Transactions of A. Razmadze
Mathematical Institute
Vol. **174** (2020), issue 3, 303–324

BOUNDARY-TRANSMISSION PROBLEMS OF THE THEORY OF ACOUSTIC WAVES FOR PIECEWISE INHOMOGENEOUS ANISOTROPIC MULTI-COMPONENT LIPSCHITZ DOMAINS

SVETA GORGISHELI[1], MAIA MREVLISHVILI[1], AND DAVID NATROSHVILI[1,2]

Abstract. We consider the time-harmonic acoustic wave scattering by a bounded anisotropic inhomogeneous obstacle embedded in an unbounded anisotropic homogeneous medium assuming that the boundary of the obstacle and the interface are Lipschitz surfaces. We assume that the obstacle contains a cavity and the material parameters may have discontinuities across the interface between the inhomogeneous interior and homogeneous exterior regions. The corresponding mathematical model is formulated as a boundary-transmission problem for a second order elliptic partial differential equation of Helmholtz type with piecewise Lipschitz-continuous variable coefficients. The problem is studied by the so-called nonlocal approach which reduces the problem to a variational-functional equation containing sesquilinear forms over a bounded region occupied by the inhomogeneous obstacle and over the interfacial surface. This is done with the help of the theory of layer potentials on Lipschitz surfaces. The coercivity properties of the corresponding sesquilinear forms are analyzed and the unique solvability of the boundary transmission acoustic problem in appropriate Sobolev-Slobodetskii and Bessel potential spaces is established.

1. Introduction

The paper deals with the time-harmonic acoustic wave scattering by a bounded *anisotropic inhomogeneous obstacle* embedded in an unbounded *anisotropic* homogeneous medium. We assume that the bounded obstacle contains an interior cavity. The boundary of the cavity will be referred to as *interior boundary of the obstacle*. We require that the interior boundary of the obstacle and the interface between the inhomogeneous interior and homogeneous exterior regions are the Lipschitz surfaces. The physical wave scattering problem with a frequency parameter $\omega \in \mathbb{R}$ is formulated mathematically as a boundary-transmission problem for a second order elliptic partial differential equation with variable Lipschitz-continuous coefficients, $A_2(x,\partial_x,\omega)\,u(x) \equiv \partial_{x_k}\big(a^{(2)}_{kj}(x)\,\partial_{x_j}u(x)\big) + \omega^2\,\kappa_2(x)\,u(x) = f_2(x)$, in the bounded region $\Omega_2 \subset \mathbb{R}^3$ occupied by an inhomogeneous anisotropic obstacle and for a Helmholtz type equation with constant coefficients, $A_1(\partial_x,\omega)u(x) \equiv a^{(1)}_{kj}\,\partial_{x_k}\partial_{x_j}u(x) + \omega^2\kappa_1\,u(x) = f_1(x)$, in the unbounded region Ω_1 occupied by the homogeneous anisotropic medium. The material parameters $a^{(q)}_{kj}$ and κ_q, $q = 1, 2$, are not assumed to be continuous across the interface. Note that in the case of *isotropic medium* occupying the domain Ω_q, we have only one material coefficient $a^{(q)}$, i.e., the corresponding material parameters satisfy the relations $a^{(q)}_{kj} = a^{(q)}\delta_{kj}$, where δ_{kj} is the Kronecker symbol.

We analyse the case when the transmission conditions relating the interior and exterior traces of the wave amplitude u and its conormal derivatives are prescribed on the interface surface, while on the interior boundary of the inhomogeneous obstacle there are given the Dirichlet or Neumann or mixed Dirichlet-Neumann boundary conditions.

The transmission problems for the Helmholtz equation in the case of *the whole piecewise homogenous isotropic space* $\mathbb{R}^3 = \overline{\Omega}_2 \cup \overline{\Omega}_1$ with *a smooth interface surface* $S = \partial\Omega_1 = \partial\Omega_2$, when $A_2(\partial) = \Delta + \kappa_2\omega^2$ and $A_1(\partial) = \Delta + \kappa_1\omega^2$, $\kappa_q = \text{const}$, $q = 1, 2$, are well studied in [14, 24–26] (see also references therein). In these papers, using the method of standard direct and indirect boundary integral equations method the transmission problem is reduced to a uniquely solvable coupled pair of boundary integral

2020 *Mathematics Subject Classification.* 35A15, 35D30, 35J20, 35P25, 35R05.

Key words and phrases. Acoustic scattering; Helmholtz operator; Lipschitz domain; Transmission problem; Layer potentials; Weak solution.

equations for a pair of unknowns. Moreover, in [25], by coupling the direct and indirect approaches, the transmission problem is reduced to a uniquely solvable single integral equation for a single unknown.

Using the harmonic analysis technique and the approach employed in the reference [26], the same transmission problem for the whole piecewise homogenous isotropic space $\mathbb{R}^3 = \overline{\Omega}_2 \cup \overline{\Omega}_1$ with a *Lipschitz interface* is considered in [40] using the potential method. Note that the harmonic analysis approach gives the optimal L_2 results, establishes the nontangential almost everywhere convergence of the solution to the boundary values, guarantees the boundedness of the corresponding nontangential maximal function, which in turn give better regularity results (see, e.g., [22]).

Similar acoustic scattering problems for the whole *isotropic composed space* $\mathbb{R}^3 = \overline{\Omega}_2 \cup \overline{\Omega}_1$ with smooth interface and with a variable continuous refractive index $\kappa(x)$, when $\kappa(x) = 1$ in the exterior domain Ω_1, are also well presented in the literature. In this case, $A_2(x, \partial_x, \omega) = \Delta + \omega^2\kappa(x)$ in the *isotropic inhomogeneous obstacle* region and $A_1(\partial_x, \omega) = \Delta + \omega^2$ in the unbounded homogeneous isotropic region. The problem is reduced to the Lippmann-Schwinger equation which is unconditionally solvable Fredholm type integral equation on the bounded obstacle region Ω_1 (see [12,35] and references therein).

Analogous acoustic transmission problem in the whole composed space $\mathbb{R}^3 = \overline{\Omega}_2 \cup \overline{\Omega}_1$ with a smooth interface, corresponding to a more general isotropic case, when $A_2(x, \partial_x, \omega) = \partial_{x_k}\big(a(x)\,\partial_{x_j}\big) + \omega^2$ with a sufficiently smooth function $a(x)$ and $A_1(\partial_x, \omega) = \Delta + \omega^2$, was analysed by the indirect boundary-domain integral equation method in the references [28, 44, 45].

The transmission problem for the whole composed anisotropic space $\mathbb{R}^3 = \overline{\Omega}_2 \cup \overline{\Omega}_1$ in the case of a smooth interface and sufficiently smooth in Ω_2 material coefficients $a^{(2)}_{kj}$ and κ_2 is studied in [10] by a special direct method based on the application of localized harmonic parametrix. This approach reduces the transmission problem to the uniquely solvable system of localized boundary-domain integral equations.

In this paper, we investigate more general anisotropic boundary-transmission problems using the so-called nonlocal approach when the interior boundary of the obstacle and the interface surface are Lipschitz manifolds, and the coefficients $a^{(2)}_{kj}$ and κ_2 are Lipschitz-continuous. Moreover, we consider in detail the case when the mixed Dirichlet-Neumann conditions are prescribed on the interior boundary.

We apply the theory of layer potentials on Lipschitz surfaces and reduce equivalently the boundary-transmission problem to the variational-functional equation containing sesquilinear forms over the interfacial surface and over a bounded domain occupied by the inhomogeneous obstacle. To substantiate our approach, we use essentially the results of [13, 21, 22], and the so-called *combined field integral equations* approach described in [6, 8, 27, 36] (see also [7]).

The paper is organized as follows. In Section 2, we introduce the generalized radiation conditions for anisotropic media, formulate the acoustic transmission problems for multi-component piecewise anisotropic structures with Lipschitz-continuous boundaries and interfaces, and prove the uniqueness theorems in appropriate function spaces. In Section 3, we construct the generalized Steklov–Poincaré type integral operator in the case of Lipschitz surfaces and derive the corresponding Dirichlet-to-Neumann relations for the acoustic equation in an unbounded anisotropic region. In Section 4, the transmission problems are equivalently reformulated as variational-functional equations containing sesquilinear forms which live on a bounded domain occupied by the obstacle and the interface surface. The boundedness and coercivity properties for the sesquilinear forms are proved in the appropriately chosen function spaces which eventually lead to the unique solvability of the original acoustic transmission problems. Finally, for the readers convenience, in Appendix we collect some auxiliary material related to anisotropic radiating layer potentials over Lipschitz surfaces.

2. Formulation of the Problems and Uniqueness Theorems

2.1. **Some auxiliary definitions and relations.** Let $\Omega_1 := \Omega^-$ be an unbounded domain in $\mathbb{R}^3$ with a simply connected compact boundary $\partial\Omega_1 = S_1$ and $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}_1$. Further, let $\Omega_2 := \Omega^+ \setminus \overline{\Omega}_3$, where Ω_3 is a subdomain of Ω^+ such that $\overline{\Omega}_3 \subset \Omega^+$. Put $S_2 = \partial\Omega_3$. Evidently, $\partial\Omega_2 = S_1 \cup S_2$. Throughout the paper, $n = (n_1, n_2, n_3)$ denotes the outward unit normal vector to S_q, $q = 1, 2$.

In what follows, we assume that the interface S_1 and the interior boundary S_2 are arbitrary Lipschitz surfaces, unless otherwise stated, and the following condition holds:

$$\text{the interface } S_1 \text{ contains a } C^2\text{-smooth open submanifold } S_1^*. \tag{2.1}$$

By $H^s(\Omega) = H_2^s(\Omega)$, $H^s_{\rm loc}(\Omega) = H^s_{2,\,\rm loc}(\Omega)$, $H^s_{\rm comp}(\Omega) = H^s_{2,\,\rm comp}(\Omega)$ and $H^s(S) = H^s_2(S)$, $s \in \mathbb{R}$, we denote the L_2-based Bessel potential spaces of complex-valued functions on an open domain $\Omega \subset \mathbb{R}^3$ and on a closed manifold S without boundary, while $\mathcal{D}(\Omega)$ stands for the space of infinitely differentiable test functions with support in Ω. Recall that $H^0(\Omega) = L_2(\Omega)$ is a space of square integrable functions on Ω.

Further, let us define the following classes of functions:

$$\begin{aligned}
&H^{1,\,0}(\Omega_2; A_2) := \{\, v \in H^1(\Omega_2) \,:\, A_2 v \in H^0(\Omega_2)\,\},\\
&H^{1,\,0}_{\rm loc}(\Omega_1; A_1) := \{\, v \in H^1_{\rm loc}(\Omega_1) \,:\, A_1 v \in H^0_{\rm loc}(\Omega_1)\,\},\\
&\widetilde{H}^s(\Omega_2) := \{v :\ v \in H^s(\mathbb{R}^3),\ \operatorname{supp} v \subset \overline{\Omega_2}\},\\
&\widetilde{H}^s(\mathcal{M}) := \{g :\ g \in H^s(S_2),\ \operatorname{supp} g \subset \overline{\mathcal{M}}\},\\
&H^s(\mathcal{M}) := \{r_{\mathcal{M}} g :\ g \in H^s(S_2)\},
\end{aligned}$$

where $\mathcal{M} \subset S_2$ is an open submanifold of the Lipschitz surface S_2 with a Lipschitz boundary curve $\partial\mathcal{M}$ and $r_{\mathcal{M}}$ stands for the restriction operator onto $\mathcal{M}$.

We assume that the propagation region of a time harmonic acoustic wave $u^{\rm tot}$ is the domain $\mathbb{R}^3 \setminus \Omega_3 = \overline{\Omega}_1 \cup \overline{\Omega}_2$, which consists of the homogeneous part Ω_1 and the inhomogeneous part Ω_2.

Acoustic wave propagation is governed by a uniformly elliptic second order scalar partial differential equation

$$A(x, \partial_x, \omega)\, u^{\rm tot}(x) \equiv \partial_k \big(a_{kj}(x)\, \partial_j u^{\rm tot}(x)\big) + \omega^2\, \kappa(x)\, u^{\rm tot}(x) = f(x), \quad x \in \Omega_1 \cup \Omega_2,$$

where $\partial_x \equiv \partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial_{x_j} = \partial/\partial x_j$, $a_{kj}(x) = a_{jk}(x)$ and $\kappa(x)$ are real-valued functions, $\omega \in \mathbb{R}$ is a frequency parameter, while f is a square integrable function in $\mathbb{R}^3$ with a compact support, $f \in L_{2,\rm comp}(\mathbb{R}^3)$. Here and in what follows, the Einstein summation by repeated indices from 1 to 3 is assumed.

