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# SOME MODULAR INEQUALITIES IN LEBESGUE SPACES WITH A VARIABLE EXPONENT

MITSUO IZUKI[1*], TAKAHIRO NOI[2], AND YOSHIHIRO SAWANO[3]

**Abstract.** Our aim is to study the modular inequalities for some operators, for example, the Bergman projection in Lebesgue spaces with a variable exponent. Under proper assumptions on the variable exponent, we prove that the modular inequalities hold, if and only if the exponent almost everywhere is equal to a constant. In order to get the main results, we establish a lower pointwise bound for these operators of a characteristic function.

## 1. Introduction

The study on variable exponent analysis has been rapidly developed after the work [18] where Kováčik and Rákosník have established fundamental properties of variable Lebesgue spaces (see also [4, 14, 21]). In particular, the theory of variable function spaces in connection with the boundedness of the Hardy–Littlewood maximal operator $M$ has been deeply studied. Cruz-Uribe, Fiorenza and Neugebauer [6, 7] and Diening [9] have independently obtained the log-Hölder continuous conditions that guarantee the boundedness of $M$ on variable Lebesgue spaces. We also note that the recent development of variable exponent analysis has the extrapolation theorem from weighted inequalities to norm inequalities on variable Lebesgue spaces [5, 8].

In general, the boundedness of $M$ on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ describes that the norm inequality

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{1.1}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, where $C$ is a positive constant independent of $f$. Lerner [19] has pointed out the crucial difference between the norm inequality (1.1) and the following modular inequality

$$\int\limits_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \le C \int\limits_{\mathbb{R}^n} |f(x)|^{p(x)}dx. \tag{1.2}$$

More precisely, Lerner has proved that $p(\cdot)$ must be a constant function whenever $1 < \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \le \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty$ and the modular inequality (1.2) holds. Izuki [11] has considered the difference for some operators arising from the wavelet theory. Izuki, Nakai and Sawano [13, 14] have given an alternative proof of Lerner's result. They have also studied the problem in the weighted case [15].

Recently, Izuki, Koyama, Noi and Sawano [12] have considered some modular inequalities for some operators. In this paper, we focus on three operators below. First, we investigate the Bergman projection operator on the unit disc $\mathbb{D}$ in the complex plane. The generalization of holomorphic function spaces in terms of variable exponent and the boundedness of Bergman projection operators on variable exponent spaces have been studied [1–3, 16, 17]. Among them we focus on the work [1] due to Chacón and Rafeiro. They defined Bergman spaces $A^{p(\cdot)}(\mathbb{D})$ with variable exponent $p(\cdot)$ on the open unit disk $\mathbb{D}$. Applying the local log-Hölder continuous condition and the extrapolation theorem, they proved the density of the set of polynomials in $A^{p(\cdot)}(\mathbb{D})$ and the boundedness of the Bergman projection $P : L^{p(\cdot)}(\mathbb{D}) \to A^{p(\cdot)}(\mathbb{D})$. In particular, Chacón and Rafeiro [1] have obtained the norm inequality

$$\|Pf\|_{L^{p(\cdot)}(\mathbb{D})} \le C\,\|f\|_{L^{p(\cdot)}(\mathbb{D})} \tag{1.3}$$

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*Corresponding author.

for all $f \in L^{p(\cdot)}(\mathbb{D})$.

Our second target operator is

$$B_{\mathbb{R}^2_+} f(z) = \frac{-1}{\pi} \int_{\mathbb{R}^2_+} \frac{f(w)}{(z-\overline{w})^2} dA(w), \quad z = x+iy \in \mathbb{R}^2_+,$$

