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COMPLEX REPRESENTATION IN THE PLANE THEORY OF VISCOELASTICITY AND ITS APPLICATIONS

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Abstract. The complex representation in the plane theory of viscoelasticity and Kolosov–Muskhelishvili's type formulas in the conditions of plane deformation and in the plane stressed state are obtained. Investigation of various possible forms of viscoelastic correlations can be found in [1–4, 6–9, 11]. Certain contact problems of viscoelastic bodies and the corresponding integro-differential equations are studied in [5, 12, 13]. The present paper considers the problem of a rigid punch on the boundary of a half-plane in the presence of fraction.

1. Introduction

Basic equations of the creep theory expressing the connection between stresses and deformations of hereditary aging media under small deformations have the form [1, 4, 11]

$$\begin{gathered} 2e_{ij}\left(t,r\right)=\frac{s_{ij}\left(t,r\right)}{G\left(t\right)}-\int\limits_{t_0}^{t}s_{ij}\left(\tau,r\right)K_1\left(t,\tau\right)d\tau \quad \left((i,j)=1,2,3\right), \\ \varepsilon\left(t,r\right)=\frac{\sigma\left(t,r\right)}{E^*\left(t\right)}-\int\limits_{t_0}^{t}\sigma\left(\tau,r\right)K_2\left(t,\tau\right)d\tau, \end{gathered} \tag{1.1}$$

where t is time, r is the radius-vector of the point, t_0 is the age of the material element at the moment of loading, $s_{ij}(t,r)$ and $e_{ij}(t,r)$ are, respectively, the tensor deviator components of stress and deformation, $G(t)$ is the instantaneous shear modulus, $E^*(t)$ is the instantaneous volumetric deformation, $\varepsilon\left(t,r\right)$ is the mean deformaton, $\sigma\left(t,r\right)$ is the mean stress, $K_1\left(t,\tau\right)$ and $K_2\left(t,\tau\right)$ are the kernels of shearing and volumetric creep deformation, respectively, which can be represented in the form

$$K_1\left(t,\tau\right)=\frac{\partial}{\partial\tau}\left[\frac{1}{G\left(\tau\right)}+\omega\left(t,\tau\right)\right],\qquad K_2\left(t,\tau\right)=\frac{\partial}{\partial\tau}\left[\frac{1}{E^*\left(\tau\right)}+C^*\left(t,\tau\right)\right],$$

where $\omega\left(t,\tau\right)$ and $C^*\left(t,\tau\right)$, are the creep measures of shearing and volumetric deformation. As is known, the components of stress and deformation tensors σ_{ij} and ε_{ij} are connected with the components of the corresponding deviator as follows:

$$s_{ij}=\sigma_{ij}-\sigma\delta_{ij},\quad \sigma=\frac{1}{3}\quad\sigma_{ii},\quad e_{ij}=\varepsilon_{ij}-\varepsilon\delta_{ij},\quad \varepsilon=\frac{1}{3}\varepsilon_{ii};$$

here, δ_{ij} is the Kronecker symbol.

For one-dimensional stressed state of tension-compression we have

$$\varepsilon_{ii}\left(t,r\right)=\frac{\sigma_{ii}\left(t,r\right)}{E\left(t\right)}-\int\limits_{t_0}^{t}\sigma_{ii}\left(\tau,r\right)K\left(t,\tau\right)d\tau, \tag{1.2}$$

$K\left(t,\tau\right)=\frac{\partial}{\partial\tau}\left[\frac{1}{E(\tau)}+C\left(t,\tau\right)\right]$ is the creep kernel of tension-compression deformation, $E(t)$ is the instantaneous Young modulus, $C(t,\tau)$ is the creep measure of tension-compression deformation. The

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following correlations are known:

$$G(t)=\frac{E(t)}{2(1+\nu_1(t))},\quad E^*(t)=\frac{E(t)}{1-2\nu_1(t)},$$
$$\omega(t,\tau)=2[1+\nu_2(t,\tau)]C(t,\tau),\quad C^*(t,\tau)=[1+2\nu_2(t,\tau)]C(t,\tau),$$

where $\nu_1(t)$ is the Poisson coefficient of elasto-instantaneous deformation, $\nu_2(t,\tau)$ is the Poisson coefficient of creep deformation.

