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A MEASURE ZERO SET IN THE PLANE WITH ABSOLUTELY NONMEASURABLE LINEAR SECTIONS

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Abstract. It is proved that there exists a translation invariant extension μ of the two-dimensional Lebesgue measure λ_2 on the plane $\mathbf{R}^2$ such that μ is metrically isomorphic to λ_2 and all linear sections of some μ-measure zero set are absolutely nonmeasurable.

Throughout this paper, we use the following fairly standard notation.
$X \triangle Y$ is the symmetric difference of two sets X and Y;
$\mathrm{dom}(f)$ is the domain of a function f;
$\mathrm{card}(X)$ is the cardinality of a set X;
ω is the least infinite ordinal (cardinal) number;
$\mathbf{R}$ is the real line equipped with the group of all its translations;
$\mathbf{c}$ is the cardinality of the continuum, i.e., $\mathbf{c}$ is $\mathrm{card}(\mathbf{R})$;
λ is the standard one-dimensional Lebesgue measure on $\mathbf{R}$;
$\mathbf{R}^n$ is the Euclidean n-dimensional space equipped with the group of all its translations;
λ_n is the standard n-dimensional Lebesgue measure on $\mathbf{R}^n$ (in particular, $\lambda_1 = \lambda$).

As is widely known, if Z is a λ_2-measure zero subset of the Euclidean plane $\mathbf{R}^2$, then almost all (with respect to λ) linear sections of Z, parallel to the coordinate axes, i.e., λ-almost all sets of the form

$$\{y : (x,y) \in Z\} \qquad (x \in \mathbf{R}),$$
$$\{x : (x,y) \in Z\} \qquad (y \in \mathbf{R}),$$

are of λ-measure zero. This fact is a direct consequence of Fubini's classical theorem. More generally, it follows from the same theorem that if l is any straight line in $\mathbf{R}^2$, then λ-almost all linear sections of Z, parallel to l, are of λ-measure zero.

The main goal of the present paper is to show that for a certain translation invariant extension μ of λ_2, which is metrically isomorphic to λ_2, the above-mentioned fact fails to be true in a very strong sense.

For our further purposes, we need some auxiliary notions from the general theory of invariant (quasi-invariant) measures (see, e.g., [1,6,11]).

Let E be an infinite ground set and let G be a group of transformations of E.

A nonzero complete σ-finite measure θ on E is called quasi-invariant with respect to G (in short, G-quasi-invariant) if the domain of θ is a G-invariant σ-algebra of subsets of E and the family of all θ-measure zero sets is a G-invariant σ-ideal of subsets of E.

A set $X \subset E$ is called almost G-invariant in E if for every transformation $g \in G$ one has

$$\mathrm{card}(g(X) \triangle X) < \mathrm{card}(E).$$

Almost G-invariant subsets of E play an important role in many topics of general topology and of the theory of invariant (quasi-invariant) measures (see, e.g., [1–4, 6, 10, 11]).

A set $Y \subset E$ is called G-absolutely nonmeasurable if for every nonzero σ-finite G-quasi-invariant measure μ on E one has $Y \notin \mathrm{dom}(\mu)$.

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In other words, $Y \subset E$ is G-absolutely nonmeasurable if Y is absolutely nonmeasurable with respect to the class of all nonzero σ-finite G-quasi-invariant measures on E.

In particular, if E is a group, then one can take as G the group of all left translations of E. In such a case, identifying E and G, one can speak of E-absolutely nonmeasurable subsets of E.

Lemma 1. *Let $(G, +)$ be an uncountable commutative group identified with the group of all its translations, and let Y be a subset of G.*

The following two assertions are equivalent:

(1) *there exists a countable family $\{g_j : j \in J\}$ of elements of G such that*

$$\cup\{g_j + Y : j \in J\} = G;$$

(2) *there exists a G-absolutely nonmeasurable set entirely contained in Y.*

For a detailed proof of Lemma 1, see [7].

We shall use this lemma in the special case where G is a group, isomorphic to the additive group of $\mathbf{R}$.

More precisely, let l be any straight line in the plane $\mathbf{R}^2$. For l, we may consider the family G_l of all those translations g of $\mathbf{R}^2$ which satisfy $g(l) = l$. In other words, G_l is the stabilizer of l in the group of all translations of $\mathbf{R}^2$. Also, l is equipped with the isomorphic image μ_l of λ and μ_l is invariant with respect to G_l. But there are many other measures on l which are invariant (or, more generally, quasi-invariant) under G_l. Let us denote by $\mathcal{M}_l$ the class of all nonzero σ-finite G_l-quasi-invariant measures on l (notice that the domains of such measures are various G_l-invariant σ-algebras of subsets of l).