where $dA(w)$ denotes the Lebesgue measure and $\mathbb{R}^2_+$ is the upper half-space over $\mathbb{R}^2 \simeq \mathbb{C}$. Via this identification of $\mathbb{R}^2$ and $\mathbb{C}$, the space $A^{p(\cdot)}(\mathbb{R}^2_+)$ is defined to be the set of all holomorphic functions which belong to $L^{p(\cdot)}(\mathbb{R}^2_+)$. Karapetyants and Samko [17] proved that $B_{\mathbb{R}^2_+}$ is a projection from $L^{p(\cdot)}(\mathbb{R}^2_+)$ onto $A^{p(\cdot)}(\mathbb{R}^2_+)$ if $p(\cdot) \in \mathcal{P}(\mathbb{R}^2_+)$, the set of all measurable functions $p(\cdot) : \mathbb{R}^2_+ \to (0,\infty)$ such that $\log\log p(\cdot) \in L^\infty(\mathbb{R}^2_+)$, satisfies the log-Hölder condition and the log-decay condition [17, Theorem 3.1 (1)]. So, they have obtained the norm inequality

$$\|B_{\mathbb{R}^2_+} f\|_{L^{p(\cdot)}(\mathbb{R}^2_+)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^2_+)} \tag{1.4}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^2_+)$.

Finally, we consider $b_{\mathbb{R}^n_+}$, the harmonic projection in $\mathbb{R}^n_+$. Let $\mathbb{R}^n_+$ stand for the upper half-space over $\mathbb{R}^n$ with $n \ge 2$. For $x = (x_1, x_2, \ldots, x_n)$, we write $x' = (x_1, x_2, \ldots, x_{n-1})$ and $\bar{x} = (x', -x_n)$. As usual, $h^p(\mathbb{R}^n_+)$ stands for the harmonic Bergman space of harmonic functions that belong to $L^p(\mathbb{R}^n_+)$. Once again $dA(x)$ denotes the Lebesgue measure. The corresponding Bergman projection $b_{\mathbb{R}^n_+}$ defined by

$$\begin{aligned} b_{\mathbb{R}^n_+} f(x) &= \int_{\mathbb{R}^n_+} R(x,y) f(y) dA(y) \\ &= \frac{2}{\pi^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}\right) \int_{\mathbb{R}^n_+} \frac{n(x_n+y_n) - |x-\bar{y}|^2}{|x-\bar{y}|^{n+2}} f(y) dA(y), \end{aligned}$$

is bounded from $L^p(\mathbb{R}^n_+)$ onto $h^p(\mathbb{R}^n_+)$ [22]. Namely, $b_{\mathbb{R}^n_+} f \in h^p(\mathbb{R}^n_+)$ and the norm inequality

$$\|b_{\mathbb{R}^n_+} f\|_{L^p(\mathbb{R}^n_+)} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{1.5}$$

hold for all $f \in L^p(\mathbb{R}^n_+)$. Karapetyants and Samko have extended (1.5) to the variable exponent settings [17, Theorem 5.1].

In the present paper, we consider the modular inequalities corresponding to the norm inequalities (1.3), (1.4) and (1.5). More precisely, for example, if $p(\cdot)$ satisfies

$$1 < \operatorname*{ess\,sup}_{z\in\mathbb{D}} p(z) \le \operatorname*{ess\,sup}_{z\in\mathbb{D}} p(z) < \infty$$

and the modular inequality

$$\int_{\mathbb{D}} |Pf(z)|^{p(z)} dA(z) \le C \int_{\mathbb{D}} |f(z)|^{p(z)} dA(z)$$

holds for all $f \in L^{p(\cdot)}(\mathbb{D})$, then the variable exponent $p(\cdot)$ must be a constant function. We can prove similar results for $B_{\mathbb{R}^n_+}$ and $b_{\mathbb{R}^n_+}$. In order to prove them, we need a lower bound for the image of the characteristic function of a certain set. We will show a key lemma for the lower bound before the statement of the main results.

In the present paper we will use the following notation.

1. Given a measurable set $E$, we denote the Lebesgue measure of $E$ by $|E|$. We define the characteristic function of $E$ by $\chi_E$.

2. A symbol $C$ always stands for a positive constant, independent of the main parameters.

## 2. Function Spaces with Variable Exponent

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, that is,

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Let also $\mathbb{R}^n_+$ be the upper half plane, that is,

$$\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0\}.$$

In the present paper we concentrate on the theory on function spaces defined on $\mathbb{D}$ or $\mathbb{R}^n_+$ with $n \geq 2$. We first define some fundamental notation on variable exponents. Let $X$ denote either $\mathbb{D}$ or $\mathbb{R}^n_+$.