Elasto-instantaneous modules are the positive, continuous, bounded and monotonically increasing functions on every $t_0\le\tau<\infty$, therefore they may satisfy the following conditions:

$$\frac{dE(\tau)}{d\tau}>0\quad(\tau<\infty),\quad E(\tau)\sim E_0<\infty\quad(\tau\to\infty),\quad E(t_0)>0,$$

where E_0 is an elastic modulus of the material, rather large in age. Creep measures are the nonnegative, continuous functions of two variables with the following properties: $t_0\le\tau\le t\le\infty$.

$$C(t,t)=0,\quad C(t,\tau)\sim\varphi(\tau)\quad(t\to\infty),$$
$$C(t,\tau)\sim\psi(t-\tau)\quad(\tau\to\infty,\quad\tau\le t),\quad\frac{\partial C(t,\tau)}{\partial t}>0,\quad\frac{\partial C(t,\tau)}{\partial\tau}<0\quad(\tau\le t<\infty).$$

$\varphi(\tau)$ defines the aging process of the material, and the function $\psi(y)$ characterizes hereditary properties of the material, moreover,

$$\frac{d\varphi(\tau)}{d\tau}<0\quad(\tau<\infty),\quad\varphi(\tau)\sim C_0>0\quad(\tau\to\infty),\quad\varphi(t_0)<\infty,$$
$$\frac{d\psi(y)}{dy}>0\quad(y<\infty),\quad\psi(y)\sim C_0\quad(y\to\infty),\quad\psi(0)=0,$$

where C_0 is the limiting creeping measure for the material, highly large in age.

In view of the above-mentioned properties, the creeping measure $C(t,\tau)$ is usually representable in the form [4]:

$$C(t,\tau)=\varphi(\tau)\left(1-e^{-\gamma(t-\tau)}\right),\qquad\gamma=\text{const}.\tag{1.3}$$

The correlations expressing stress components through deformation components are obtained from (1.1) and (1.2) by solving the Volterra integral equations. From (1.2) we get [11]:

$$\frac{\sigma_{ii}(t,r)}{E(t)}=\varepsilon_{ii}(t,r)+\int_{t_0}^{t}\varepsilon_{ii}(\tau,r)R(t,\tau)d\tau.$$

Here, $R(t,\tau)$ is called a kernel of relaxation, or in other words, the resolvent of creeping kernel $K(t,\tau)$.

2. Complex Representations in the Plane Theory of Viscoelasticity

(a) For a plane stressed state $\sigma_{13}=\sigma_{23}=\sigma_{33}=0$, all the rest components of stresses together with the components of deformation are the functions of variables (t,x,y), therefore correlations (1.1) take the form

$$\varepsilon_{ij}(t,x,y)=\frac{1+\nu_1(t)}{E(t)}\sigma_{ij}-\int_{t_0}^{t}\sigma_{ij}\frac{\partial}{\partial\tau}\left[\frac{1+\nu_1(\tau)}{E(\tau)}+(1+\nu_2(t,\tau))C(t,\tau)\right]d\tau$$
$$-\delta_{ij}\frac{\nu_1(t)}{E(t)}(\sigma_{11}+\sigma_{22})+\delta_{ij}\int_{t_0}^{t}\frac{\partial}{\partial\tau}\left[\frac{\nu_1(\tau)}{E(\tau)}+\nu_2(t,\tau)C(t,\tau)\right](\sigma_{11}+\sigma_{22})d\tau,\quad i,j=1,2,\tag{2.1}$$
$$\varepsilon_{33}(t,x,y)=-\frac{\nu_1(t)}{E(t)}(\sigma_{11}+\sigma_{22})+\int_{t_0}^{t}\frac{\partial}{\partial\tau}\left[\frac{\nu_1(\tau)}{E(\tau)}+\nu_2(t,\tau)C(t,\tau)\right](\sigma_{11}+\sigma_{22})d\tau.$$

(b) For a plane deformation, ε_{11} and ε_{22} are independent of z, $\varepsilon_{33}(t,x,y)=0$. Assuming $E(t)=E=\text{const}$, $\nu_1(t)=\nu_2(t,\tau)=\nu=\text{const}$, we obtain $\sigma_{33}=\nu(\sigma_{11}+\sigma_{22})$, and equalities (1.1) take the form

$$\varepsilon_{ij}(t,x,y)=\frac{1+\nu}{E}\sigma_{ij}-(1+\nu)\int\limits_{t_0}^{t}\sigma_{ij}\frac{\partial}{\partial\tau}C(t,\tau)\,d\tau-\delta_{ij}\frac{\nu(1+\nu)}{E}(\sigma_{11}+\sigma_{22})$$
$$+\delta_{ij}\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)(\sigma_{11}+\sigma_{22})\,d\tau,\quad i,j=1,2. \tag{2.2}$$

Expressions (2.1) and (2.2) are the analogues of Hook's law in the theory of viscoelasticity, i.e., they establish a connection between the components of deformation and stress tensors in the conditions of plane deformation and plane stressed state, respectively.

