{n}p_\rho y_\varrho}{\zeta_2-\zeta_1}\right)$$
$$-\sum_{\rho=1}^{n}p_\rho\Theta_l\left(\frac{\zeta_2-x_\rho}{\zeta_2-\zeta_1}\right)+\Theta_l\left(\frac{\zeta_2-\sum_{\rho=1}^{n}p_\rho x_\rho}{\zeta_2-\zeta_1}\right), \tag{20}$$

$$J_2\Big(\ddot{\Theta}_l(\cdot)\Big)=\sum_{\rho=1}^{n}p_\rho\Theta_l\left(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\right)-\Theta_l\left(\frac{\sum_{\rho=1}^{n}p_\rho y_\varrho-\zeta_1}{\zeta_2-\zeta_1}\right)$$
$$-\sum_{\rho=1}^{n}p_\rho\Theta_l\left(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1}\right)+\Theta_l\left(\frac{\frac{1}{P_n}\sum_{\rho=1}^{n}p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1}\right), \tag{21}$$

$$J_2\Big(G_p(t,\cdot)\Big)=\sum_{\rho=1}^{n}p_\rho G_p\left(\frac{y_\varrho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)-G_p\left(\frac{\sum_{\rho=1}^{n}p_\rho y_\varrho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)$$
$$-\sum_{\rho=1}^{n}p_\rho G_p\left(\frac{x_\rho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right)+G_p\left(\frac{\sum_{\rho=1}^{n}p_\rho x_\rho-\zeta_1}{\zeta_2-\zeta_1},\frac{t-\zeta_1}{\zeta_2-\zeta_1}\right), \tag{22}$$

$$J_2\Big(G_p(t,\cdot)\Big)\geq 0, \tag{23}$$

and

$$J_2(f(\cdot)) \geq \sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_2) J_2\Big(\Theta_l(\cdot)\Big) + \sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_2) J_2\Big(\ddot{\Theta}_l(\cdot)\Big). \tag{24}$$

Theorem 2.3. *Let $f:\mathbb{I}_1=[\zeta_1,\zeta_2]\to\mathbb{R}$ be a $2p$ $(p>2)$-convex function. Also, let $(p_1,\dots,p_n)$ be positive real numbers such that $\sum_{\rho=1}^{n} p_\rho = 1$. Then for the functional $J_2(\cdot)$ defined in (5), we have the following:*

(i) (24) *holds for every $2p$-convex function if p is odd.*

(ii) *Let* (24) *hold. If the function*

$$F(x)=\sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_1)\Theta_l\left(\frac{\zeta_2-x}{\zeta_2-\zeta_1}\right) + \sum_{l=1}^{p-1}(\zeta_2-\zeta_1)^{2l} f^{(2l)}(\zeta_2)\Theta_l\left(\frac{x-\zeta_1}{\zeta_2-\zeta_1}\right) \tag{25}$$

is 3-convex, then the right-hand side of (24) *is non-negative and we have the inequality*

$$J_2(f(\cdot)) \geq 0. \tag{26}$$

Proof.

(i) By (7), $G_1(t,s)\leq 0$, for $0\leq t,\ s\leq 1$. By using (8), we have $G_p(t,s)\leq 0$ for odd p and $G_p(t,s)\geq 0$ for even p. Now, as G_1 is 3-convex and G_{p-1} is positive for odd p, therefore by using (8), G_p is 3-convex in the first variable if p is odd. Similarly, G_p is 3-concave in the first variable if p is even.
Hence if p is odd, then by Remark 1.1,

$$J_2\Big(G_p(t,\cdot)\Big) \geq 0,$$

therefore (24) holds.

(ii) $J_2(\cdot)$ is a linear functional, so we can write the right-hand side of (24) in the form $J_2(F(x))$, where F is defined in (25). Since F is assumed to be 3-convex, therefore using the given conditions and by Remark 1.1, the non-negativity of the right-hand side of (24) is immediate and we have (26) for n-tuples. □

In the next result we give generalization of Levinson's type inequality given in [21] (see also [20]).