According to the general definition presented above, we say that a set $Y \subset l$ is G_l-absolutely nonmeasurable in l if Y is nonmeasurable with respect to each measure from the class $\mathcal{M}_l$.

Using Lemma 1, it is not hard to show the validity of the next auxiliary statement.

Lemma 2. *Let l be a straight line in the plane $\mathbf{R}^2$ and let X be a set in l such that* $\mathrm{card}(l \setminus X) < \mathbf{c}$*.*

Then X contains a G_l-absolutely nonmeasurable subset of l.

Proof. Since $\mathrm{card}(l \setminus X) < \mathbf{c}$, there is an element $g \in G_l$ such that

$$(g + (l \setminus X)) \cap (l \setminus X) = \emptyset$$

or, equivalently,

$$(g + X) \cup X = l.$$

Now, taking into account Lemma 1, we conclude that X contains some G_l-absolutely nonmeasurable set. □

Lemma 3. *There exists a set $Z \subset \mathbf{R}^2$ which satisfies the following three conditions:*

(1) *Z is almost $\mathbf{R}^2$-invariant, i.e.,* $\mathrm{card}((h + Z)\triangle Z) < \mathbf{c}$ *for every $h \in \mathbf{R}^2$;*

(2) *the inner λ_2-measure of the set Z is equal to zero;*

(3) *for any straight line l in $\mathbf{R}^2$, the set $l \setminus Z$ has cardinality strictly less than* $\mathbf{c}$.

Proof. We follow the argument used in [5].

Let α be the least ordinal number of cardinality $\mathbf{c}$. We introduce the following notation.

$\{l_\xi : \xi < \alpha\}$ is the injective family of all straight lines in $\mathbf{R}^2$.

$\{F_\xi : \xi < \alpha\}$ is the family of all closed subsets of $\mathbf{R}^2$ having strictly positive λ_2-measure.

$\{G_\xi : \xi < \alpha\}$ is a family of groups of translations of $\mathbf{R}^2$ such that:

(a) $\{G_\xi : \xi < \alpha\}$ is increasing by the standard inclusion relation;

(b) $\mathrm{card}(G_\xi) \le \mathrm{card}(\xi) + \omega$ for each ordinal $\xi < \alpha$;

(c) $\cup\{G_\xi : \xi < \alpha\}$ is the group of all translations of $\mathbf{R}^2$.

Further, we construct by transfinite recursion a family $\{z'_\xi : \xi < \alpha\}$ of points of $\mathbf{R}^2$.

Suppose that for an ordinal $\xi < \alpha$, the partial family $\{z'_\zeta : \zeta < \xi\}$ has already been defined. Let us put

$$L_\xi = G_\xi(\cup\{l_\zeta : \zeta < \xi\}) \cup G_\xi(\{z'_\zeta : \zeta < \xi\}).$$

Keeping in mind the fact that $\lambda_2(F_\xi) > 0$, it is not hard to show that there exists a point $z' \in F_\xi \setminus L_\xi$. Then we define $z'_\xi = z'$.

Proceeding in this manner, we obtain the required α-sequence $\{z'_\xi : \xi < \alpha\}$ of points of $\mathbf{R}^2$. It follows from the above construction that the set

$$Z' = \cup\{G_\xi(z'_\xi) : \xi < \alpha\}$$

is almost $\mathbf{R}^2$-invariant and λ_2-thick in $\mathbf{R}^2$. Moreover, it is not difficult to check that

$$\mathrm{card}(Z' \cap l) < \mathbf{c}$$

for every straight line l in $\mathbf{R}^2$. These properties of Z' imply that the set

$$Z = \mathbf{R}^2 \setminus Z'$$

satisfies all conditions (1), (2) and (3) of Lemma 3, so is as required. □

Lemma 4. *Let Z be a subset of $\mathbf{R}^2$ as in Lemma* 3.

There exists a complete translation invariant measure μ on $\mathbf{R}^2$ such that:

(1) *μ is an extension of λ_2;*

(2) *$Z \in \mathrm{dom}(\mu)$ and $\mu(Z) = 0$;*

(3) *every μ-measurable set $X \subset \mathbf{R}^2$ admits a representation in the form*

$$X = (X_0 \cup A) \setminus B,$$

where $X_0 \in \mathrm{dom}(\lambda_2)$ and $\mu(A) = \mu(B) = 0$ (in particular, the measures μ and λ_2 are metrically isomorphic).