In the absence of body forces, the equilibrium equations take the form

$$\frac{\partial\sigma_{11}(t,x,y)}{\partial x}+\frac{\partial\sigma_{12}(t,x,y)}{\partial y}=0,\qquad\frac{\partial\sigma_{21}(t,x,y)}{\partial x}+\frac{\partial\sigma_{22}(t,x,y)}{\partial y}=0.$$

As is known [10], these equalities result in

$$\sigma_{11}(t,x,y)=\frac{\partial^2U(t,x,y)}{\partial y^2},\quad\sigma_{22}(t,x,y)=\frac{\partial^2U(t,x,y)}{\partial x^2},\quad\sigma_{12}(t,x,y)=-\frac{\partial^2U(t,x,y)}{\partial x\partial y},$$

where $U(t,x,y)$ is the stress function or the Airy function. $\Delta\Delta U=0$, $\Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

Equalities (2.2) yield

$$\varepsilon_{11}(t,x,y)=\frac{(1+\nu)\left(\Delta U-\frac{\partial^2U}{\partial x^2}\right)}{E}-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(\Delta U-\frac{\partial^2U}{\partial x^2}\right)d\tau-\frac{\nu(1+\nu)}{E}\Delta U$$
$$+\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\,\Delta U\,d\tau,$$
$$\varepsilon_{22}(t,x,y)=\frac{(1+\nu)\left(\Delta U-\frac{\partial^2U}{\partial y^2}\right)}{E}-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(\Delta U-\frac{\partial^2U}{\partial y^2}\right)d\tau-\frac{\nu(1+\nu)}{E}\Delta U$$
$$+\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\,\Delta U\,d\tau. \tag{2.3}$$

Introducing the notation $\Delta U\equiv P$ and considering holomorphic functions $F(z,t)=P+iQ$ ($\Delta P=0$, $\Delta Q=0$) and $\varphi(z,t)=p+iq=\frac{1}{4}\int F(z,t)\,dz$, we have $P=4\frac{\partial p}{\partial x}=4\frac{\partial q}{\partial y}$, whence (2.3) takes the form

$$\begin{aligned}\varepsilon_{11}(t,x,y)=\frac{\partial u_1}{\partial x}&=\frac{(1+\nu)\left(4\frac{\partial p}{\partial x}-\frac{\partial^2 U}{\partial x^2}\right)}{E}-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(4\frac{\partial p}{\partial x}-\frac{\partial^2 U}{\partial x^2}\right)d\tau\\&-\frac{4\nu(1+\nu)}{E}\frac{\partial p}{\partial x}+4\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\frac{\partial p}{\partial x}d\tau,\\\varepsilon_{22}(t,x,y)=\frac{\partial u_2}{\partial y}&=\frac{(1+\nu)\left(4\frac{\partial q}{\partial y}-\frac{\partial^2 U}{\partial y^2}\right)}{E}-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(4\frac{\partial q}{\partial y}-\frac{\partial^2 U}{\partial y^2}\right)d\tau\\&-\frac{4\nu(1+\nu)}{E}\frac{\partial q}{\partial y}+4\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\frac{\partial q}{\partial y}d\tau,\end{aligned}\tag{2.4}$$

where u_1, u_2 are displacement components.