Note that in the mathematical model of an inhomogeneous absorbing medium the function κ is complex-valued, with nonzero real and imaginary parts, in general (see, e.g., [12, Ch. 8]). Here we treat only the case when the refractive index κ is a real-valued function, but it should be mentioned that the complex-valued case can also be considered by the approach developed in the present paper.

In our further analysis, it is assumed that the real-valued variable coefficients a_{kj} and κ are the constants in the homogeneous unbounded region Ω_1,

$$a_{kj}(x) = a_{jk}(x) = \begin{cases} a^{(1)}_{kj} & \text{for } \ x \in \Omega_1,\\ a^{(2)}_{kj}(x) & \text{for } \ x \in \Omega_2, \end{cases} \qquad \kappa(x) = \begin{cases} \kappa_1 > 0 & \text{for } \ x \in \Omega_1,\\ \kappa_2(x) > 0 & \text{for } \ x \in \Omega_2, \end{cases} \tag{2.2}$$

where $a^{(1)}_{kj}$ and κ_1 are the constants, while $a^{(2)}_{kj}$ and κ_2 are the Lipschitz-continuous functions in $\overline{\Omega}_2$,

$$a^{(2)}_{kj},\ \kappa_2 \in C^{0,\,1}(\overline{\Omega}_2), \quad j,k = 1,2,3. \tag{2.3}$$

Moreover, the matrices $\mathbf{a}_q = \big[\, a^{(q)}_{kj} \,\big]_{k,j=1}^3$ are uniformly positive definite, i.e., there are positive constants c_1 and c_2 such that

$$c_1\, |\xi|^2 \le a^{(q)}_{kj}(x)\, \xi_k\, \xi_j \le c_2\, |\xi|^2 \qquad \forall\, x \in \overline{\Omega}_q,\ \ \forall\, \xi \in \mathbb{R}^3,\ \ q = 1,2. \tag{2.4}$$

We do not assume that the coefficients a_{kj} and κ are continuous across the interface S_1, in general, i.e., the case $a^{(2)}_{kj}(x) \ne a^{(1)}_{kj}$ and $\kappa_2(x) \ne \kappa_1$ for $x \in S_1$ is covered by our analysis.

Further, we denote

$$\begin{aligned}
&r_{\Omega_1} A(x, \partial_x, \omega) u(x) \equiv A_1(\partial_x, \omega) u(x) := a^{(1)}_{kj}\, \partial_k \partial_j u(x) + \omega^2 \kappa_1\, u(x) \ \text{ for } \ x \in \Omega_1, \qquad (2.5)\\
&r_{\Omega_2} A(x, \partial_x, \omega) u(x) \equiv A_2(x, \partial_x, \omega) u(x) := \partial_{x_k} \big(a^{(2)}_{kj}(x)\, \partial_j u(x)\big) + \omega^2\, \kappa_2(x)\, u(x) \ \text{ for } \ x \in \Omega_2.
\end{aligned}$$

We will often drop the arguments and write A_1 and A_2 instead of $A_1(\partial_x,\omega)$ and $A_2(x,\partial_x,\omega)$, respectively, when this does not lead to misunderstanding.

For a function u_q, sufficiently smooth in Ω_q (say, $u_1\in H^2_{\rm loc}(\Omega_1)$ or $u_2\in H^2(\Omega_2)$), the *classical conormal derivative operators* $T_q^\pm$ are well defined as

$$T_q^\pm u_q(x):=a_{kj}^{(q)}\,n_k(x)\,\gamma^\pm_{S_m}(\partial_j u_q(x)),\ \ x\in S_m,\ \ q,m=1,2, \tag{2.6}$$

where the symbols $\gamma^+_{S_m}$ and $\gamma^-_{S_m}$ denote one-sided boundary trace operators on S_m from the interior and exterior domains, respectively.

Motivated by the first Green identity, the classical conormal derivative operators (2.6) can be extended by continuity to the functions $u_1\in H^{1,0}_{\rm loc}(\Omega_1;A_1)$ and $u_2\in H^{1,0}(\Omega_2;A_2)$ giving well defined *canonical conormal derivatives* $T_1^- u_1\in H^{-\frac12}(S_1)$, $T_2^+u_2\in H^{-\frac12}(S_1)$, and $T_2^-u_2\in H^{-\frac12}(S_2)$, defined for arbitrary $g_1\in H^{\frac12}(S_1)$ and $g_2\in H^{\frac12}(S_2)$ by the following relations:

$$\big\langle T_1^- u_1\,,\,\overline{g_1}\big\rangle_{S_1}:=-\int\limits_{\Omega_1}A_1u_1(x)\,\overline{w_1(x)}\,dx-\int\limits_{\Omega_1}[E_1(u_1,\overline{w_1})-\omega^2\kappa_1u_1(x)\,\overline{w_1(x)}]\,dx, \tag{2.7}$$

$$\begin{aligned}\big\langle T_2^+ u_2\,,\,\overline{g_1}\big\rangle_{S_1}-\big\langle T_2^- u_2\,,\,\overline{g_2}\big\rangle_{S_2}:=&\int\limits_{\Omega_2}A_2u_2(x)\ \overline{w_2(x)}\,dx\\&+\int\limits_{\Omega_2}\big[E_2(u_2,\overline{w_2})-\omega^2\kappa_2(x)u_2(x)\,\overline{w_2(x)}\,\big]\,dx,\end{aligned} \tag{2.8}$$

where the angular brackets $\langle\cdot\,,\,\cdot\rangle_{S_m}$ are understood as duality pairing of $H^{-\frac12}(S_m)$ with $H^{\frac12}(S_m)$ which extends the usual bilinear $L_2(S_m)$ inner product, $w_1\in H^1_{\rm comp}(\Omega_1)$ with $\gamma^-_{S_1}w_1=g_1$, $w_2\in H^1(\Omega_2)$ with $\gamma^+_{S_1}w_2=g_1$ and $\gamma^-_{S_2}w_2=g_2$, and

$$E_1(u_1,\overline{w_1}):=a_{kj}^{(1)}\,\partial_ju_1(x)\,\overline{\partial_kw_1(x)},\qquad E_2(u_2,\overline{w_2}):=a_{kj}^{(2)}(x)\,\partial_ju_2(x)\,\overline{\partial_kw_2(x)}. \tag{2.9}$$

Evidently, there is a constant $C>0$ such that

$$\begin{aligned}\|T_1^-u_1\|_{H^{-\frac12}(S_1)}&\leqslant C\left(\|A_1u_1\|_{H^0(\Omega_1^*)}+\|u_1\|_{H^1(\Omega_1^*)}\right),\\ \|T_2^+u_2\|_{H^{-\frac12}(S_1)}&\leqslant C\left(\|A_2u_2\|_{H^0(\Omega_2)}+\|u_2\|_{H^1(\Omega_2)}\right),\\ \|T_2^-u_2\|_{H^{-\frac12}(S_2)}&\leqslant C\left(\|A_2u_2\|_{H^0(\Omega_2)}+\|u_2\|_{H^1(\Omega_2)}\right),\end{aligned} \tag{2.10}$$

where Ω_1^* is an arbitrary one-sided exterior neighbourhood of the surface $S_1=\partial\Omega_1$ located in Ω_1. For the properties of the trace operator in the case of Lipschitz domains and for the corresponding conormal derivatives see [13,15], [29, Ch. 4], [30].

Recall that for arbitrary functions $u_1\in H^{1,0}_{\rm loc}(\Omega_1;A_1)$ and $u_2\in H^{1,0}(\Omega_2;A_2)$, the Green first identities associated with the operators A_1 and A_2 (see, e.g., [13, Section 3], [29, Ch. 4], [30, Theorem 3.9])

$$\begin{aligned}&\int\limits_{\Omega_1(R)}A_1u_1(x)\,\overline{v_1(x)}\,dx+\int\limits_{\Omega_1(R)}[E_1(u_1,\overline{v_1})-\omega^2\kappa_1u_1(x)\,\overline{v_1(x)}]\,dx\\&\qquad=\big\langle T_1^+u_1\,,\,\overline{\gamma^+_{\Sigma(R)}v_1}\big\rangle_{\Sigma(R)}-\big\langle T_1^-u_1\,,\,\overline{\gamma^-_{S_1}v_1}\big\rangle_{S_1}\qquad\forall\ v_1\in H^1_{\rm loc}(\Omega_1),\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\int\limits_{\Omega_2}A_2u_2(x)\ \overline{v_2(x)}\,dx+\int\limits_{\Omega_2}\big[E_2(u_2,\overline{v_2})-\omega^2\kappa_2(x)u_2(x)\,\overline{v_2(x)}\,\big]\,dx\\&\qquad=\big\langle T_2^+u_2\,,\,\overline{\gamma^+_{S_1}v_2}\big\rangle_{S_1}-\big\langle T_2^-u_2\,,\,\overline{\gamma^-_{S_2}v_2}\big\rangle_{S_2}\qquad\forall\ v_2\in H^1(\Omega_2)\end{aligned} \tag{2.12}$$

hold, where $\Omega_1(R):=\Omega_1\cap B(R)$ with $B(R)$ being a ball centered at the origin and radius R such that $\overline{\Omega}_2\subset B(R)$, $\Sigma(R):=\partial B(R)$, $E_q(u_q,\overline{v_q})$, $q=1,2$, are defined in (2.9).

By $Z(\Omega_1)$ we denote the sub-class of complex-valued functions from $H^1_{\rm loc}(\Omega_1)$ satisfying the Sommerfeld radiation conditions at infinity (see [12,37,42] for the Helmholtz operator and [19,20,34,41] for the "anisotropic" operator A_1 defined by (2.5)). Denote by S_ω the characteristic ellipsoid associated with the operator $A_1(\partial_x, \omega)$,

$$a^{(1)}_{kj}\xi_k\,\xi_j - \omega^2\kappa_1 = 0, \quad \xi \in \mathbb{R}^3, \quad \omega \neq 0.$$

For an arbitrary vector $\eta \in \mathbb{R}^3$ with $|\eta| = 1$ there exists only one point $\xi(\eta) \in S_\omega$ such that the outward unit normal vector $n(\xi(\eta))$ to S_ω at the point $\xi(\eta)$ has the same direction as η, i.e., $n(\xi(\eta)) = \eta$. Note that $\xi(-\eta) = -\xi(\eta) \in S_\omega$ and $n(-\xi(\eta)) = -\eta$.

It can easily be verified that

$$\xi(\eta) = \omega\,\sqrt{\kappa_1}\,(\mathbf{a}_1^{-1}\eta\cdot\eta)^{-\frac{1}{2}}\ \mathbf{a}_1^{-1}\eta, \tag{2.13}$$

where $\mathbf{a}_1^{-1}$ is the matrix, inverse to $\mathbf{a}_1 := \big[\,a^{(1)}_{kj}\,\big]_{k,j=1}^3$, and the central dot denotes the scalar product in $\mathbb{R}^3$.

Definition 2.1. A complex-valued function v belongs to the class $Z(\Omega_1)$ if there exists a ball $B(R)$ of radius R centered at the origin such that $v \in C^1(\Omega_1 \setminus B(R))$, and v satisfies the Sommerfeld radiation conditions associated with the operator $A_1(\partial_x, \omega)$ for sufficiently large $|x|$,

$$v(x) = \mathcal{O}(|x|^{-1}), \quad \partial_k v(x) - i\xi_k(\eta)v(x) = \mathcal{O}(|x|^{-2}), \quad k = 1, 2, 3, \tag{2.14}$$

where $\xi(\eta) \in S_\omega$ corresponds to the vector $\eta = x/|x|$ (i.e., $\xi(\eta)$ is given by (2.13) with $\eta = x/|x|$).

Notice that due to the ellipticity of the operator $A_1(\partial_x, \omega)$, any solution to the constant coefficient homogeneous equation $A_1(\partial_x, \omega)v(x) = 0$ in an open region $\Omega \subset \mathbb{R}^3$ is a real analytic function of x in Ω.