**Definition 2.1.**

1. Given a measurable function $p(\cdot) : X \to [1, \infty)$, we define

$$p_+ := \operatorname*{ess\,sup}_{z \in X} p(z), \quad p_- := \operatorname*{ess\,inf}_{z \in X} p(z).$$

2. The set $\mathcal{P}(X)$ consists of all measurable functions $p(\cdot) : X \to [1, \infty)$ satisfying $1 < p_-$ and $p_+ < \infty$.

Chacón and Rafeiro [1] defined generalized Lebesgue spaces and Bergman spaces on $\mathbb{D}$ with a variable exponent.

**Definition 2.2.** Let $dA(z)$ be the normalized Lebesgue measure on $X$ and $p(\cdot) \in \mathcal{P}(X)$. The Lebesgue space $L^{p(\cdot)}(X)$ consists of all measurable functions $f$ on $X$ satisfying that the modular

$$\rho_p(f) := \int_X |f(z)|^{p(z)}\, dA(z)$$

is finite. The Bergman space $A^{p(\cdot)}(\mathbb{D})$ is the set of all holomorphic functions $f$ on $\mathbb{D}$ such that $f \in L^{p(\cdot)}(\mathbb{D})$.

We note that $L^{p(\cdot)}(X)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(X)} := \inf\{\lambda > 0 : \rho_p(f/\lambda) \leq 1\}.$$

The projection $P : L^2(\mathbb{D}) \to A^2(\mathbb{D})$ is called the Bergman projection and given by

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \bar{w}z)^2} dA(w).$$

It is known that $P : L^p(\mathbb{D}) \to A^p(\mathbb{D})$ is bounded in the case where $p(\cdot) = p \in (0, \infty)$ is a constant exponent [10, 22]. See also [20] for the case of $p = 2$.

Chacón and Rafeiro [1, Theorem 4.4] proved the following boundedness

**Theorem 2.3.** Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{D})$ satisfies the local log-Hölder continuous condition

$$|p(z_1) - p(z_2)| \leq \frac{C}{\log(\mathrm{e} + 1/|z_1 - z_2|)} \quad (z_1,\, z_2 \in \mathbb{D}).$$

Then the Bergman projection $P$ is bounded from $L^{p(\cdot)}(\mathbb{D})$ to $A^{p(\cdot)}(\mathbb{D})$, in particular, the norm inequality

$$\|Pf\|_{L^{p(\cdot)}(\mathbb{D})} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{D})}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{D})$.

In the following sections, we consider the modular inequalities corresponding to the norm inequalities (1.3), (1.4) and (1.5).

## 3. Bergman Projection on $\mathbb{D}$

**Theorem 3.1.** Let $p(\cdot) \in \mathcal{P}(\mathbb{D})$. If the modular inequality

$$\int_{\mathbb{D}} |Pf(z)|^{p(z)} dA(z) \leq C \int_{\mathbb{D}} |f(z)|^{p(z)} dA(z) \tag{3.1}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{D})$, then $p(z)$ equals to a constant for almost every $z \in \mathbb{D}$.

In order to prove this theorem, we apply the following lower pointwise estimate for the Bergman projection.

**Lemma 3.2.** Let $\tau \in \mathbb{D}$. Then there exists a compact neighborhood $K_\tau$ of $\tau$ such that

$$\mathrm{Re}(P\chi_E(z)) \geq c_\tau |E|$$

for all measurable sets $E \subset K_\tau$, where $c_\tau$ is a positive constant depending only on $\tau$.

*Proof.* Note that there exists a compact neighborhood $K_\tau$ of $\tau$ such that

$$c_\tau := \inf_{z,w \in K_\tau} \mathrm{Re}\left(\frac{1}{(1-\bar{w}z)^2}\right) > 0.$$

Thus,

$$\mathrm{Re}(P\chi_E(z)) = \int_E \mathrm{Re}\left(\frac{1}{(1-\bar{w}z)^2}\right) dA(w) \geq c_\tau \int_E dA(w) = c_\tau |E|,$$

as required. □

Now we prove Theorem 3.1.