As a result of integration of each of the correlations (2.4), we get

$$\begin{aligned}u_1&=\frac{(1+\nu)\left(4p-\frac{\partial U}{\partial x}\right)}{E}-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(4p-\frac{\partial U}{\partial x}\right)d\tau\\&-\frac{4\nu(1+\nu)}{E}p+4\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)p\,d\tau+f_1(y,t),\\u_2&=\frac{(1+\nu)\left(4q-\frac{\partial U}{\partial y}\right)}{E}-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left(4q-\frac{\partial U}{\partial y}\right)d\tau\\&-\frac{4\nu(1+\nu)}{E}q+4\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)q\,d\tau+f_2(x,t).\end{aligned}\tag{2.5}$$

Taking into account the third equality of (2.2), it follows that

$$f'_{1y}(y,t)+f'_{2x}(x,t)=0,$$

from which $f_1(y,t)=\varepsilon yt+\alpha$, $f_2(x,t)=-\varepsilon xt+\beta$, i.e., $f_1(y,t)$ and $f_2(x,t)$ provide a rigid displacement of the body which can be neglected. From equality (2.5) we have

$$\begin{aligned}u_1+iu_2&=\frac{(1+\nu)}{E}\left(4\varphi(z,t)-\left(\frac{\partial U}{\partial x}+i\frac{\partial U}{\partial y}\right)\right)\\&-(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\left[4\varphi(z,\tau)-\left(\frac{\partial U}{\partial x}+i\frac{\partial U}{\partial y}\right)\right]d\tau\\&-\frac{4\nu(1+\nu)}{E}\varphi(z,t)+4\nu(1+\nu)\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C(t,\tau)\varphi(z,\tau)d\tau.\end{aligned}\tag{2.6}$$

As is known [10], from a general solution of biharmonic equation, the Goursat formula $U=\mathrm{Re}\,[\overline{z}\varphi(z,t)+\chi(z,t)]$, we find that $\frac{\partial U}{\partial x}+i\frac{\partial U}{\partial y}=\varphi(z,t)+z\overline{\varphi'(z,t)}+\overline{\psi(z,t)}$, where $\frac{\partial\chi(z,t)}{\partial z}=\psi(z,t)$, $\varphi(z,t)$ and $\psi(z,t)$ are holomorphic functions of the variable $z=x+iy$.

If we introduce the notation

$$(I-L)\,g\,(t)=\frac{g\,(t)}{E}-\int\limits_{t_0}^{t}\frac{\partial}{\partial\tau}C\,(t,\tau)\,g\,(\tau)\,d\tau,$$

then expression (2.6) can be written as

$$u_1+iu_2=(1+\nu)\,(I-L)\left((3-4\nu)\varphi\,(z,t)-(z\overline{\varphi'\,(z,t)}+\overline{\psi\,(z,t)})\right), \tag{2.7}$$

and for a plane stressed state, an analogous reasoning results in

$$u_1+iu_2=(I-L)\left((3-\nu)\varphi\,(z,t)-(1+\nu)\,(z\overline{\varphi'\,(z,t)}+\overline{\psi\,(z,t)})\right). \tag{2.8}$$

Correlations (2.7) and (2.8) together with the relations

$$\sigma_{11}+\sigma_{22}=4\left[\Phi\,(z,t)+\overline{\Phi\,(z,t)}\right],\qquad \sigma_{22}-\sigma_{11}+2i\sigma_{12}=2\left[\overline{z}\Phi'\,(z,t)+\Psi\,(z,t)\right],$$

where $\Phi\,(z,t)=\varphi'\,(z,t)$, $\Psi\,(z,t)=\psi'\,(z,t)$, are the analogues of the well-known Kolosov–Muskhelishvili's formulas in the theory of viscoelasticity.

3. Solution of the Punch Problem for a Half-plane

Let in the conditions of plane deformation a viscoelastic body occupy a half-plane $y<0$ which we denote by S^-, so the body S^- leaves on the right when moving along the ox-axis in a positive direction. We denote the upper half-plane by S^+ and the ox-axis by L.

Assume also that the principal vector (X,Y) of outer forces applied to the boundary is finite, stresses and rotations vanish at infinity. Thus, for large $|z|$, we have

$$\Phi\,(z,t)=\frac{X+iY}{2\pi z}+o\left(\frac{1}{z}\right),\quad \Phi'\,(z,t)=-\frac{X+iY}{2\pi z^2}+o\left(\frac{1}{z^2}\right),\quad \Psi\,(z,t)=\frac{X-iY}{2\pi z}+o\left(\frac{1}{z}\right).$$

For a half-plane, Kolosov–Muskhelishvili's formulas take the form [10]:

$$\sigma_{12}-\sigma_{11}+2i\sigma_{12}=2\left[\Phi'\,(z,t)\,(\overline{z}-z)-\overline{\Phi}\,(z,t)-\Phi\,(z,t)\right],$$

$$\sigma_{22}-i\sigma_{12}=\Phi\,(z,t)-\Phi\,(\overline{z},t)+(z-\overline{z})\,\overline{\Phi'\,(z,t)}, \tag{3.1}$$

$$u'_1+iu'_2=(1+\nu)\,(I-L)\left[(3-4\nu)\,\Phi\,(z,t)+\Phi\,(\overline{z},t)+(\overline{z}-z)\,\overline{\Phi'\,(z,t)}\right]. \tag{3.2}$$

Equality (3.2) is written for the case of plane deformation. The prime denotes the derivative with respect to the variable z, and in the sequel, the dot will denote the derivative with respect to the variable t.