Conditions (2.14) are equivalent to the classical Sommerfeld radiation conditions for the Helmholtz equation if $A_1(\partial_x, \omega) = \Delta(\partial) + \omega^2$, i.e., if $\kappa_1 = 1$ and $a^{(1)}_{kj} = \delta_{kj}$, where δ_{kj} is the Kronecker delta. The following analogue of the classical Rellich-Vekua lemma holds (for details see [19, 34]).

Lemma 2.2. *Let $v \in Z(\Omega_1)$ be a solution of the equation $A_1(\partial_x, \omega)v = 0$ in Ω_1 and let*

$$\lim_{R\to+\infty} \operatorname{Im}\Bigg\{\ \int\limits_{\Sigma(R)} \overline{v(x)}\ T_1(x, \partial_x)v(x)\,d\Sigma(R)\Bigg\} = 0, \tag{2.15}$$

where $\Sigma(R)$ is the sphere of radius R centered at the origin. Then $v = 0$ in Ω_1.

Remark 2.3. For $x \in \Sigma(R)$ and $\eta = x/|x|$, we have $n(x) = \eta$ and, in view of (2.6) and (2.14), for a function $v \in Z(\Omega_1)$, we get

$$T_1(x, \partial_x)v(x) = a^{(1)}_{kj} n_k(x)\,[\,i\,\xi_j(\eta)\,v(x)] + \mathcal{O}(|x|^{-2}) = i\,a^{(1)}_{kj}\eta_k\,\xi_j(\eta)\,v(x) + \mathcal{O}(|x|^{-2})\,.$$

Therefore, by (2.13) and the symmetry condition $a^{(1)}_{kj} = a^{(1)}_{jk}$, we arrive at the relation

$$\begin{aligned}\overline{v(x)}\,T_1(x, \partial_x)v(x) &= i\,\omega\,\sqrt{\kappa_1}\,|v(x)|^2\,(\mathbf{a}_1^{-1}\eta\cdot\eta)^{-\frac{1}{2}}\,\mathbf{a}_1\eta\cdot\mathbf{a}^{-1}\eta + \mathcal{O}(|x|^{-3})\\ &= i\,\omega\,\sqrt{\kappa_1}\,(\mathbf{a}_1^{-1}\eta\cdot\eta)^{-\frac{1}{2}}\,|v(x)|^2 + \mathcal{O}(|x|^{-3}).\end{aligned}$$

On the other hand, the matrix $\mathbf{a}_1$ is positive definite (cf. (2.4)), which implies positive definiteness of the inverse matrix $\mathbf{a}_1^{-1}$. Hence there are positive constants δ_0 and δ_1 such that for all $\eta \in \Sigma(1)$,

$$0 < \delta_0 \leqslant (\mathbf{a}_1^{-1}\eta\cdot\eta)^{-\frac{1}{2}} \leqslant \delta_1 < \infty\,.$$

Consequently, for $\omega \neq 0$, condition (2.15) is equivalent to the following relation:

$$\lim_{R\to+\infty} \int\limits_{\Sigma(R)} |v(x)|^2\,d\Sigma(R) = 0,$$

which is the well known Rellich-Vekua condition in the theory of Helmholtz equation (for details see [12, 37, 42]).

2.2. **Formulation of the transmission problems.** In the unbounded region Ω_1, we have a total wave field $u^{\rm tot} = u^{\rm inc} + u^{\rm sc}$, where $u^{\rm inc}$ is a wave motion initiating the known incident field and u^{sc} is a radiating unknown scattered field. It is often assumed that the incident field is defined in the whole of $\mathbb{R}^3$, being, for example, a corresponding plane wave which solves the homogeneous equation $A_1 u^{\rm inc} = 0$ in $\mathbb{R}^3$ but does not satisfy the Sommerfeld radiation conditions at infinity. Motivated by relations (2.2), we set $u_1(x) := u^{sc}(x)$ for $x \in \Omega_1$ and $u_2(x) := u^{\rm tot}(x)$ for $x \in \Omega_2$.

Now we formulate the transmission problem associated with the time-harmonic acoustic wave scattering by a bounded anisotropic inhomogeneity embedded in an unbounded anisotropic homogeneous medium:

Find complex-valued functions $u_1 \in H^{1,0}_{\rm loc}(\Omega_1; A_1) \cap Z(\Omega_1)$ *and* $u_2 \in H^{1,0}(\Omega_2; A_2)$ *satisfying the differential equations*

$$A_1(\partial_x, \omega) u_1(x) = f_1(x) \ \ \textit{for} \ \ x \in \Omega_1, \tag{2.16}$$

$$A_2(x, \partial_x, \omega) u_2(x) = f_2(x) \ \ \textit{for} \ \ x \in \Omega_2, \tag{2.17}$$

the transmission conditions on the interface S_1*,*

$$\gamma^+_{S_1} u_2 - \gamma^-_{S_1} u_1 = \varphi_1 \ \ \textit{on} \ \ S_1, \tag{2.18}$$

$$T_2^+ u_2 - T_1^- u_1 = \psi_1 \ \ \textit{on} \ \ S_1, \tag{2.19}$$

and one of the following boundary conditions on S_2*:*
The Dirichlet condition

$$\gamma^-_{S_2} u_2 = 0 \ \ \textit{on} \ \ S_2, \tag{2.20}$$

The Neumann condition

$$T_2^- u_2 = \psi_2 \ \ \textit{on} \ \ S_2, \tag{2.21}$$

The mixed type conditions

$$\gamma^-_{S_{2D}} u_2 = 0 \ \ \textit{on} \ \ S_{2D}, \qquad T_2^- u_2 = \psi_{2N} \ \ \textit{on} \ \ S_{2N}, \tag{2.22}$$

where $S_{2D} \cap S_{2N} = \varnothing$, $\overline{S}_{2D} \cup \overline{S}_{2N} = S_2$, *and*

$$\begin{gathered} f_2 := r_{\Omega_2} f \in H^0(\Omega_2), \quad f_1 := r_{\Omega_1} f \in H^0_{\rm comp}(\Omega_1), \quad f \in H^0_{\rm comp}(\mathbb{R}^3), \\ \varphi_1 \in H^{\frac12}(S_1), \quad \psi_1 \in H^{-\frac12}(S_1), \quad \psi_2 \in H^{-\frac12}(S_2), \quad \psi_{2N} \in H^{-\frac12}(S_{2N}). \end{gathered} \tag{2.23}$$

In the above setting, equations (2.16) and (2.17) are understood in the distributional sense, the Dirichlet type conditions in (2.18), (2.20) and (2.22) are understood in the usual trace sense, while the Neumann type conditions in (2.19), (2.21) and (2.22) are understood in the canonical conormal derivative sense defined by relations (2.7)–(2.8).

If the total field $u^{\rm tot}$ and its conormal derivative are continuous across the interface, then $\varphi_1 = \gamma^-_{S_1} u^{\rm inc}$ and $\psi_1 = T_1^- u^{\rm inc}$.

The above-formulated boundary-transmission problems with the Dirichlet, Neumann, and mixed type conditions will be referred to as Problem (TD), (TN) and (TM), respectively.

2.3. **Uniqueness theorems.** Here we prove the uniqueness theorem.

Theorem 2.4. *The boundary-transmission problems (TD), (TN) and (TM) possess at most one solution.*

Proof. Due to the linearity of the problems, we have to show that the corresponding homogeneous problems possess only the trivial solution.

Let a pair (u_2, u_1) with $u_2 \in H^{1,0}(\Omega_2; A_2)$ and $u_1 \in H^{1,0}_{\rm loc}(\Omega_1; A_1) \cap Z(\Omega_1)$ be a solution to the homogeneous boundary-transmission problem (TD) or (TN) or (TM). Note that $u_1 \in C^\infty(\Omega_1)$ due to ellipticity of the constant coefficient operator A_1.

Let R be an arbitrary positive number such that $\overline{\Omega}_2 \subset B(R)$. We can write Green's first identities (2.11) and (2.12) for the functions u_1 and u_2 in the domains $\Omega_1(R) := \Omega_1 \cap B(R)$ and Ω_2. In view of the homogeneity of the boundary conditions on S_2, we arrive at the relations

$$\int\limits_{\Omega_1(R)} [a_{kj}^{(1)}\,\partial_j u_1(x)\,\overline{\partial_k u_1(x)} - \omega^2\kappa_1 |u_1(x)|^2]\,dx = -\langle T_1^- u_1\,,\,\overline{\gamma_{S_1}^- u_1}\rangle_{S_1} + \langle T_1^+ u_1\,,\,\overline{\gamma_{\Sigma(R)}^+ u_1}\rangle_{\Sigma(R)}, \tag{2.24}$$

$$\int\limits_{\Omega_2} [a_{kj}^{(2)}(x)\,\partial_j u_2(x)\,\overline{\partial_k u_2(x)} - \omega^2\kappa_2(x) |u_2(x)|^2]\,dx = \langle T_2^+ u_2\,,\,\overline{\gamma_{S_1}^+ u_2}\rangle_{S_1}. \tag{2.25}$$

Due to the homogeneous transmission conditions and since the matrices $\mathbf{a}_q = [a_{kj}^{(q)}]_{k,j=1}^3$ are symmetric and positive definite, after adding (2.24) and (2.25) and separating the imaginary part, we get

$$\operatorname{Im}\Big\{\int\limits_{\Sigma(R)} \overline{u_1(x)}\,T_1(x,\partial_x)u_1(x)\,d\Sigma(R)\Big\} = 0,$$

whence by Lemma 2.2, we deduce that $u_1 = 0$ in Ω_1.

Therefore, in view of (2.16)–(2.22), the function $u_2 \in H^{1,0}(\Omega_2; A_2)$ satisfies the homogeneous differential equation

$$A_2(x,\partial_x,\omega)u_2(x) = 0 \ \text{ in } \ \Omega_2,$$

the homogeneous Cauchy type conditions

$$\gamma_{S_1}^+ u_2 = 0 \quad \text{and} \quad T_2^+ u_2 = 0 \quad \text{on} \quad S_1,$$

and one of the homogeneous boundary conditions (2.20)–(2.22) on S_2.

Keeping in mind the relations (2.1) and (2.3), by the interior and boundary regularity properties of solutions to a strongly elliptic partial differential equation, we deduce $u_2 \in C^2(\Omega_2 \cup S_1^*)$ (see, e.g., [18, Lemmas 6.16, 6.18], [29, Theorem 4.18]). Thus, the Cauchy data of the function u_2 vanish continuously on $S_1^* \subset S_1$ and due to [39, Theorem 2.9], we conclude that $u_2 = 0$ in Ω_2, which completes the proof. □

3. Integral Relations for Radiating Function in the Domain Ω_1

For any radiating solution $u_1 \in H^{1,0}_{\mathrm{loc}}(\Omega_1; A_1) \cap Z(\Omega_1)$ with $A_1 u_1 \in H^0_{\mathrm{comp}}(\Omega_1)$ the Green third identity (for details see [13, 19, 29, 34])

$$u_1 + V(T_1^- u_1) - W(\gamma_{S_1}^- u_1) = \mathcal{P}(A_1 u_1) \quad \text{in} \quad \Omega_1 \tag{3.1}$$

holds, where V, W, and $\mathcal{P}$ denote, respectively, the single layer potential, double layer potential and volume potential associated with the operator $A_1(\partial_x, \omega)$,

$$Vg(y) := -\int\limits_{S_1} \Gamma(x-y,\omega)\,g(x)\,dS_x, \quad y \in \mathbb{R}^3 \setminus S_1, \tag{3.2}$$

$$Wg(y) := -\int\limits_{S_1} [T_1(x,\partial_x)\Gamma(x-y,\omega)]\,g(x)\,dS_x, \quad y \in \mathbb{R}^3 \setminus S_1, \tag{3.3}$$

$$\mathcal{P}h(y) := \int\limits_{\Omega_1} \Gamma(x-y,\omega)\,h(x)\,dx, \quad y \in \mathbb{R}^3. \tag{3.4}$$

Here g and h are densities of the potentials, $T_1(x,\partial_x) = a_{kj}^{(1)} n_k(x)\partial_{x_j}$, $n(x)$ is the outward unit normal vector to S_1 at the point $x \in S_1$, and

$$\Gamma(x,\omega) = -\frac{\exp\{i\omega\sqrt{\kappa_1}\,(\mathbf{a}_1^{-1}x\cdot x)^{1/2}\}}{4\pi(\det\mathbf{a}_1)^{1/2}\,(\mathbf{a}_1^{-1}x\cdot x)^{1/2}} \tag{3.5}$$

is a radiating fundamental solution of the operator $A_1(\partial_x,\omega)$ (see, e.g., Lemma 1.1 in [20]).