Theorem 2.4. *Let $f\in C^{2p}[\zeta_1,\zeta_1]$ $(p>2)$, $(p_1,\dots,p_n)$ be positive real numbers such that $\sum\limits_{\rho=1}^{n} p_\rho = 1$. Also, let $x_1,\dots,x_n$ and $y_1,\dots,y_n\in\mathbb{I}_1$ be such that $x_\rho+y_\rho=2\check{c}$, $x_\rho+x_{n-\rho+1}\leq 2\check{c}$ and $\frac{p_\rho x_\rho+p_{n-\rho+1}x_{n-\rho+1}}{p_\rho+p_{n-\rho+1}}\leq\check{c}$. Moreover, let $\Theta_p(t)$ be the same as defined in Lemma* 1.1, *then* (19) *holds.*

Proof. The Proof is similar to that of Theorem 2.1 by assuming the conditions given in the statement. □

As an application, we give generalizations of Levinson's type inequalities for $2p$-convex functions $(p>2)$.

Theorem 2.5. *Let $f\in C^{2p}[\zeta_1,\zeta_2]$ $(p>2)$, $(p_1,\dots,p_n)$ be positive real numbers such that $\sum\limits_{\rho=1}^{n} p_\rho = 1$. Also, let $x_1,\dots,x_n$ and $y_1,\dots,y_n\in\mathbb{I}_1$ be such that $x_\rho+y_\rho=2\check{c}$, $x_\rho+x_{n-\rho+1}\leq 2\check{c}$ and $\frac{p_\rho x_\rho+p_{n-\rho+1}x_{n-\rho+1}}{p_\rho+p_{n-\rho+1}}\leq\check{c}$. Moreover, let $\Theta_p(t)$ be the same as defined in Lemma* 1.1. *If* (23) *is valid, then* (24) *is also valid.*

Proof. Proof is similar to that of Theorem 2.2. □

Theorem 2.6. *Let $f\in C^{2p}[\zeta_1,\zeta_2]$ $(p>2)$, $(p_1,\dots,p_n)$ be positive real numbers such that $\sum\limits_{\rho=1}^{n} p_\rho = 1$. Also, let $x_1,\dots,x_n$ and $y_1,\dots,y_n\in\mathbb{I}_1$ such that $x_\rho+y_\rho=2\check{c}$ and $x_\rho+x_{n-\rho+1}$, $\frac{p_\rho x_\rho+p_{n-\rho+1}x_{n-\rho+1}}{p_\rho+p_{n-\rho+1}}\leq\check{c}$. Moreover, let $\Theta_p(t)$ be the same as defined in Lemma* 1.1. *Then:*

(i) *If p is odd, then for every $2p$-convex function $f:[\zeta_1,\zeta_2]\to\mathbb{R}$,* (24) *holds.*

(ii) *Let inequality* (24) *be satisfied. If the function* (25) *is* 3*-convex, the R.H.S of* (24) *is non-negative, we have inequality* (26).

Proof. Proof is similar to that of Theorem 2.5. □

In the next result, Levinson's type inequality is given (for positive weights) under Mercer's condition.

Corollary 2.1. *Let* $f : \mathbb{I}_1 = [\zeta_1, \zeta_2] \to \mathbb{R}$ *be a* $2p$*-convex function,* x_ρ*,* y_ϱ *satisfy* (6) *and the* $\max\{x_1 \dots x_n\} \leq \min\{y_1 \dots y_n\}$*. Also, let* $(p_1, \dots, p_n) \in \mathbb{R}^n$ *such that* $\sum_{\rho=1}^{n} p_\rho = 1$*. Then* (19) *is valid.*

Remark 2.2. Cebyšev, Grüss and Ostrowski-type new bounds related to the obtained generalizations can also be discussed. Moreover, we can also give the related mean value theorems by using non-negative functional (13) to construct new families of n-exponentially convex functions and Cauchy means related to these functionals such as given in Section 4 of [10].