A punch with a base of given shape, with a force directed vertically downwards, acts along the segment $L'=[a;b]$ of the boundary. Let the punch displacement along the boundary normal be translational (vertically downwards) in the conditions of friction. The boundary conditions have the form

$$T\,(x,t)=kP\,(x,t)\,,\qquad x\in L', \tag{3.3}$$

$$v^-\,(x,t)=f\,(x,t)+\text{const},\qquad x\in L', \tag{3.4}$$

$$T\,(x,t)=P\,(x,t)=0,\qquad x\in L-L', \tag{3.5}$$

where $f\,(x,t)$ is the given function defining the punch profile at the moment $t=t_0$, i.e., $y=f\,(x,t_0)$ is the punch profile equation.

Let $f'\,(x,t)$ satisfy the Hölder (H) condition with respect to the variable x, and $P_0\,(t)=\int\limits_a^b P\,(x,t)\,dx$, $T_0\,(t)=kP_0\,(t)$. From (3.1) and (3.2), passing to the boundary values as $y\to 0-$, we obtain

$$Y_y-iX_y=\Phi^-\,(x,t)-\Phi^+\,(x,t)\,,$$

$$u'+iv'=(1+\nu)\,(I-L)\left[(3-4\nu)\,\Phi^-\,(x,t)+\Phi^+\,(x,t)\right],$$

whence, in view of the boundary conditions (3.3)–(3.5), we have

$$(1-ik)\,\Phi^{+}(x,t)+(1+ik)\,\overline{\Phi}^{+}(x,t)=(1-ik)\,\Phi^{-}(x,t)+(1+ik)\,\overline{\Phi}^{-}(x,t)\,, \tag{3.6}$$

$$(1+\nu)\,(I-L)\left[(3-4\nu)\,\Phi^{-}(x,t)+\Phi^{+}(x,t)-(3-4\nu)\,\overline{\Phi}^{+}(x,t)-\overline{\Phi}^{-}(x,t)\right]=2if'(x,t)\,. \tag{3.7}$$

From (3.6), according to the Liouville theorem, $(1-ik)\,\Phi(z,t)+(1+ik)\,\overline{\Phi}(z,t)=0$. Taking into account the last correlation in (3.7), we obtain

$$(I-L)\left[\Phi^{+}(x,t)-g\Phi^{-}(x,t)\right]=f_0(x,t)\,, \tag{3.8}$$

where

$$g=-\frac{(3-4\nu)\,(1+ik)+1-ik}{1+ik+(3-4\nu)\,(1-ik)},\quad f_0(x,t)=\frac{2i\,(1+ik)}{(1+\nu)\,(1+ik+(3-4\nu)\,(1-ik))}f'(x,t)\,.$$

Introducing the notation

$$\Gamma(x,t)=\Phi^{+}(x,t)-g\Phi^{-}(x,t)\,, \tag{3.9}$$

the Volterra integral equation (3.8) takes the form

$$(I-L)\,\Gamma(x,t)=f_0(x,t)\,. \tag{3.10}$$

Based on (1.3), the integral equation (3.10) reduces to the ordinary differential equation of second order

$$\ddot{\Gamma}(x,t)+\gamma\alpha(t)\,\dot{\Gamma}(x,t)=A(x,t) \tag{3.11}$$

with the following initial conditions

$$\begin{cases}\Gamma(x,t_0)=Ef_0(x,t_0)\,,\\ \dot{\Gamma}(x,t_0)=E\dot{f}_0(x,t_0)-\gamma E^2\varphi(t_0)\,f_0(x,t_0)\,,\end{cases} \tag{3.12}$$

where $\alpha(t)\equiv 1+E\varphi(t)$, $A(x,t)\equiv E\left[\ddot{f}_0(x,t)+\gamma\dot{f}_0(x,t)\right]$.