Remark 3.1. In a neighbourhood of the origin, e.g., for $|x| < 1$, we have the decomposition

$$\Gamma(x,\omega) = -\frac{1}{4\pi\,(\det \mathbf{a}_1)^{1/2}}\Big\{\frac{1}{(\mathbf{a}_1^{-1}x\cdot x)^{1/2}} + i\omega\sqrt{\kappa_1} - \frac{1}{2}\,\omega^2\kappa_1\,(\mathbf{a}_1^{-1}x\cdot x)^{1/2} + \cdots\Big\}, \tag{3.6}$$

while for sufficiently large $|y|$, we have the following asymptotic formula:

$$\Gamma(y-x,\omega) = c(\xi)\frac{\exp\{i\xi\cdot(y-x)\}}{|y|} + O(|y|^{-2}), \quad c(\xi) = -\frac{|\mathbf{a}_1\,\xi|}{4\pi\omega\sqrt{\kappa_1}\,(\det \mathbf{a}_1)^{1/2}}, \tag{3.7}$$

where x varies in a bounded subset of $\mathbb{R}^3$, $\xi = \xi(\eta) \in S_\omega$ corresponds to the direction $\eta = y/|y|$ and is given by (2.13). The asymptotic formula (3.7) can be differentiated arbitrarily many times with respect to x and y. Both formulas, (3.5) and (3.6), hold true for an arbitrary complex parameter $\omega = \omega_1 + i\omega_2$ with $\omega_j \in \mathbb{R}$, $j = 1, 2$. Evidently, the function $\Gamma(x) := \Gamma(x, 0)$ is a fundamental solution of the operator $A_1(\partial_x) := A_1(\partial_x, 0)$, while $\Gamma(x, i)$ is the exponentially decaying real-valued fundamental solution of the operator $A_1(\partial_x, i)$. In view of (3.6), we have

$$\Gamma(x,\omega) - \Gamma(x,i) = -\frac{1}{4\pi\,(\det \mathbf{a}_1)^{1/2}}\Big\{(1+i\omega)\sqrt{\kappa_1} - \frac{1}{2}(\omega^2+1)\,\kappa_1\,(\mathbf{a}_1^{-1}x\cdot x)^{1/2} + \cdots\Big\}, \tag{3.8}$$

implying for $|x| < 1$ the following relations:

$$\frac{\partial}{\partial x_k}\big[\Gamma(x,\omega) - \Gamma(x,i)\big] = \mathcal{O}(1), \quad \frac{\partial^2}{\partial x_j \partial x_k}\big[\Gamma(x,\omega) - \Gamma(x,i)\big] = \mathcal{O}(|x|^{-1}), \quad k,j = 1,2,3. \tag{3.9}$$

The mapping properties of these potentials and the boundary operators generated by them in the case of Lipschitz surface S_1 are collected in Appendix A. Note that the mapping properties of layer potentials associated with Lipschitz and smooth surfaces are essentially different (cf., e.g., [3–5, 13, 29, 43] and references cited therein).

Evidently, the layer potentials Vg and Wg solve the homogeneous differential equation (2.16), i.e.,

$$A_1 Vg = A_1 Wg = 0 \ \text{ in } \ \mathbb{R}^3 \setminus S_1, \tag{3.10}$$

while for $f_1 \in H^0_{comp}(\Omega_1)$, the volume potential $\mathcal{P}f_1 \in H^2_{\rm loc}(\mathbb{R}^3)$ solves the following nonhomogeneous equation (see Lemma A.1)

$$A_1\mathcal{P}f_1 = \begin{cases} f_1 & \text{in} \quad \Omega_1, \\ 0 & \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}_1. \end{cases} \tag{3.11}$$

Using the properties of layer and volume potentials (see Lemma A.1(iii)), for the exterior traces of Green's third identity (3.1) and its conormal derivative on S_1, we get

$$\mathcal{V}(T_1^- u_1) + (2^{-1}I - \mathcal{W})(\gamma^-_{S_1} u_1) = \gamma^-_{S_1}\mathcal{P}(A_1 u_1) \quad \text{on} \quad S_1, \tag{3.12}$$

$$(2^{-1}I + \mathcal{W}')(T_1^- u_1) - \mathcal{L}(\gamma^-_{S_1} u_1) = T_1^-\mathcal{P}(A_1 u_1) \quad \text{on} \quad S_1, \tag{3.13}$$

where the integral operators $\mathcal{V}$, $\mathcal{W}$, $\mathcal{W}'$ and $\mathcal{L}$ are defined in Appendix A by formulas (A.2)–(A.5). Recall that the operators $\mathcal{V}$, $2^{-1}I - \mathcal{W}$, $2^{-1}I + \mathcal{W}'$ and $\mathcal{L}$ involved in (3.12)–(3.13) are not invertible for resonant values of the frequency parameter ω. The set of these resonant values is countable and consists of eigenfrequencies of the interior Dirichlet and Neumann boundary value problems for the operator A_1 in the bounded domain surrounded by the surface S_1 (see [42, Section 4], [11, Ch. 3], [9, Section 7.7], [7]). Therefore, to obtain Dirichlet-to-Neumann or Neumann-to-Dirichlet mappings for arbitrary values of the frequency parameter ω, we apply *the combined-field integral equations* approach and proceed as follows. Multiply equation (3.12) by $i\,\alpha$ with some fixed positive α and add to equation (3.13) to obtain (cf., [6, 8, 27, 36])

$$\mathcal{K}(T_1^- u_1) - \mathcal{M}(\gamma^-_{S_1} u_1) = \Phi(A_1 u_1) \quad \text{on} \quad S_1, \tag{3.14}$$

where

$$\mathcal{K}g := \Big(\frac{1}{2}\,I + \mathcal{W}' + i\,\alpha\,\mathcal{V}\Big)g = \big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big)V\,g \quad \text{on} \quad S_1, \tag{3.15}$$

$$\mathcal{M}h := \Big[\mathcal{L} + i\,\alpha\Big(-\frac{1}{2}\,I + \mathcal{W}\Big)\Big]h = \big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big)W\,h \quad \text{on} \quad S_1, \tag{3.16}$$

$$\Phi\,f_1 := \big(T_1^- + i\,\alpha\,\gamma^-_{S_1}\big)\mathcal{P}\,f_1 = \big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big)\mathcal{P}\,f_1 \quad \text{on} \quad S_1, \tag{3.17}$$

for $f_1 \in H^0_{\mathrm{comp}}(\Omega_1)$, $g \in H^{-\frac{1}{2}}(S_1)$ and $h \in H^{\frac{1}{2}}(S_1)$. The relation (3.17) follows from the imbedding $\mathcal{P}\,f_1 \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ for $f_1 \in H^0_{\mathrm{comp}}(\mathbb{R}^3)$.

In view of Lemma A.2, from (3.14), for arbitrary $u_1 \in H^{1,\,0}_{\mathrm{loc}}(\Omega_1;A_1)\cap Z(\Omega_1)$, we derive the following analogue of the Steklov-Poincaré type relation:

$$T_1^- u_1 = \mathcal{K}^{-1}\big[\mathcal{M}\,(\gamma^-_{S_1}u_1) + \Phi(A_1u_1)\big] \quad \text{on} \quad S_1, \tag{3.18}$$

where $\mathcal{K}^{-1} : H^{-\frac{1}{2}}(S_1) \to H^{-\frac{1}{2}}(S_1)$ is the inverse to the operator $\mathcal{K} : H^{-\frac{1}{2}}(S_1) \to H^{-\frac{1}{2}}(S_1)$.

4. Weak Formulation of the Mixed Boundary-transmission Problems and the Existence Results

Here we apply the so-called non-local approach to obtain the variational-functional formulation of the transmission problem under consideration. To this end, let us assume that a pair $(u_2,u_1) \in H^{1,\,0}(\Omega_2;A_2)\times\big(H^{1,0}_{\mathrm{loc}}(\Omega_1;A_1)\cap Z(\Omega_1)\big)$ solves the mixed transmission problem (TM) (see (2.16)–(2.19)) and (2.22). Applying relation (3.18), transmission conditions (2.18)–(2.19) and mixed boundary conditions (2.22) in the Green first identity (2.12), for the domain Ω_2, we arrive at the equation

$$\begin{gathered}\mathfrak{B}(u_2,v) = \mathfrak{F}(v)\\ \forall\ v \in H^1(\Omega_2;S_{2D}) := \{w \in H^1(\Omega_2)\ :\ r_{S_{2D}}\gamma^-_{S_{2D}}w = 0\},\end{gathered} \tag{4.1}$$

where $\mathfrak{B}$ is a sesquilinear form and $\mathfrak{F}$ is an antilinear functional defined, respectively, as

$$\mathfrak{B}(u_2,v) := \mathfrak{B}^{(1)}(u_2,v) + \mathfrak{B}^{(2)}(u_2,v), \tag{4.2}$$

$$\mathfrak{B}^{(1)}(u_2,v) := \int\limits_{\Omega_2}\big[a^{(2)}_{kj}(x)\,\partial_j u_2(x)\,\overline{\partial_k v(x)} - \omega^2\kappa_2(x)u_2(x)\,\overline{v(x)}\,\big]\,dx, \tag{4.3}$$

$$\mathfrak{B}^{(2)}(u_2,v) := -\big\langle\mathcal{K}^{-1}\,\mathcal{M}(\gamma^+_{S_1}u_2)\,,\,\overline{\gamma^+_{S_1}v}\big\rangle_{S_1}, \tag{4.4}$$

$$\begin{aligned}\mathfrak{F}(v) := -\int\limits_{\Omega_2} f_2(x)\,\overline{v(x)}\,dx + \langle\mathcal{K}^{-1}\Phi f_1\,,\,\overline{\gamma^+_{S_1}v}\rangle_{S_1} + \big\langle\psi_1,\overline{\gamma^+_{S_1}v}\big\rangle_{S_1} - \big\langle\mathcal{K}^{-1}\,\mathcal{M}\varphi_1\,,\,\overline{\gamma^+_{S_1}v}\big\rangle_{S_1}\\ - \big\langle\psi_{2N},\overline{\gamma^-_{S_{2N}}v}\big\rangle_{S_{2N}},\end{aligned} \tag{4.5}$$

with the operators $\mathcal{K}$, $\mathcal{M}$, and Φ defined by relations (3.15)–(3.17). We associate with equation (4.1) the following variational-functional problem (in a wider space).

Problem (VMT1)**.** Find a function $u_2 \in H^1(\Omega_2;S_{2D})$ satisfying variational-functional equation (4.1) for all $v \in H^1(\Omega_2;S_{2D})$.

Now, let us first prove the following equivalence

Theorem 4.1. *Let conditions* (2.23) *be fulfilled.*

(i) *If a pair* $(u_2,u_1) \in H^{1,\,0}(\Omega_2;A_2)\times\big(H^{1,\,0}_{\mathrm{loc}}(\Omega_1;A_1)\cap Z(\Omega_1)\big)$ *solves the mixed transmission problem (TM), then the function* u_2 *solves variational-functional equation* (4.1) *and the following relation holds:*

$$u_1(y) = \mathcal{P}f_1(y) - V(T_2^+u_2 - \psi_1)(y) + W(\gamma^+_{S_1}u_2 - \varphi_1)(y), \quad y \in \Omega_1. \tag{4.6}$$

(ii) *Vice versa, if a function* $u_2 \in H^1(\Omega_2;S_{2D})$ *solves variational-functional equation* (4.1)*, then the pair* (u_2,u_1) *with* u_1 *defined by* (4.6) *belongs to the class* $H^{1,\,0}(\Omega_2;A_2)\times\big(H^{1,\,0}_{\mathrm{loc}}(\Omega_1;A_1)\cap Z(\Omega_1)\big)$ *and solves the mixed transmission problem (TM).*

Proof. (i) The first part of the theorem follows from the derivation of variational-functional equation (4.1).