Proof. Since Z satisfies conditions (1), (2) and (3) of Lemma 3, the required measure μ is obtained in the standard manner, by applying Marczewski's method of extending measures (see, e.g., [8,9,11]). Moreover, slightly modifying the transfinite construction of Z, it can be established that μ is a measure invariant under the group of all isometric transformations of $\mathbf{R}^2$. □

Using the above lemmas, we can prove the following statement.

Theorem 1. *For the measure μ indicated in Lemma* 4*, there exists a set $W \subset \mathbf{R}^2$ such that:*

(1) *$W \subset Z$ and, consequently, $\mu(W) = 0$;*

(2) *for any straight line l in $\mathbf{R}^2$, the set $l \cap W$ is G_l-absolutely nonmeasurable.*

Let α be the least ordinal number of cardinality $\mathbf{c}$. We again denote by $\{l_\xi : \xi < \alpha\}$ the injective family of all straight lines in $\mathbf{R}^2$.

Using the method of transfinite recursion, we construct a disjoint family $\{W_\xi : \xi < \alpha\}$ of sets which fulfil the following two conditions:

(a) $W_\xi \subset l_\xi \cap Z$ for each ordinal $\xi < \alpha$;

(b) W_ξ is G_{l_ξ}-absolutely nonmeasurable for each ordinal $\xi < \alpha$.

Assume that, for an ordinal $\xi < \alpha$, the partial disjoint family $\{W_\zeta : \zeta < \xi\}$ of sets has already been constructed so that

$$W_\zeta \subset l_\zeta \ (\zeta < \xi).$$

Take the straight line l_ξ and consider the set

$$P_\xi = (Z \cap l_\xi) \setminus \cup\{l_\zeta : \zeta < \xi\}.$$

Since $\mathrm{card}(l_\xi \setminus Z) < \mathbf{c}$, it is not difficult to verify that

$$\mathrm{card}(l_\xi \setminus P_\xi) < \mathbf{c}.$$

According to Lemma 2, there exists a set $T \subset P_\xi$ which is G_{l_ξ}-absolutely nonmeasurable. We then define $W_\xi = T$.

Proceeding in this manner, we get the disjoint family of sets $\{W_\xi : \xi < \alpha\}$. Finally, putting

$$W = \cup\{W_\xi : \xi < \alpha\},$$

we obtain the set W satisfying conditions (1) and (2) of Theorem 1.

The next auxiliary statement generalizes Lemma 2 to the case of $\mathbf{R}^n$.

Lemma 5. *Let $n \geq 1$ be a natural number and let $\{\Gamma_j : j \in J\}$ be a family of affine hyperplanes in the Euclidean space $\mathbf{R}^n$ such that* $\mathrm{card}(J) < \mathbf{c}$.

Then the set $\mathbf{R}^n \setminus \cup\{\Gamma_j : j \in J\}$ contains an $\mathbf{R}^n$-absolutely nonmeasurable subset.

This lemma can be deduced from the general Lemma 1.

Using Lemma 5, we obtain an analog of Theorem 1 for the space $\mathbf{R}^n$ and for the Lebesgue measure λ_n, where $n \geq 3$.

Theorem 2. *For any natural number $n \geq 3$, there exist a complete measure ν on $\mathbf{R}^n$ and a set $V \subset \mathbf{R}^n$ such that:*

(1) *ν extends λ_n and is invariant under the group of all isometric transformations of $\mathbf{R}^n$;*

(2) *ν is metrically isomorphic to λ_n;*

(3) $\nu(V) = 0$*;*

(4) *for every affine hyperplane Γ in $\mathbf{R}^n$, the set $V \cap \Gamma$ is absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation quasi-invariant measures on Γ.*

A set $U \subset \mathbf{R}^n$ is called $\mathbf{R}^n$-negligible in $\mathbf{R}^n$ if U satisfies the following two relations:

(i) there exists at least one nonzero σ-finite $\mathbf{R}^n$-quasi-invariant measure θ such that $U \in \mathrm{dom}(\theta)$ (equivalently, U is not $\mathbf{R}^n$-absolutely nonmeasurable);

(ii) for every σ-finite $\mathbf{R}^n$-quasi-invariant measure θ' such that $U \in \mathrm{dom}(\theta')$, the equality $\theta'(U) = 0$ holds true.

Some structural properties of $\mathbf{R}^n$-negligible sets are considered in [4] and [6].

It would be interesting to study the question of whether there exists an $\mathbf{R}^n$-negligible set $U \subset \mathbf{R}^n$ such that, for any affine hyperplane Γ in $\mathbf{R}^n$, the set $U \cap \Gamma$ is absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation quasi-invariant measures on Γ.

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