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ON THE GENERALIZED NONMEASURABILITY OF SOME CLASSICAL POINT SETS

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Abstract. The generalized nonmeasurability of certain classical point sets (such as Vitali sets, Bernstein sets, and Hamel bases) is considered in connection with **CH** and **MA**.

This short note is a continuation of our paper [7]. It was shown in [7] that the nonmeasurability in Ulam's sense (i.e., the non-real-valued measurability) of the cardinality continuum is equivalent to some generalized nonmeasurability of Vitali subsets and Bernstein subsets of the real line $\mathbf{R}$. Here it is demonstrated that, assuming the Continuum Hypothesis (**CH**), it becomes possible to essentially strengthen the result obtained in [7], concerning the generalized nonmeasurability of Vitali sets and Bernstein sets.

According to the classical theorem of Erdös and Kakutani [3], the Continuum Hypothesis is equivalent to the following assertion:

There exists a countable family $\{H_i : i \in I\}$ of Hamel bases of $\mathbf{R}$ such that

$$\cup\{H_i : i \in I\} = \mathbf{R} \setminus \{0\}.$$

Starting with this result and using the Banach-Kuratowski matrix [1] (or Ulam's $(\omega\times\omega_1)$-matrix [12] or a countable base of a Luzin subspace of $\mathbf{R}$), one can prove the following statement.

Theorem 1. *Under* **CH**, *there exists a countable family $\{H_j : j \in J\}$ of Hamel bases of $\mathbf{R}$ such that, for every nonzero σ-finite diffused measure μ on $\mathbf{R}$, at least one member of $\{H_j : j \in J\}$ is nonmeasurable with respect to μ.*

In fact, the existence of $\{H_j : j \in J\}$ with the above property implies **CH** (cf., [4], where an analogous result in terms of nonzero σ-finite translation invariant measures on $\mathbf{R}$ is formulated and proved).

It makes sense to examine analogues of Theorem 1 for some other classical point sets. First of all, we mean here the Vitali subsets and Bernstein subsets of $\mathbf{R}$.

Recall that a Vitali set in $\mathbf{R}$ is any selector of the quotient group $\mathbf{R}/\mathbf{Q}$, where $\mathbf{Q}$ denotes the rational subgroup of the additive group $(\mathbf{R}, +)$.

Recall also that a Bernstein set in $\mathbf{R}$ is any set $B \subset \mathbf{R}$ which has the property that, for every nonempty perfect set $P \subset \mathbf{R}$, the relations

$$P \cap B \neq \emptyset, \ P \cap (\mathbf{R} \setminus B) \neq \emptyset$$

hold true.

In many works, the Vitali sets and Bernstein sets are discussed from the measure-theoretical and topological viewpoints (see, e.g., [2,5,6,8–11,13]). Usually, these sets are treated as pathological ones.

In particular, it is well known within **ZFC** set theory that:

(a) if μ is a measure on $\mathbf{R}$ extending the standard Lebesgue measure and invariant under all rational translations of $\mathbf{R}$, then no Vitali set is measurable with respect to μ (cf., [13]);

(b) if μ is the completion of a nonzero σ-finite diffused Borel measure on $\mathbf{R}$, then no Bernstein set is measurable with respect to μ.

Notice that μ in (a) carries some algebraic structure and μ in (b) carries some topological structure. At the same time, suppose that ν is an arbitrary σ-finite measure on $\mathbf{R}$ without any additional structure, and let $\{V_k : k \in K\}$ (respectively, $\{B_k : k \in K\}$) be a finite family of Vitali sets

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(respectively, Bernstein sets). Then there exists a measure ν' on $\mathbf{R}$ extending ν and such that all sets V_k (respectively, all sets B_k) become ν'-measurable. Actually, the same fact remains valid for ν and for an arbitrary finite family $\{Z_k : k \in K\}$ of subsets of $\mathbf{R}$ (see, for example, [5]).

Remark 1. There exists a Vitali set which is measurable with respect to some translation quasi-invariant extension of the Lebesgue measure on $\mathbf{R}$ (see [5, 6]).

Remark 2. If μ is a nonzero σ-finite diffused measure on $\mathbf{R}$ containing in its domain some Bernstein set, then μ cannot be a Radon measure.

For the class $\mathcal{M}(\mathbf{R})$ of all nonzero σ-finite diffused measures on $\mathbf{R}$, we have the next two results (similar to Theorem 1) which show us the generalized nonmeasurability of Vitali sets and Bernstein sets with respect to $\mathcal{M}(\mathbf{R})$.

Theorem 2. *Under* **CH***, there exists a countable family* $\{V_j : j \in J\}$ *of Vitali subsets of* $\mathbf{R}$ *such that, for every nonzero* σ*-finite diffused measure* μ *on* $\mathbf{R}$*, at least one member of* $\{V_j : j \in J\}$ *is nonmeasurable with respect to* μ*.*

Theorem 3. *Under* **CH***, there exists a countable family* $\{B_j : j \in J\}$ *of Bernstein subsets of* $\mathbf{R}$ *such that, for every nonzero* σ*-finite diffused measure* μ *on* $\mathbf{R}$*, at least one member of* $\{B_j : j \in J\}$ *is nonmeasurable with respect to* μ*.*

Both proofs of Theorems 2 and 3 are based on the following auxiliary statement.

Lemma 1. *Let* $\{X_i : i \in I\}$ *be a partition of a ground set* E *such that*

$$(\forall i \in I)(2 \leq \mathrm{card}(X_i) \leq \omega),$$

where ω *denotes the least infinite cardinal number.*

Then the union of any subfamily of $\{X_i : i \in I\}$ *belongs to the* σ*-algebra generated by a countable family of selectors of* $\{X_i : i \in I\}$*.*

Also, the following auxiliary statement is used in the proof of Theorem 3.

Lemma 2. *There exists a partition* $\{Y_t : t \in T\}$ *of* $\mathbf{R}$ *such that:*
(1) $2 \leq \mathrm{card}(Y_t) \leq \omega$ *for each index* $t \in T$*;*
(2) *all selectors of* $\{Y_t : t \in T\}$ *are Bernstein subsets of* $\mathbf{R}$*.*

Remark 3. The assertions of Theorems 2 and 3 can also be established under Martin's Axiom (**MA**). As widely known, **MA** is much weaker than the Continuum Hypothesis, because the conjunction **MA** & ¬**CH** is consistent with **ZFC** set theory. The proofs of the modified versions of Theorems 2 and 3 are based on Lemmas 1 and 2 and on some properties of so-called generalized Luzin subsets of $\mathbf{R}$. As indicated after Theorem 1, **CH** is equivalent to the existence of a countable family $\{H_j : j \in J\}$ of Hamel bases of $\mathbf{R}$ such that, for every nonzero σ-finite diffused measure μ on $\mathbf{R}$, at least one member of $\{H_j : j \in J\}$ is nonmeasurable with respect to μ. We thus see that the case of Hamel bases of $\mathbf{R}$ essentially differs from the cases of Vitali and Bernstein sets in $\mathbf{R}$.

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