(ii) To prove the second part, we proceed as follows. If u_2 solves variational-functional equation (4.1), then for $v \in \mathcal{D}(\Omega_2)$ the equation

$$\int\limits_{\Omega_2} \big[a_{kj}^{(2)}(x)\,\partial_j u_2(x)\,\overline{\partial_k v(x)} - \omega^2 \kappa_2(x) u_2(x)\,\overline{v(x)}\,\big]\,dx = -\int\limits_{\Omega_2} f_2(x)\,\overline{v(x)}\,dx,$$

particularly holds and implies that u_2 is a solution of equation (2.17), $A_2(x,\partial_x,\omega)u_2 = f_2$ in Ω_2 in the sense of distributions and, evidently, $u_2 \in H^{1,\,0}(\Omega_2; A_2)$ in view of (2.23). Therefore the canonical conormal derivatives $T_2^+ u_2 \in H^{-\frac12}(S_1)$ and $T_2^- u_2 \in H^{-\frac12}(S_2)$ are well-defined in the sense of (2.8).

Further, it is easy to see that function (4.6) is well-defined, solves the differential equation (2.16) in Ω_1 due to (3.10)–(3.11), and belongs to the space $H^{1,\,0}_{\rm loc}(\Omega_1; A_1)\cap Z(\Omega_1)$ in view of (2.23) and properties of the volume and layer potentials. Therefore, the canonical conormal derivative $T_1^- u_1 \in H^{-\frac12}(S_1)$ is also well-defined in the sense of (2.7).

Now we show that mixed boundary conditions (2.22) on S_2 and transmission conditions (2.18)–(2.19) on S_1 are satisfied. To this end, we write Green's identity (2.12) for u_2 and arbitrary $v \in H^1(\Omega_2; S_{2D})$,

$$\int\limits_{\Omega_2} \big[E_2(u_2,\overline{v}) - \omega^2 \kappa_2(x) u_2(x)\,\overline{v(x)}\,\big]\,dx = -\int\limits_{\Omega_2} f_2(x)\,\overline{v(x)}\,dx$$
$$+\big\langle T_2^+ u_2\,,\,\overline{\gamma^+_{S_1} v}\big\rangle_{S_1} - \big\langle T_2^- u_2\,,\,\overline{\gamma^-_{S_2} v}\big\rangle_{S_{2N}}. \tag{4.7}$$

Comparing (4.7) and (4.1) leads to the relation

$$\big\langle \mathcal{K}^{-1}\,\mathcal{M}(\gamma^+_{S_1} u_2)\,,\,\overline{\gamma^+_{S_1} v}\big\rangle_{S_1} + \langle \mathcal{K}^{-1}\Phi f_1\,,\,\overline{\gamma^+_{S_1} v}\rangle_{S_1} + \big\langle \psi_1, \overline{\gamma^+_{S_1} v}\big\rangle_{S_1} - \big\langle \mathcal{K}^{-1}\,\mathcal{M}\varphi_1\,,\,\overline{\gamma^+_{S_1} v}\big\rangle_{S_1}$$
$$-\big\langle \psi_{2N}, \overline{\gamma^-_{S_{2N}} v}\big\rangle_{S_{2N}} = \langle T_2^+ u_2\,,\,\overline{\gamma^+_{S_1} v}\rangle_{S_1} - \langle T_2^- u_2\,,\,\overline{\gamma^-_{S_2} v}\rangle_{S_{2N}} \tag{4.8}$$

for all $v \in H^1(\Omega_2; S_{2D})$.

If we take an arbitrary function $v \in H^1(\Omega_2; S_{2D})$ such that $\gamma^+_{S_1} v = 0$, from (4.8), we get

$$\big\langle \psi_{2N}, \overline{\gamma^-_{S_{2N}} v}\big\rangle_{S_{2N}} = \langle T_2^- u_2\,,\,\overline{\gamma^-_{S_2} v}\rangle_{S_{2N}} \tag{4.9}$$

implying the boundary condition $T_2^- u_2 = \psi_{2N}$ on S_{2N}. Consequently, due to the inclusion $u_2 \in H^1(\Omega_2; S_{2D})$, it is evident that the mixed boundary conditions (2.22) on S_2 are satisfied.

In view of (4.9), from (4.8), we deduce

$$\mathcal{K}^{-1}\,\mathcal{M}(\gamma^+_{S_1} u_2) + \mathcal{K}^{-1}\Phi f_1 + \psi_1 - \mathcal{K}^{-1}\,\mathcal{M}\varphi_1 = T_2^+ u_2 \ \text{ on } \ S_1.$$

Applying the operator $\mathcal{K}$ to this equation and taking into account (3.17), we arrive at the relation

$$\mathcal{M}(\gamma^+_{S_1} u_2 - \varphi_1) - \mathcal{K}(T_2^+ u_2 - \psi_1) = -\big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big)\mathcal{P}\,f_1 \ \text{ on } \ S_1.$$

By (3.15), (3.16) and (3.17), the later equation can be rewritten as

$$\big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big)\big[W(\gamma^+_{S_1} u_2 - \varphi_1) - V(T_2^+ u_2 - \psi_1) + \mathcal{P}\,f_1\big] = 0 \ \text{ on } \ S_1. \tag{4.10}$$

Let us introduce the function

$$w := W(\gamma^+_{S_1} u_2 - \varphi_1) - V(T_2^+ u_2 - \psi_1) + \mathcal{P}\,f_1 \ \text{ in } \ \mathbb{R}^3 \setminus S_1.$$

Evidently, in view of the mapping properties of the layer and volume potentials (see Lemma A.1), on the one hand, $r_{\Omega_1} w = u_1 \in H^{1,0}_{\rm loc}(\Omega_1; A_1) \cap Z(\Omega_1)$ due to (4.6), and on the other hand, $r_{\Omega^+} w \in H^{1,\,0}(\Omega^+; A_1)$, where $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}_1$.

Further, by (3.10), (3.11) and (4.10), we deduce that w solves the homogeneous interior Robin's problem

$$A_1(\partial,\omega)w=0 \ \text{ in } \ \Omega^+=\mathbb{R}^3\setminus\overline{\Omega}_1,$$
$$\big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big)w=0 \ \text{ on } \ S_1=\partial\Omega^+,$$

where α is a positive number. Therefore, by the uniqueness theorem, for the interior Robin's problem we infer $w=0$ in Ω^+. Thus,

$$w=W(\gamma^+_{S_1}u_2-\varphi_1)-V(T_2^+u_2-\psi_1)+\mathcal{P}\,f_1=\begin{cases}u_1 & \text{in} \quad \Omega_1,\\ 0 & \text{in} \quad \Omega^+.\end{cases} \tag{4.11}$$

Using the inclusion $\mathcal{P}\,f_1\in H^2_{\rm loc}(\mathbb{R}^3)$, relation (3.17) and the jump relations, for the layer potentials across the surface S_1 (see Lemma A.1), we find from (4.11) that

$$\gamma^-_{S_1}w-\gamma^+_{S_1}w=\gamma^+_{S_1}u_2-\varphi_1=\gamma^-_{S_1}u_1 \ \text{ on } \ S_1,$$
$$T_1^-w-T_1^+w=T_2^+u_2-\psi_1=T_1^-u_1 \ \text{ on } \ S_1,$$

which show that the transmission conditions (2.18)–(2.19) hold. This completes the proof. □

Theorem 4.2. *The homogeneous variational-functional Problem (VMT1) possesses only the trivial solution in the space* $H^1(\Omega_2;S_{2D})$.

Proof. Let $u_2\in H^1(\Omega_2;S_{2D})$ be a solution of the homogeneous variational-functional Problem (VMT1),

$$\begin{aligned}\mathfrak{B}(u_2,v)\equiv&\int\limits_{\Omega_2}\big[a^{(2)}_{kj}(x)\,\partial_j u_2(x)\,\overline{\partial_k v(x)}-\omega^2\kappa_2(x)u_2(x)\,\overline{v(x)}\,\big]\,dx\\ &-\big\langle\mathcal{K}^{-1}\,\mathcal{M}(\gamma^+_{S_1}u_2)\,,\,\overline{\gamma^+_{S_1}v}\big\rangle_{S_1}=0\qquad\forall\,v\in H^1(\Omega_2;S_{2D}).\end{aligned} \tag{4.12}$$

By the word for word arguments applied in the proof of Theorem 4.1, we can show that u_2 is a solution of the homogeneous equation $A_2(x,\partial_x,\omega)u_2=0$ in Ω_2 in the distributional sense and, evidently, $u_2\in H^{1,\,0}(\Omega_2;A_2)$. Therefore the canonical conormal derivatives $T_2^+u_2\in H^{-\frac12}(S_1)$ and $T_2^-u_2\in H^{-\frac12}(S_2)$ are well-defined in the sense of (2.8) and for u_2 and arbitrary $v\in H^1(\Omega_2;S_{2D})$, Green's identity

$$\int\limits_{\Omega_2}\big[E_2(u_2,\overline{v})-\omega^2\kappa_2(x)u_2(x)\,\overline{v(x)}\,\big]\,dx=\big\langle T_2^+u_2\,,\,\overline{\gamma^+_{S_1}v}\big\rangle_{S_1}-\big\langle T_2^-u_2\,,\,\overline{\gamma^-_{S_2}v}\big\rangle_{S_{2N}} \tag{4.13}$$

holds. Comparing (4.12) and (4.13) leads to the relation

$$\big\langle\mathcal{K}^{-1}\,\mathcal{M}(\gamma^+_{S_1}u_2)\,,\,\overline{\gamma^+_{S_1}v}\big\rangle_{S_1}=\langle T_2^+u_2\,,\,\overline{\gamma^+_{S_1}v}\rangle_{S_1}-\langle T_2^-u_2\,,\,\overline{\gamma^-_{S_2}v}\rangle_{S_{2N}} \tag{4.14}$$

for all $v\in H^1(\Omega_2;S_{2D})$. If we take an arbitrary function $v\in H^1(\Omega_2;S_{2D})$ such that $\gamma^+_{S_1}v=0$, from (4.14), we find

$$T_2^-u_2=0 \ \text{ on } \ S_{2N}.$$

Therefore from (4.14), we deduce

$$\mathcal{M}(\gamma^+_{S_1}u_2)-\mathcal{K}(T_2^+u_2)=0 \ \text{ on } \ S_1,$$

which can be rewritten as

$$\big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big)\big[W(\gamma^+_{S_1}u_2)-V(T_2^+u_2)\big]=0 \ \text{ on } \ S_1. \tag{4.15}$$

Let

$$u_1:=W(\gamma^+_{S_1}u_2)-V(T_2^+u_2) \ \text{ in } \ \mathbb{R}^3\setminus S_1. \tag{4.16}$$

Note that in view of Lemma A.1, $r_{\Omega_1} u_1 \in H^{1,0}_{\rm loc}(\Omega_1; A_1) \cap Z(\Omega_1)$ and $r_{\Omega^+} u_1 \in H^{1,0}(\Omega^+; A_1)$ with $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}_1$. Moreover, by (3.10), (4.15) and (4.16), we see that u_1 solves the homogeneous interior Robin's problem

$$A_1(\partial, \omega) u_1 = 0 \ \text{ in } \ \Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}_1,$$
$$\big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big) u_1 = 0 \ \text{ on } \ S_1 = \partial\Omega^+,$$

where α is a positive number. Consequently, $u_1 = 0$ in Ω^+ and due to the jump relations, for the layer potentials, from (4.16), we deduce

$$\gamma^-_{S_1} u_1 = \gamma^-_{S_1} u_1 - \gamma^+_{S_1} u_1 = \gamma^+_{S_1} u_2 \ \text{ on } \ S_1,$$
$$T_1^- u_1 = T_1^- u_1 - T_1^+ u_1 = T_2^+ u_2 \ \text{ on } \ S_1.$$

Combining the above obtained results, we finally see that the pair $(u_2, u_1) \in H^{1,0}(\Omega_2; A_2) \times \big(H^{1,0}_{\rm loc}(\Omega_1; A_1) \cap Z(\Omega_1)\big)$ solves the mixed homogeneous transmission problem and by the uniqueness Theorem 2.4, we have $u_2 = 0$ in Ω^+, which completes the proof. □

Now let us consider the following variational-functional problem.