3. Application to Information Theory

The idea of Shannon entropy is the central job of information speculation now and again implied as measure of uncertainty. The entropy of a random variable is described with respect to the probability distribution and can be shown to be a decent measure of randomness. Shannon entropy grants to assess the typical least number of bits expected to encode a progression of pictures subject to the letters all together size and the repeat of the symbols.
Divergences between probability distributions have been familiar with measure of the difference between them. An assortment of sorts of divergences exist, for example the $\mathfrak{f}$-divergences (especially, Kullback–Leibler divergences, Hellinger distance and total variation distance), Rényi divergences, Jensen–Shannon divergences, etc. (see [18, 27]). There are a lot of papers overseeing inequalities and entropies, see, e.g., [1, 14, 16, 26] and references therein. The Jensen inequality is an essential job in a bit of these inequalities. Regardless, Jensen's inequality manages one kind of data points and Levinson's inequality deals with two types of data points.

3.1. Csiszár divergence.

In [12, 13], Csiszár gave the following

Definition 3.1. *Let* f *be a convex function from* $\mathbb{R}^+$ *to* $\mathbb{R}^+$*. Let* $\tilde{\mathbf{r}}, \tilde{\mathbf{k}} \in \mathbb{R}^n_+$ *be such that* $\sum_{\rho=1}^{n} r_\rho = 1$ *and* $\sum_{\rho=1}^{n} k_\rho = 1$*. Then the* f*-divergence functional is defined by*

$$\mathbb{I}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{\rho=1}^{n} k_\rho f\left(\frac{r_\rho}{k_\rho}\right).$$

By defining

$$f(0) := \lim_{x \to 0^+} f(x), \qquad 0f\left(\frac{0}{0}\right) := 0, \qquad 0f\left(\frac{a}{0}\right) := \lim_{x \to 0^+} xf\left(\frac{a}{x}\right), \quad a > 0,$$

he stated that non-negative probability distributions can also be used.
Using the definition of the f-divergence functional, Horv́ath *et al.* [15] gave the following functional.

Definition 3.2. *Let* $\mathbb{I}$ *be an interval contained in* $\mathbb{R}$ *and* $f : \mathbb{I} \to \mathbb{R}$ *be a function. Also, let* $\tilde{\mathbf{r}} = (r_1, \dots, r_n) \in \mathbb{R}^n$ *and* $\tilde{\mathbf{k}} = (k_1, \dots, k_n) \in (0, \infty)^n$ *be such that*

$$\frac{r_\rho}{k_\rho} \in \mathbb{I}, \qquad \rho = 1, \dots, n.$$

Then

$$\hat{\mathbb{I}}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{\rho=1}^{n} k_\rho f\left(\frac{r_\rho}{k_\rho}\right). \tag{27}$$

We apply Theorem 2.2 for the $2p$-convex functions to $\hat{\mathbb{I}}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}})$.

Theorem 3.1. *Let* $\tilde{\mathbf{r}} = (r_1, \dots, r_n) \in \mathbb{R}^n$, $\tilde{\mathbf{w}} = (w_1, \dots, w_m) \in \mathbb{R}^m$, $\tilde{\mathbf{k}} = (k_1, \dots, k_n) \in (0, \infty)^n$ *and* $\tilde{\mathbf{t}} = (t_1, \dots, t_m) \in (0, \infty)^m$ *be such that*

$$\frac{r_\rho}{k_\rho} \in \mathbb{I}, \qquad \rho = 1, \dots, n,$$

and

$$\frac{w_\varrho}{t_\varrho} \in \mathbb{I}, \qquad \varrho = 1, \dots, m.$$

Also, let $f \in C^{2p}[\zeta_1, \zeta_2]$ *be such that* f *is* $2p$*-convex function* (*for odd* p), *then*

$$J_{cis}(f(\cdot)) \geq \sum_{l=1}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_1) J\Big(\Theta_l(\cdot)\Big) + \sum_{l=1}^{p-1} (\zeta_2 - \zeta_1)^{2l} f^{(2l)}(\zeta_2) J\Big(\ddot{\Theta}_l(\cdot)\Big), \tag{28}$$