A solution of equations (3.11) and (3.12) is represented in the form

$$\Gamma(x,t)=C(x)\int_{t_0}^{t}\delta(\tau)\,d\tau+\int_{t_0}^{t}\delta(\tau)\left(\int_{t_0}^{\tau}\frac{A(x,s)\,ds}{\delta(s)}\right)d\tau+C_1(x)\,, \tag{3.13}$$

where

$$C(x)=E\dot{f}_0(x,t_0)-\gamma E^2\varphi(t_0)\,f_0(x,t_0)\,,\quad C_1(x)=Ef_0(x,t_0)\,,$$

$$\delta(t)=\exp\left\{-\gamma\int_{t_0}^{t}\alpha(\tau)\,d\tau\right\}.$$

Respectively, from (3.9) we obtain the following problem of linear conjugation:

$$\Phi^{+}(x,t)=g\Phi^{-}(x,t)+\Gamma(x,t)\,, \tag{3.14}$$

where $\Gamma(x,t)$ is defined by equality (3.13).

Introducing the constant α defined by the equality

$$tg\pi\alpha=k\frac{1-2\nu}{2\,(1-\nu)}\quad 0<\alpha<\frac{1}{2},\quad \text{we get}\quad g=-e^{2\pi i\alpha}.$$

Any solution of the homogeneous problem will be [10]

$$\chi_0(z)=(z-a)^{-\frac{1}{2}-\alpha}(b-z)^{-\frac{1}{2}+\alpha}.$$

Finally, a general solution of problem (3.14) takes the form

$$\Phi(z,t)=\frac{\chi_0(z)}{2\pi i}\int_a^b\frac{\Gamma(x,t)\,dx}{\chi_0^{+}(x)\,(x-z)}+\chi_0(z)\,\widetilde{C}(t), \tag{3.15}$$

where the function $\widetilde{C}(t)$ to be determined.

Under the expression $(z-a)^{-\frac{1}{2}-\alpha}(b-z)^{-\frac{1}{2}+\alpha}$ we mean a branch which is holomorphic on the segment $[a,b]$ and takes a real positive value $(x-a)^{\frac{1}{2}+\alpha}(b-x)^{\frac{1}{2}-\alpha}$ on the upper boundary of that segment. This branch is characterized by the fact that

$$\lim_{z\to\infty}\frac{(z-a)^{-\frac{1}{2}-\alpha}(b-z)^{-\frac{1}{2}+\alpha}}{z}=-ie^{\pi i\alpha}.$$

$\widetilde{C}(t)$ can be defined from the following formula:

$$\lim_{z\to\infty} z\Phi\left(z,t\right)=\frac{-T_0\left(t\right)+iP_0\left(t\right)}{2\pi}=\frac{iP_0\left(t\right)\left(1+ik\right)}{2\pi},$$

whence by virtue of (3.15), we get $\widetilde{C}\left(t\right)=\dfrac{P_0\left(t\right)\left(1+ik\right)e^{\pi i\alpha}}{2\pi}$.

Finally,

$$\Phi\left(z,t\right)=\frac{\chi_0\left(z\right)}{2\pi i}\int\limits_a^b\frac{\Gamma\left(x,t\right)dx}{\chi_0^+\left(x\right)\left(x-z\right)}+\chi_0\left(z\right)\frac{P_0\left(t\right)\left(1+ik\right)e^{\pi i\alpha}}{2\pi}.$$

It can be easily verified that all the conditions of the problem will be satisfied if $\Gamma\left(x,t\right)$ satisfies Hölder's condition condition (H) with respect to the variable x on the segment $[a,b]$.

Since

$$P\left(x,t\right)+iT\left(x,t\right)=P\left(x,t\right)\left(1+ik\right)=\Phi^+\left(x,t\right)-\Phi^-\left(x,t\right),$$

therefore the pressure under the punch is calculated by the formula

$$P\left(x,t\right)=\frac{\chi_0\left(x\right)}{\pi i}\int\limits_a^b\frac{\Gamma\left(y,t\right)dy}{\chi_0^+\left(y\right)\left(y-x\right)}+\chi_0\left(x\right)\frac{2P_0\left(t\right)\left(1+ik\right)e^{\pi i\alpha}}{2\pi}.$$

For $k=0$ $(\alpha=0)$, we obtain a solution corresponding to the case without friction.

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