Problem (VMT2)**.** Find a pair $(u_2, u_1) \in H^1(\Omega_2; S_{2D}) \times \big(H^1_{\rm loc}(\Omega_1) \cap Z(\Omega_1)\big)$ satisfying the system of equations

$$\mathfrak{B}(u_2, v) = \mathfrak{F}(v) \quad \text{for all} \quad v \in H^1(\Omega_2; S_{2D}), \tag{4.17}$$

$$u_1(y) + V(T_2^+ u_2)(y) - W(\gamma^+_{S_1} u_2)(y) = \mathcal{P} f_1(y) + V\psi_1(y) - W\varphi_1(y), \quad y \in \Omega_1, \tag{4.18}$$

where $\mathfrak{B}$ and $\mathfrak{F}$ are defined in (4.2)–(4.5) and conditions (2.23) are satisfied.

Corollary 4.3. *System* (4.17)–(4.18) *is equivalent to the mixed transmission problem (TM) in the following sense: if a pair* $(u_2, u_1) \in H^1(\Omega_2; S_{2D}) \times \big(H^1_{\rm loc}(\Omega_1) \cap Z(\Omega_1)\big)$ *solves system* (4.17)–(4.18)*, then it is unique and solves the mixed transmission problem (TM), and vice versa.*

Proof. In view of Theorems 2.4, 4.1 and 4.2, it suffices to show that the right-hand sides of system (4.17)–(4.18) vanish if and only if

$$f_1 = 0, \quad f_2 = 0, \quad \varphi_1 = 0, \quad \psi_1 = 0, \quad \psi_{2N} = 0. \tag{4.19}$$

Let

$$\mathfrak{F}(v) = 0 \quad \forall\, v \in H^1(\Omega_2; S_{2D}), \tag{4.20}$$

$$\mathcal{P} f_1 + V\psi_1 - W\varphi_1 = 0 \ \text{ in } \ \Omega_1. \tag{4.21}$$

By the same arguments as in the proof of Theorem 4.1 (see the derivation of formula (4.11)), from (4.20), we obtain

$$\mathcal{P}\, f_1 + V\psi_1 - W\varphi_1 = 0 \ \text{ in } \ \Omega_2. \tag{4.22}$$

From relations (4.21) and (4.22) the equalities $f_1 = 0$, $\varphi_1 = 0$, and $\psi_1 = 0$ follow immediately in view of Lemma A.1. In accordance with (4.5), then (4.20) takes the form

$$-\int\limits_{\Omega_2} f_2(x)\,\overline{v(x)}\,dx - \big\langle \psi_{2N}, \overline{\gamma^-_{S_{2N}} v} \big\rangle_{S_{2N}} = 0, \quad \forall\, v \in H^1(\Omega_2; S_{2D}),$$

implying $f_2 = 0$ and $\psi_{2N} = 0$, which completes the proof. □

Remark 4.4. Note that only equality (4.20) separately leads to (4.22) and does not imply relations (4.19).

Now we prove the following boundedness and coercivity theorem.

Theorem 4.5. *For the sesquilinear form $\mathfrak{B}$ defined by* (4.2)–(4.4) *and the antilinear functional $\mathfrak{F}$ defined in* (4.5) *under conditions* (2.23)*, there are real constants $C_j^* > 0$, $j = 1, 2, 3, 4$ such that*

$$|\mathfrak{B}(u_2, v)| \le C_1^* \|u_2\|_{H^1(\Omega_2)} \|v\|_{H^1(\Omega_2)} \qquad \forall\, u_2,\, v \in H^1(\Omega_2; S_{2D}), \tag{4.23}$$

$$|\mathfrak{F}(v)| \le C_2^* \|v\|_{H^1(\Omega_2)} \qquad \forall\, v \in H^1(\Omega_2; S_{2D}),$$

$$\operatorname{Re}\mathfrak{B}(u_2, u_2) \ge C_3^* \|u_2\|^2_{H^1(\Omega_2)} - C_4^* \|u_2\|^2_{H^0(\Omega_2)} \qquad \forall\, u_2 \in H^1(\Omega_2; S_{2D}). \tag{4.24}$$

Proof. The boundedness of the sesquilinear form $\mathfrak{B}^{(1)}(u_2, v)$ follows directly from the Cauchy-Schwartz inequality, $\big|\mathfrak{B}^{(1)}(u_2, v)\big| \leqslant C_5 \|u_2\|_{H^1(\Omega_2)} \|v\|_{H^1(\Omega_2)}$, while the boundedness of the sesquilinear form $\mathfrak{B}^{(2)}(u_2, v)$ can be shown by the duality inequality, Lemma A.2, and trace theorem,

$$\begin{aligned}
\big|\mathfrak{B}^{(2)}(u_2, v)\big| &= \big|\big\langle \mathcal{K}^{-1}\,\mathcal{M}(\gamma^+_{S_1} u_2)\,,\, \overline{\gamma^+_{S_1} v}\big\rangle_{S_1}\big| \\
&\leqslant C_1 \|\mathcal{K}^{-1}\,\mathcal{M}(\gamma^+_{S_1} u_2)\|_{H^{-\frac12}(S_1)} \|\gamma^+_{S_1} v\|_{H^{\frac12}(S_1)} \\
&\leqslant C_2 \|\mathcal{M}(\gamma^+_{S_1} u_2)\|_{H^{-\frac12}(S_1)} \|v\|_{H^1(\Omega_2)} \leqslant C_3 \|\gamma^+_{S_1} u_2\|_{H^{\frac12}(S_1)} \|v\|_{H^1(\Omega_2)} \\
&\leqslant C_4 \|u_2\|_{H^1(\Omega_2)} \|v\|_{H^1(\Omega_2)},
\end{aligned}$$

where C_j, $j = 1, \dots, 4$, are some positive constants. Consequently, (4.23) holds.

Keeping in mind conditions (2.23), relations (2.10), (3.11), (3.17), (4.5) and the estimate

$$\begin{aligned}
\big|\langle \mathcal{K}^{-1}\Phi f_1\,,\, \overline{\gamma^+_{S_1} v}\rangle_{S_1}\big| &\leqslant C_5 \|\big(T_1^+ + i\,\alpha\,\gamma^+_{S_1}\big)\mathcal{P} f_1\|_{H^{-\frac12}(S_1)} \|\gamma^+_{S_1} v\|_{H^{\frac12}(S_1)} \\
&\leqslant C_6 \,\big(\|A_1 \mathcal{P} f_1\|_{H^0(\Omega_2)} + \|\mathcal{P} f_1\|_{H^1(\Omega_2)}\big)\, \|v\|_{H^1(\Omega_2)} \\
&\leqslant C_7 \|f_1\|_{H^0(\Omega_1)} \|v\|_{H^1(\Omega_2)},
\end{aligned}$$

the boundedness of the functional $\mathfrak{F}$ can be proved by the arguments similar to the above ones,

$$\begin{aligned}
|\mathfrak{F}(v)| \le{} & C_8 \Big(\|f_1\|_{L_2(\Omega_1)} + \|f_2\|_{L_2(\Omega_2)} + \|\varphi_1\|_{H^{\frac12}(S_1)} + \|\psi_1\|_{H^{-\frac12}(S_1)} \\
& + \|\psi_{S_{2N}}\|_{H^{-\frac12}(S_{2N})}\Big) \|v\|_{H^1(\Omega_2)} \quad \text{for all} \quad v \in H^1(\Omega_2; S_{2D}).
\end{aligned}$$

Now we prove inequality (4.24). In view of the positive definiteness of the matrix $\mathbf{a}_2 = \big[\, a^{(2)}_{kj} \,\big]_{k,j=1}^3$, we have

$$\operatorname{Re}\mathfrak{B}^{(1)}(u_2, u_2) \geqslant C_9 \|u_2\|^2_{H^1(\Omega_2)} - C_{10} \|u_2\|^2_{H^0(\Omega_2)},$$

where $C_9 > 0$ and $C_{10} = \omega^2 \max\limits_{\overline{\Omega}_2} \kappa_2(x)$.

Further, by Lemma A.6, we deduce

$$\begin{aligned}
\operatorname{Re}\mathfrak{B}^{(2)}(u_2, u_2) &= -\operatorname{Re}\big\langle \mathcal{K}^{-1}\,\mathcal{M}(\gamma^+_{S_1} u_2)\,,\, \overline{\gamma^+_{S_1} u_2}\big\rangle_{S_1} \\
&\geqslant C_1' \|\gamma^+_{S_1} u_2\|^2_{H^{\frac12}(S_1)} - C_2' \|\gamma^+_{S_1} u_2\|^2_{H^0(S_1)} \\
&\geqslant C_1' \|\gamma^+_{S_1} u_2\|^2_{H^{\frac12}(S_1)} - C_2' \|\gamma^+_{S_1} u_2\|^2_{H^\delta(S_1)} \\
&\geqslant C_1' \|\gamma^+_{S_1} u_2\|^2_{H^{\frac12}(S_1)} - C_2' \|u_2\|^2_{H^{\frac12+\delta}(\Omega_2)} \\
&\geqslant -C_2' \|u_2\|^2_{H^{\frac12+\delta}(\Omega_2)},
\end{aligned} \tag{4.25}$$

where $C_1' > 0$, $C_2' > 0$, and δ is an arbitrarily small positive number. By Ehrling's lemma (see, e.g., [38, Theorem 7.30]), for an arbitrarily small positive number ε there is a positive constant $C(\varepsilon)$

such that

$$\|w\|^2_{H^{\frac{1}{2}+\delta}(\Omega_2)} \leqslant \varepsilon\|w\|^2_{H^1(\Omega_2)} + C(\varepsilon)\|w\|^2_{H^0(\Omega_2)} \quad \text{for all} \quad w \in H^1(\Omega_2), \quad 0<\delta<\frac{1}{2}.$$

Therefore from (4.25), we have

$$\operatorname{Re}\mathfrak{B}^{(2)}(u_2,u_2) \geqslant -C_2' \left(\varepsilon\,\|u_2\|^2_{H^1(\Omega_2)} + C(\varepsilon)\|u_2\|^2_{H^0(\Omega_2)}\right),$$

with ε such that $C_9 - \varepsilon\, C_2' > 0$, which completes the proof. □

Now we can prove the following existence results.

Theorem 4.6. *Let conditions* (2.23) *be fulfilled.*

(i) *Variational-functional equation* (4.1) *is uniquely solvable in the space* $H^1(\Omega_2; S_{2D})$ *for arbitrary antilinear bounded functional* $\mathfrak{F}$ *defined on* $H^1(\Omega_2; S_{2D})$.

(ii) *System* (4.17)–(4.18) *with* $\mathfrak{F}$ *defined in* (4.5)*, is uniquely solvable with respect to the unknown pair* $(u_2,u_1) \in H^1(\Omega_2; S_{2D}) \times \big(H^1_{\rm loc}(\Omega_1)\cap Z(\Omega_1)\big)$.

(iii) *The mixed transmission problem (TM) is uniquely solvable in the space* $H^1(\Omega_2; S_{2D}) \times \big(H^1_{\rm loc}(\Omega_1)\cap Z(\Omega_1)\big)$.

Proof. Item (i) follows directly from Theorem 4.2, Theorem 4.5 and the Lax–Milgram lemma (see, e.g., [29, Theorem 2.33]).

Further, as we have already shown, equation (4.17) with $\mathfrak{F}$ given by (4.5) uniquely defines the sought function u_2 and, consequently, equation (4.18) defines explicitly and uniquely the sought function u_1 in Ω_1 which proves Item (ii).

Item (iii) follows from the uniqueness Theorem 2.4, Corollary 4.3 and Item (ii). □

Remark 4.7. Investigation of the transmission problems with Dirichlet or Neumann boundary conditions on the interior surface S_2 can be carried out quite similarly by using the above-employed arguments. Under conditions (2.23), they are uniquely solvable in the spaces $H^1(\Omega_2; S_2) \times \big(H^1_{\rm loc}(\Omega_1)\cap Z(\Omega_1)\big)$ and $H^1(\Omega_2) \times \big(H^1_{\rm loc}(\Omega_1)\cap Z(\Omega_1)\big)$ respectively.

5. Appendix A: Properties of Radiating Potentials

Here, we present some results concerning the properties of the layer potentials defined by (3.2), (3.3), and the volume potential (cf. (3.4))

$$\mathbf{P}f(y) := \int\limits_{\mathbb{R}^3} \Gamma(x-y,\omega)\, f(x)\, dx, \quad y \in \mathbb{R}^3,$$

in the case of Lipschitz domains which are employed in the main text of the paper. Evidently, $\mathcal{P}f_1(y) = \mathbf{P}\widetilde{f}_1(y)$, where $\widetilde{f}_1$ is the extension by zero of the function f_1 form Ω_1 onto its complement $\mathbb{R}^3\setminus\overline{\Omega}_1$.