where

$$\begin{aligned} J_{cis}(f(\cdot)) = & \frac{1}{\sum_{\varrho=1}^m t_\varrho} \hat{\mathbb{I}}_f(\tilde{\mathbf{w}}, \tilde{\mathbf{t}}) - f\left(\sum_{\varrho=1}^m \frac{w_\varrho}{\sum_{\varrho=1}^m t_\varrho} \right) - \frac{1}{\sum_{\rho=1}^n k_\rho} \hat{\mathbb{I}}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) \\ & + f\left(\sum_{\rho=1}^n \frac{r_\rho}{\sum_{\rho=1}^n k_\rho} \right), \end{aligned} \tag{29}$$

$$\begin{aligned} J\Big(\Theta_l(\cdot)\Big) = & \sum_{\rho=1}^m \frac{t_\varrho}{\sum_{\varrho=1}^m t_\varrho} \Theta_l\left(\frac{\zeta_2 - \frac{w_\varrho}{t_\varrho}}{\zeta_2 - \zeta_1} \right) - \Theta_l\left(\frac{\zeta_2 - \sum_{\rho=1}^m \frac{w_\varrho}{\sum_{\varrho=1}^m t_\varrho}}{\zeta_2 - \zeta_1} \right) \\ & - \sum_{\rho=1}^n \frac{k_\rho}{\sum_{\rho=1}^n k_\rho} \Theta_l\left(\frac{\zeta_2 - \frac{r_\rho}{k_\rho}}{\zeta_2 - \zeta_1} \right) + \Theta_l\left(\frac{\zeta_2 - \sum_{\rho=1}^n \frac{r_\rho}{\sum_{\rho=1}^n k_\rho}}{\zeta_2 - \zeta_1} \right), \end{aligned} \tag{30}$$

$$\begin{aligned} J\Big(\ddot{\Theta}_l(\cdot)\Big) = & \sum_{\rho=1}^m \frac{t_\varrho}{\sum_{\varrho=1}^m t_\varrho} \Theta_l\left(\frac{\frac{w_\varrho}{t_\varrho} - \zeta_1}{\zeta_2 - \zeta_1} \right) - \Theta_l\left(\frac{\sum_{\rho=1}^m \frac{w_\varrho}{\sum_{\varrho=1}^m t_\varrho} - \zeta_1}{\zeta_2 - \zeta_1} \right) \\ & - \sum_{\rho=1}^n \frac{k_\rho}{\sum_{\rho=1}^n k_\rho} \Theta_l\left(\frac{\frac{r_\rho}{k_\rho} - \zeta_1}{\zeta_2 - \zeta_1} \right) + \Theta_l\left(\frac{\sum_{\rho=1}^n \frac{r_\rho}{\sum_{\rho=1}^n k_\rho} - \zeta_1}{\zeta_2 - \zeta_1} \right) \end{aligned} \tag{31}$$

and

$$\begin{aligned} J\Big(G_p(t, \cdot)\Big) = & \sum_{\rho=1}^m \frac{t_\varrho}{\sum_{\varrho=1}^m t_\varrho} G_p\left(\frac{\frac{w_\varrho}{t_\varrho} - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) - G_p\left(\frac{\sum_{\rho=1}^m \frac{w_\varrho}{\sum_{\varrho=1}^m t_\varrho} - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) \\ & - \sum_{\rho=1}^n \frac{k_\rho}{\sum_{\rho=1}^n k_\rho} G_p\left(\frac{\frac{r_\rho}{k_\rho} - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right) + G_p\left(\frac{\sum_{\rho=1}^n \frac{r_\rho}{\sum_{\rho=1}^n k_\rho} - \zeta_1}{\zeta_2 - \zeta_1}, \frac{t - \zeta_1}{\zeta_2 - \zeta_1} \right). \end{aligned} \tag{32}$$