*Proof of Theorem* 3.1. Let $\tau \in \mathbb{D}$ and $K_\tau$ be the compact neighborhood appearing in Lemma 3.2. Assume that $p(z)$ does not equal to any constant for almost every $z \in K_\tau$. Then we can find subsets $K_\tau^\pm$ of $K_\tau$ such that

$$\sup_{z \in K_\tau^-} p(z) < \inf_{z \in K_\tau^+} p(z). \tag{3.2}$$

Using Lemma 3.2 and modular inequality (3.1), we have

$$\int_{K_\tau^+} (kc_\tau |K_\tau^-|)^{p(z)}\, dA(z) \leq \int_{K_\tau^+} |kP\chi_{K_\tau^-}(z)|^{p(z)}\, dA(z) \leq C \int_{\mathbb{D}} (k\chi_{K_\tau^-})^{p(z)}\, dA(z)$$

for all $k > 0$. Consequently, if $kc_\tau |K_\tau^-| > 1$ and $k > 1$, then we obtain

$$|K_\tau^+| (kc_\tau |K_\tau^-|)^{\operatorname{ess\,inf}_{z \in K_\tau^+} p(z)} \leq C |K_\tau^-| k^{\operatorname{ess\,sup}_{z \in K_\tau^-} p(z)}.$$

This contradicts (3.2). Consequently, it follows that for all $\tau \in \mathbb{D}$ there exists a compact neighborhood $K_\tau$ such that $p(z)$ is equal to a constant for almost every $z \in K_\tau$. Since $\mathbb{D}$ is connected, it follows that $p(z)$ is equal to a constant for almost every $z \in \mathbb{D}$. □

## 4. Bergman Projection onto $\mathbb{R}^2_+$

As the following lemma shows, $B_{\mathbb{R}^2_+}$ is not degenerate.

**Lemma 4.1.** Let $\tau \in \mathbb{R}^2_+$. Then there exists a compact neighborhood $K_\tau$ of $\tau$ such that

$$\mathrm{Re}\left(B_{\mathbb{R}^2_+}(\chi_E)(z)\right) \geq C_\tau |E|$$

for all measurable sets $E \subset K_\tau$.

*Proof.* Let $\tau = \alpha + \beta i \in \mathbb{C} \simeq \mathbb{R}^2_+$. Firstly, we prove that there exist $C_\tau$ and a compact neighborhood $K_\tau$ of $\tau$ such that

$$\mathrm{Re}\left(\frac{1}{(z-\overline{w})^2}\right) \le -C_\tau < 0$$

holds for any $z, w \in K_\tau$. To do this, we consider the real part of $(\overline{z}-w)^2$ keeping in mind that

$$\mathrm{Re}\left(\frac{1}{(z-\overline{w})^2}\right) = \mathrm{Re}\left(\frac{(\overline{z}-w)^2}{|z-\overline{w}|^4}\right).$$

We can take $\gamma > 0$ so that $\beta - \gamma > 0$ because $\beta > 0$. We learn that

$$K_\tau = \{x + yi \,:\, \alpha - (\beta-\gamma)/2 \le x \le \alpha + (\beta-\gamma)/2,\, \beta - \gamma \le y \le \beta + \gamma\} (\subset \mathbb{R}^2_+)$$

makes the job. In fact, let $z = a + bi$, $w = c + di \in K_\tau$. It is easy to see that $\mathrm{Re}(\overline{z}-w)^2 < 0$, since

$$(\overline{z}-w)^2 = (a-c)^2 - (b+d)^2 - 2(a-c)(b+d)i$$

and $|a-c| \le \beta - \gamma < 2(\beta-\gamma) \le |b+d|$.

Consequently, from the property of $K_\tau$, we have

$$\mathrm{Re}(B_{\mathbb{R}^2_+}(\chi_E(z))) = \frac{-1}{\pi}\int_E \mathrm{Re}\left(\frac{1}{(z-\overline{w})^2}\right) dA(w) \ge C_\gamma \int_E dA(w) = c_\gamma |E|$$

for any $E \subset K_\tau$. □

Using Lemma 4.1 and an argument similar to the proof of Theorem 3.1, we obtain the following theorem. So we omit the proof.