We start with the following well known results (for more specific properties see [2–5, 13, 16, 17, 29, 32, 43] and references cited therein).

Lemma A.1. (i) **[13, Theorem 1(i),(ii)]** *For all* $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, *the following layer potential operators*

$$V : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{1+\sigma}(\mathbb{R}^3\setminus\overline{\Omega}_1),$$

$$V : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{1+\sigma}_{\rm loc}(\Omega_1)\cap Z(\Omega_1),$$

$$W : H^{\frac{1}{2}+\sigma}(S_1) \to H^{1+\sigma}(\mathbb{R}^3\setminus\overline{\Omega}_1),$$

$$W : H^{\frac{1}{2}+\sigma}(S_1) \to H^{1+\sigma}_{\rm loc}(\Omega_1)\cap Z(\Omega_1)$$

are continuous.

(ii) **[31, Ch.XI, Theorem 11.2]; [16, Proposition 2.1]** *If* $f \in H^0_{\rm comp}(\mathbb{R}^3)$, *then* $\mathbf{P}f \in H^2_{\rm loc}(\mathbb{R}^3) \cap Z(\mathbb{R}^3)$ *and*

$$A_1 \mathbf{P} f = f \ \ in \ \ \mathbb{R}^3, \qquad \|\mathbf{P}f\|_{H^2(\Omega^*)} \leqslant C^* \, \|f\|_{H^0(\Omega_f)},$$

where Ω^* *is an arbitrary bounded domain in* $\mathbb{R}^3$, $\Omega_f := \operatorname{supp} f$, *and* $C^* > 0$ *is a constant which depends on the diameter of the domain* Ω^*.

(iii) **[13, Lemma 4.1]; [17, Theorem 1.1]** *For* $h \in H^{-\frac{1}{2}}(S_1)$ *and* $g \in H^{\frac{1}{2}}(S_1)$, *the following jump relations*

$$\gamma^{+}_{S_1} V\, h = \gamma^{-}_{S_1} V(h) = \mathcal{V}(h), \quad T_1^{\pm} V\, h = \Big(\pm \frac{1}{2} I + \mathcal{W}' \Big) h \quad on \quad S_1, \tag{A.1}$$

$$\gamma^{\pm}_{S_1} W\, g = \Big(\mp \frac{1}{2} I + \mathcal{W} \Big) g, \quad T_1^{+} W\, g = T_1^{-} W\, g =: \mathcal{L}\, g \quad on \quad S_1, \tag{A.2}$$

hold true, where I *stands for the identity operator, and*

$$\mathcal{V}\, h(y) := - \int\limits_{S_1} \Gamma(x-y,\omega)\, h(x)\, dS_x, \quad y \in S_1, \tag{A.3}$$

$$\mathcal{W}\, g(y) := - \int\limits_{S_1} [T_1(x,\partial_x) \Gamma(x-y,\omega))]\, g(x)\, dS_x, \quad y \in S_1, \tag{A.4}$$

$$\mathcal{W}'\, h(y) := - \int\limits_{S_1} [T_1(y,\partial_y) \Gamma(x-y,\omega))]\, h(x)\, dS_x, \quad y \in S_1, \tag{A.5}$$

$\Gamma(x,\omega)$ *is the radiating fundamental solution defined by* (3.5). *The operators* (A.4) *and* (A.5) *are to be understood in the Cauchy principal value sense, while* (A.3) *is a weakly singular integral operator.*

(iv) **[13, Theorem 1(iii)–(vi)]; [32, Theorems 7.1, 7.2]; [16, Theorems 3.1 & 4.1]; [5, Corollary 3.6, Theoem 3.10]** *For all* $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, *the operators*

$$\mathcal{V} : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{\frac{1}{2}+\sigma}(S_1), \qquad \pm\tfrac{1}{2}\, I + \mathcal{W}' : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1),$$
$$\pm\tfrac{1}{2}\, I + \mathcal{W} : H^{\frac{1}{2}+\sigma}(S_1) \to H^{\frac{1}{2}+\sigma}(S_1), \qquad \mathcal{L} : H^{\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1),$$

are continuous Fredholm operators with zero index.

Lemma A.2. *Let* $\mathcal{K}$ *and* $\mathcal{M}$ *be defined by* (3.15) *and* (3.16) *with* $\alpha > 0$. *For all* $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$ *the following operators*

$$\mathcal{K} \equiv \frac{1}{2}\, I + \mathcal{W}' + i\, \alpha\, \mathcal{V} : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1), \tag{A.6}$$

$$\mathcal{M} \equiv \mathcal{L} + i\, \alpha \Big(- \frac{1}{2}\, I + \mathcal{W} \Big) : H^{\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1), \tag{A.7}$$

are invertible.

Proof. Due to Lemma A.1(iv), we need only to prove that the operators (A.6) and (A.7) have the trivial null-spaces. First, we consider the case $\sigma = 0$ and let $g \in H^{-\frac{1}{2}}(S_1)$ be a solution of the homogeneous equation

$$\mathcal{K}\, g = 0 \quad \text{on} \ \ S_1, \tag{A.8}$$

and construct the function $v := Vg$ in $\mathbb{R}^3$, where Vg is the single layer potential defined by (3.2). Evidently, $v \in H^{1,0}(\mathbb{R}^3 \setminus \overline{\Omega}_1; A_1)$, $v \in H^{1,0}_{\rm loc}(\Omega_1; A_1) \cap Z(\Omega_1)$, $A_1(\partial_x,\omega) v = A_1(\partial_x,\omega) V(g) = 0$ in $\mathbb{R}^3 \setminus S_1$, and $T_1^{\pm} v = T_1^{\pm} V g \in H^{-\frac{1}{2}}(S_1)$ is well defined. In accordance with relation (3.15), equation (A.8) is equivalent to the condition

$$\big(T_1^{+} + i\, \alpha\, \gamma^{+}_{S_1}\big) v = 0 \quad \text{on} \quad S_1, \quad \alpha > 0. \tag{A.9}$$

Therefore v solves the homogeneous interior Robin problem in $\mathbb{R}^3\setminus\overline{\Omega}_1$. Boundary condition (A.9) and Green's formula

$$\int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_1} A_1 v\,\overline{v}\,dx + \int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_1} \big[E_1(v,\overline{v}) - \omega^2\kappa_1|v|^2\,\big]\,dx = \big\langle T_1^+ v\,,\,\overline{\gamma_{S_1}^+ v}\big\rangle_{S_1}$$

lead to the equality

$$\int\limits_{\mathbb{R}^3\setminus\overline{\Omega}_1} \big[E_1(v,\overline{v}) - \omega^2\kappa_1|v|^2\,\big]\,dx + i\,\alpha\int\limits_{S_1}|\gamma_{S_1}^+ v|^2\,dS = 0.$$

By separating imaginary part, we deduce $\gamma_{S_1}^+ v = 0$ on S_1, implying $T_1^+ v = 0$ on S_1. Therefore, with the help of the general integral representation formula of solutions of the homogeneous differential equation $A_1 v = 0$, $v = V(T_1^+ v) - W(\gamma_{S_1}^+ v)$, we finally deduce $v = V(g) = 0$ in $\mathbb{R}^3\setminus\overline{\Omega}_1$. By the continuity property of the single layer potential across the surface S_1 (see the first equation in (A.1)), we have $\gamma_{S_1}^+ v = \gamma_{S_1}^- v = 0$ on S_1. Consequently, the radiating function $v = Vg$ solves the homogeneous exterior Dirichlet problem for the operator $A_1(\partial_x,\omega)$ and therefore vanishes identically in Ω_1. Consequently, by the jump relations (A.1) for the conormal derivative of the single layer potential, we find that $g = 0$ on S_1 implying that the null-space of the operator (A.6) is trivial.

Now let $\sigma\in[-\frac{1}{2},\,\frac{1}{2}]$. Recall that for $-\frac{1}{2}\leqslant\sigma_1<\sigma_2\leqslant\frac{1}{2}$, the inclusion $H^{-\frac{1}{2}+\sigma_2}(S_1)\subset H^{-\frac{1}{2}+\sigma_1}(S_1)$ is continuous and dense. Therefore the null-space of the Fredholm operator (A.6) is the same for all $\sigma\in[-\frac{1}{2},\,\frac{1}{2}]$ (see, e.g., [33, Lemma 11.40], [1, Proposition 10.6]). This completes the proof for operator (A.6).

The proof for operator (A.7) is quite similar. □

Introduce the boundary operators $\widetilde{\mathcal{V}}$, $\widetilde{\mathcal{W}}$, $\widetilde{\mathcal{W}}'$, $\widetilde{\mathcal{L}}$, $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{M}}$ generated by the single and double layer potentials $\widetilde{V}$ and $\widetilde{W}$ constructed by the exponentially decaying real-valued fundamental solution $\Gamma(x-y,i)$ (see (3.5)). Evidently, they are defined by the same formulas as their counterpart operators $\mathcal{V}$, $\mathcal{W}$, $\mathcal{W}'$, $\mathcal{L}$, $\mathcal{K}$, $\mathcal{M}$, V and W with $\Gamma(x-y,i)$ for $\Gamma(x-y,\omega)$ and have all the mapping and jump properties described in Lemmas A.1 and A.2. In addition, for these "tilde" operators we have the following assertion.

Lemma A.3. *For all $\sigma\in[-\frac{1}{2},\,\frac{1}{2}]$ and $\alpha>0$, the following operators*

$$\widetilde{\mathcal{V}}: H^{-\frac{1}{2}+\sigma}(S_1)\to H^{\frac{1}{2}+\sigma}(S_1),$$

$$\pm\frac{1}{2}\,I+\widetilde{\mathcal{W}}': H^{-\frac{1}{2}+\sigma}(S_1)\to H^{-\frac{1}{2}+\sigma}(S_1),$$

$$\pm\frac{1}{2}\,I+\widetilde{\mathcal{W}}: H^{\frac{1}{2}+\sigma}(S_1)\to H^{\frac{1}{2}+\sigma}(S_1),$$

$$\widetilde{\mathcal{L}}: H^{\frac{1}{2}+\sigma}(S_1)\to H^{-\frac{1}{2}+\sigma}(S_1),$$

$$\widetilde{\mathcal{K}}\equiv\frac{1}{2}\,I+\widetilde{\mathcal{W}}'+i\,\alpha\,\widetilde{\mathcal{V}}: H^{-\frac{1}{2}+\sigma}(S_1)\to H^{-\frac{1}{2}+\sigma}(S_1),$$

$$\widetilde{\mathcal{M}}\equiv\widetilde{\mathcal{L}}+i\,\alpha\Big(-\frac{1}{2}\,I+\widetilde{\mathcal{W}}\Big): H^{\frac{1}{2}+\sigma}(S_1)\to H^{-\frac{1}{2}+\sigma}(S_1)$$

are invertible.

Proof. All the operators stated in the lemma are Fredholm ones with zero index and their null-spaces are the same for all $\sigma\in[-\frac{1}{2},\,\frac{1}{2}]$ (see, e.g., [33, Lemma 11.40], [1, Proposition 10.6]). Therefore it suffices to show that the null-spaces of the operators are trivial for $\sigma=0$.

Recall that $\Omega_1:=\Omega^-$ and $\Omega^+=\mathbb{R}^3\setminus\overline{\Omega}_1$.

First, let us prove that the null-space of the operator $\widetilde{\mathcal{V}}$ is trivial. Let $g\in H^{-\frac{1}{2}}(S_1)$ be a solution to the homogeneous equation $\widetilde{\mathcal{V}}g=0$ on S_1. Then the single layer potential $u=\widetilde{V}(g)$ belongs to

$H^1(\Omega^\pm, \widetilde{A}_1)$ with $\widetilde{A}_1 := A_1(\partial_x, i)$, exponentially decays at infinity, and solves the homogeneous interior and exterior Dirichlet problems

$$A_1(\partial_x, i)u = a_{kj}^{(1)}\,\partial_k\partial_j u - \kappa_1\, u = 0 \text{ in } \Omega^\pm, \qquad \gamma_{S_1}^\pm u = 0 \text{ on } S_1 = \partial\Omega^\pm.$$

Consequently, with the help of Green's formulas (cf. (2.11))

$$\int\limits_{\Omega^\pm} A_1(\partial_x, i)u(x)\,\overline{u(x)}\,dx + \int\limits_{\Omega^\pm}[\,E_1(u,\overline{u}) + \kappa_1|u(x)|^2\,]\,dx = \pm\big\langle T_1^\pm u\,,\,\overline{\gamma_{S_1}^\pm u}\big\rangle_{S_1}, \tag{A.10}$$

we deduce $u = 0$ in $\Omega^\pm$, whence $g = 0$ on S_1 follows due to the jump relations for the conormal derivative of the single layer potential (see Lemma A.1(iii)) which completes the proof for the operator $\widetilde{\mathcal{V}}$.