Proof. Since G_1 is 3-convex and G_{p-1} is positive for odd p, therefore by using (8), G_p is 3-convex in first variable if p is odd. Hence (17) holds. So using $p_\rho = \frac{k_\rho}{\sum_{\rho=1}^n k_\rho}$, $x_\rho = \frac{r_\rho}{k_\rho}$, $q_\varrho = \frac{t_\varrho}{\sum_{\varrho=1}^m t_\varrho}$, $y_\varrho = \frac{w_\varrho}{t_\varrho}$ in Theorem 2.2, (18) becomes (28), where $\hat{\mathbb{I}}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}})$ is defined in (27) and

$$\hat{\mathbb{I}}_f(\tilde{\mathbf{w}}, \tilde{\mathbf{t}}) := \sum_{\varrho=1}^m t_\varrho f\left(\frac{w_\varrho}{t_\varrho} \right). \tag{33}$$

The theorem is proved. □

3.2. Shannon Entropy.

Definition 3.3 (see [15]). *The Shannon entropy of the positive probability distribution* $\tilde{\mathbf{k}}=(k_1,\dots,k_n)$ *is defined by*

$$\mathcal{S} := -\sum_{\rho=1}^{n} k_\rho \log(k_\rho). \tag{34}$$

Corollary 3.1. *Let* $\tilde{\mathbf{k}} = (k_1,\dots,k_n)$ *and* $\tilde{\mathbf{t}} = (t_1,\dots,t_m)$ *be the positive probability distributions. Also, let* $\tilde{\mathbf{r}} = (r_1,\dots,r_n) \in (0,\infty)^n$ *and* $\tilde{\mathbf{w}} = (w_1,\dots,w_m) \in (0,\infty)^m$.
If the base of log *is greater than* 1 *and p=odd* $(n = 3, 5, \dots)$, *then*

$$\begin{aligned} J_s(\cdot) \le &\sum_{l=1}^{p-1} (\zeta_2-\zeta_1)^{2l} \frac{(-1)^{2l-1}(2l-1)!}{(\zeta_1)^{2l}} J\Big(\Theta_l(\cdot)\Big) \\ &+ \sum_{l=1}^{p-1} (\zeta_2-\zeta_2)^{2l} \frac{(-1)^{2l-1}(2l-1)!}{(\zeta_2)^{2l}} J\Big(\ddot{\Theta}_l(\cdot)\Big), \end{aligned} \tag{35}$$

where

$$\begin{aligned} J_s(\cdot) = &\sum_{\varrho=1}^{m} t_\varrho \log(w_\varrho) + \tilde{\mathcal{S}} - \log\left(\sum_{\varrho=1}^{m} w_\varrho\right) - \sum_{\rho=1}^{n} k_\rho \log(r_\rho) - \mathcal{S} \\ &+ \log\left(\sum_{\rho=1}^{n} r_\rho\right) \end{aligned} \tag{36}$$

and $J\Big(\Theta_l(\cdot)\Big)$, $J\Big(\ddot{\Theta}_l(\cdot)\Big)$, $J\Big(G_p(t,\cdot)\Big)$ *are the same as defined in* (30), (31) *and* (32), *respectively.*

Proof. The function $f(x) = \log(x)$ is $2p$-concave for odd p $(p > 2)$ and the base of log is greater than 1. So, by using Remark 2.1(ii), (18) holds in reverse direction. Therefore using $f(x) = \log(x)$ and $p_\rho = \frac{k_\rho}{\sum_{\rho=1}^{n} k_\rho}$, $x_\rho = \frac{r_\rho}{k_\rho}$, $q_\varrho = \frac{t_\varrho}{\sum_{\varrho=1}^{m} t_\varrho}$, $y_\varrho = \frac{w_\varrho}{t_\varrho}$ in reversed inequality (18), we have (35), where $\mathcal{S}$ is defined in (34) and

$$\tilde{\mathcal{S}} = -\sum_{\varrho=1}^{m} t_\varrho \log(t_\varrho). \qquad \square$$

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