**Theorem 4.2.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^2_+)$. If the modular inequality

$$\int_{\mathbb{R}^2_+} \left|B_{\mathbb{R}^2_+} f(z)\right|^{p(z)} dA(z) \le C \int_{\mathbb{R}^2_+} |f(z)|^{p(z)}\, dA(z)$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^2_+)$, then $p(z)$ is equal to a constant for almost every $z \in \mathbb{R}^2_+$.

## 5. Harmonic Projection in $\mathbb{R}^n_+$

The same technique can be applied to the harmonic projection over $\mathbb{R}^n_+$.

**Theorem 5.1.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_+)$. If the modular inequality

$$\int_{\mathbb{R}^n_+} \left|b_{\mathbb{R}^n_+} f(z)\right|^{p(z)} dA(z) \le C \int_{\mathbb{R}^n_+} |f(z)|^{p(z)}\, dA(z)$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n_+)$, then $p(z)$ is equal to a constant for almost every $z \in \mathbb{R}^n_+$.

*Proof.* Let $x = (x', x_n) \in \mathbb{R}^n_+$ be fixed. Then we have

$$\frac{n(x_n+z_n) - |x-\bar{z}|^2}{|x-\bar{z}|^{n+2}} = \frac{n-2x_n}{2^{n+1}} {x_n}^{-n-1}$$

for $z = (z', z_n) = x = (x', x_n)$. Based on this equality, we will prove that $p(z)$ is equal to a constant for almost every $z \in \mathbb{R}^n_+$ via three steps.

1. If $x_n < \frac{n}{2}$, then we obtain

$$\frac{n(x_n+y_n) - |x-\bar{y}|^2}{|x-\bar{y}|^{n+2}} > \frac{n-2x_n}{2^{n+3}} {x_n}^{-n-1} > 0$$

as long as $y = (y', y_n)$ belongs to an open neighborhood $U$ of $x$. Thus, if we go through the same argument as before, we see that $p(z)$ is equal to a constant $p_1$ for almost every $z \in \mathbb{R}^n_+$ with $z_n > \frac{n}{2}$.

2. If $x_n > \frac{n}{2}$ instead, then we obtain

$$\frac{n(x_n+y_n) - |x-\bar{y}|^2}{|x-\bar{y}|^{n+2}} < \frac{n-2x_n}{2^{n+3}} {x_n}^{-n-1} < 0$$

as long as $y = (y', y_n)$ belongs to an open neighborhood $U$ of $x$. Thus, if we go through the same argument as before, we see that $p(z)$ equals to a constant $p_2$ for almost every $z \in \mathbb{R}^n_+$ with $z_n < \frac{n}{2}$.

3. Finally, we prove that $p_1 = p_2$. To this end, we consider a small neighborhood $U$ at $(0, \frac{n}{4})$ and a small neighborhood $V$ at $(0, 3n)$. Since

$$\frac{n(x_n + z_n) - |x - \bar{z}|^2}{|x - \bar{z}|^{n+2}} < 0$$

if $x = (0, \frac{n}{4})$ and $z = (0, 3n)$,

$$\frac{n(x_n + z_n) - |x - \bar{z}|^2}{|x - \bar{z}|^{n+2}} \leq -c_n$$

for any $x \in U$ and $z \in V$ for some $c_n > 0$. Thus, we can through the same argument as before, to conclude that $p_1 = p_2$. □

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[1]Faculty of Liberal Arts and Sciences, Tokyo City University, 1-28-1, Tamadutsumi, Setagaya-ku, Tokyo, 158-8557, Japan

[2]Department of Mathematics and Information Science, Tokyo Metropolitan University, Hachioji, Tokyo, 192-0397, Japan

[3]Yoshihiro Sawano, Department of Mathematics, Chuo University, 1-13-27, Kasuga, Bunkyo-ku, Tokyo, 112-8551, Japan

*E-mail address*: izuki@tcu.ac.jp
*E-mail address*: taka.noi.hiro@gmail.com
*E-mail address*: yoshihiro-sawano@celery.ocn.ne.jp