Now, let us consider the operator $\widetilde{\mathcal{M}}$ and let $h \in H^{\frac{1}{2}}(S_1)$ be a solution to the homogeneous equation $\widetilde{\mathcal{M}}h = 0$ on S_1. Then the double layer potential $w = \widetilde{W}(h)$ belongs to $H^1(\Omega^\pm; \widetilde{A}_1)$, exponentially decays at infinity and solves the homogeneous interior Robin's problem

$$A_1(\partial_x, i)w = a_{kj}^{(1)}\,\partial_k\partial_j w - \kappa_1\, w = 0 \text{ in } \Omega^+, \qquad T_1^+ w + i\,\alpha\,\gamma_{S_1}^+ w = 0 \text{ on } S_1.$$

Therefore by Green's formula (A.10), we deduce $w = 0$ in Ω^+ and by Lemma A.1(iii), we have $T_1^+ w = T_1^- w = 0$. Thus, w solves the homogeneous exterior Neumann problem and, consequently, $w = 0$ in Ω^- in view of (A.10). The jump properties of the double layer potential complete the proof for the operator $\widetilde{\mathcal{M}}$. □

For the other operators stated in the lemma the proofs are word for word.

Lemma A.4. *For $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, the operators*

$$\mathcal{V} - \widetilde{\mathcal{V}} : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{\frac{1}{2}+\sigma}(S_1), \tag{A.11}$$

$$\mathcal{W}' - \widetilde{\mathcal{W}}' : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1), \tag{A.12}$$

$$\mathcal{W} - \widetilde{\mathcal{W}} : H^{\frac{1}{2}+\sigma}(S_1) \to H^{\frac{1}{2}+\sigma}(S_1), \tag{A.13}$$

$$\mathcal{L} - \widetilde{\mathcal{L}} : H^{\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1), \tag{A.14}$$

$$\mathcal{K} - \widetilde{\mathcal{K}} : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1), \tag{A.15}$$

$$\mathcal{M} - \widetilde{\mathcal{M}} : H^{\frac{1}{2}+\sigma}(S_1) \to H^{-\frac{1}{2}+\sigma}(S_1). \tag{A.16}$$

are compact.

Proof. In view of Remark 3.1 and relations (3.8) and (3.9), the potential type operators $V - \widetilde{V}$ and $W - \widetilde{W}$ for $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$ have the following mapping properties:

$$V - \widetilde{V} : H^{-\frac{1}{2}+\sigma}(S_1) \to H^{3+\sigma}(\Omega_2),$$

$$W - \widetilde{W} : H^{\frac{1}{2}+\sigma}(S_1) \to H^{3+\sigma}(\Omega_2).$$

Therefore the traces on S_1 of the functions $V(h) - \widetilde{V}(h)$ and $W(g) - \widetilde{W}(g)$ with $h \in H^{-\frac{1}{2}+\sigma}(S_1)$ and $g \in H^{\frac{1}{2}+\sigma}(S_1)$ belong to $H^1(S_1)$ in view of the Lipschitz character of the surface S_1. Recall that in the case of Lipschitz surfaces, the space $H^s(S_1)$ is well-defined only for $-1 \leqslant s \leqslant 1$. Moreover, in general, the trace of a function from the space $H^s(\Omega^\pm)$ belongs either to the space $H^{s-\frac{1}{2}}(\partial\Omega^\pm)$ if $\frac{1}{2} < s < \frac{3}{2}$, or to the space $H^1(\partial\Omega^\pm)$ if $s > \frac{3}{2}$ (see, e.g., [13, 15], [23, Section 3]).

Consequently, for $\sigma \in (-\frac{1}{2}, \frac{1}{2})$, the operators (A.11) and (A.13) are smoothing operators with the range in $H^1(S_1)$ which is compactly imbedded in $H^{\frac{1}{2}+\sigma}(S_1)$, while operators (A.12), (A.14), (A.15) and (A.16) are smoothing operators with the range in $H^0(S_1)$ which is compactly imbedded in $H^{-\frac{1}{2}+\sigma}(S_1)$ for arbitrary $\sigma \in (-\frac{1}{2}, \frac{1}{2})$.

For $\sigma = \pm\frac{1}{2}$, the claim can be proved again using relations (3.8) and (3.9). For illustration, we consider operator (A.11) for $\sigma = \frac{1}{2}$, i.e., we show the compactness of the operator

$$\mathcal{V} - \widetilde{\mathcal{V}} \, : \, H^0(S_1) \to H^1(S_1).$$

Let M_0 be a bounded subset in $H^0(S_1)$, i.e. $\|g\|_{H^0(S_1)} \leqslant C_0$ for all $g \in M_0$. Let $\{g_n\}_{n=1}^{\infty} \in M_0$ be an arbitrary sequence and $Q(y,x) := \Gamma(y-x,\omega) - \Gamma(y-x,i)$ be defined by (3.8). Then the sequence

$$v_n(y) = \mathcal{V}(g_n)(y) - \widetilde{\mathcal{V}}(g_n)(y) \equiv \mathcal{Q}\, g_n(y) := \int\limits_{S_1} Q(y,x)\, g_n(x)\, dS_x, \quad y \in S_1,$$

contains a fundamental subsequence in the norm of the space $H^0(S_1)$ since the Hilbert-Schmidt integral operator $\mathcal{Q} : H^0(S_1) \to H^0(S_1)$ is compact. We denote the fundamental subsequence by $v_n^{(1)} = \mathcal{Q}\, g_n^{(1)}$. It is evident that the same arguments can be applied to the sequence

$$D_{y_j} v_n^{(1)}(y) = D_{y_j} \mathcal{Q}\, g_n^{(1)}(y) = \int\limits_{S_1} D_{y_j} Q(y,x)\, g_n(x)\, dS_x, \quad y \in S_1,$$

where D_{y_j} denotes a tangential differentiation. We again conclude that this sequence contains a fundamental subsequence in the norm of the space $H^0(S_1)$. Denote this subsequence by $v_n^{(2)} = \mathcal{Q}\, g_n^{(2)}$. Thus we have shown that the sequence $v_n = \mathcal{Q}\, g_n$ contains a fundamental subsequence $v_n^{(2)}$ in the norm of the space $H^{(1)}(S_1)$ which implies that the operator $\mathcal{Q} : H^0(S_1) \to H^1(S_1)$, i.e., operator (A.11) for $\sigma = \frac{1}{2}$ is compact. For $\sigma = -\frac{1}{2}$, the claim follows from the duality arguments.

Now let us consider operator (A.13) for $\sigma = \frac{1}{2}$,

$$\mathcal{R} := \mathcal{W} - \widetilde{\mathcal{W}} : H^1(S_1) \to H^1(S_1). \tag{A.17}$$

Further, let $M_1 \subset H^1(S_1)$ be a bounded set and $\{g_n\}_{n=1}^{\infty} \in M_1$ be an arbitrary sequence. It is evident that the kernel function $T_1(x,\partial_x)Q(y,x)$ of the weakly singular integral operator

$$\mathcal{R}\, g_n(y) := \int\limits_{S_1} T_1(x,\partial_x) Q(y,x)\, g_n(x)\, dS_x, \quad y \in S_1, \tag{A.18}$$

is bounded on $S_1 \times S_1$ in view of (3.8)–(3.9). Moreover, the kernel function $D_{y_j} T_1(x,\partial_x)Q(y,x)$ of the operator $D_{y_j} \mathcal{R}\, g_n(y)$ has a weak singularity of type $\mathcal{O}(|x-y|^{-1})$. Therefore, by the same arguments as above, we again can show that the sequence $\{\mathcal{R}\, g_n\}_{n=1}^{\infty}$ contains a fundamental subsequence in the norm of the space $H^1(S_1)$ which completes the proof for operator (A.17), i.e., for operator (A.13) for $\sigma = \frac{1}{2}$.

The duality arguments imply the compactness of operator (A.12) for $\sigma = -\frac{1}{2}$.

The compactness of operator (A.12) for $\sigma = \frac{1}{2}$ and operator (A.13) for $\sigma = -\frac{1}{2}$ is trivial.

Next, we consider operator (A.14) for $\sigma = \frac{1}{2}$,

$$\mathcal{N} := \mathcal{L} - \widetilde{\mathcal{L}} : H^1(S_1) \to H^0(S_1).$$

We have

$$\mathcal{N}\, g(y) := \int\limits_{S_1} T_1(y,\partial_y) T_1(x,\partial_x) Q(y,x)\, g(x)\, dS_x, \quad y \in S_1.$$

It is evident that the kernel function $T_1(y,\partial_y)T_1(x,\partial_x)Q(y,x)$ is symmetric and possesses a weak singularity of type $\mathcal{O}(|x-y|^{-1})$ due to (3.8)–(3.9). Therefore the Hilbert-Schmidt operator $\mathcal{N} : H^0(S_1) \to H^0(S_1)$ is compact, implying the compactness of operator (A.14). By the duality arguments, we conclude the compactness of operator (A.14) for $\sigma = -\frac{1}{2}$.

The above results along with relations (A.6)–(A.7) and their counterparts for the "tilde" operators imply directly the compactness of operators (A.15) and (A.16) for $\sigma = \pm\frac{1}{2}$, which completes the proof. $\square$

Remark A.5. Actually, in the proof of Lemma A.4 we have shown the following mapping properties (cf. [5]):

$$\mathcal{V} - \widetilde{\mathcal{V}} : H^{-\frac{1}{2}}(S_1) \to H^1(S_1),$$

$$\mathcal{W}' - \widetilde{\mathcal{W}}' : H^{-\frac{1}{2}}(S_1) \to H^0(S_1),$$

$$\mathcal{W} - \widetilde{\mathcal{W}} : H^{\frac{1}{2}}(S_1) \to H^1(S_1),$$

$$\mathcal{L} - \widetilde{\mathcal{L}} : H^{\frac{1}{2}}(S_1) \to H^0(S_1),$$

$$\mathcal{K} - \widetilde{\mathcal{K}} : H^{-\frac{1}{2}}(S_1) \to H^0(S_1),$$

$$\mathcal{M} - \widetilde{\mathcal{M}} : H^{\frac{1}{2}}(S_1) \to H^0(S_1).$$

For the operator $\mathcal{K}$ defined by (A.6), we have the following representation $\mathcal{K} = \widetilde{\mathcal{T}} + \mathcal{C}$ with $\widetilde{\mathcal{T}} = \frac{1}{2}I + \widetilde{\mathcal{W}}'$ and $\mathcal{C} = \mathcal{W}' - \widetilde{\mathcal{W}}' + i\alpha\mathcal{V}$ and by Lemmas A.2 and A.3, we deduce

$$\begin{aligned} \mathcal{K}^{-1} &= \widetilde{\mathcal{T}}^{-1} - \mathcal{K}^{-1}\mathcal{C}\widetilde{\mathcal{T}}^{-1}, \\ \mathcal{K}^{-1}\mathcal{M} &= \widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}} + \mathcal{G}, \end{aligned} \tag{A.19}$$

where

$$\mathcal{G} := -\mathcal{K}^{-1}\mathcal{C}\widetilde{\mathcal{T}}^{-1}\mathcal{M} + \widetilde{\mathcal{T}}^{-1}\Big[\mathcal{L} - \widetilde{\mathcal{L}} + i\alpha\Big(-\frac{1}{2}I + \mathcal{W}\Big)\Big].$$

By Lemmas A.2, A.3 and A.4, the following operators

$$\begin{aligned} \mathcal{K}^{-1}\mathcal{C}\widetilde{\mathcal{T}}^{-1} &: H^{-\frac{1}{2}}(S_1) \to H^0(S_1), \\ \mathcal{G} &: H^{\frac{1}{2}}(S_1) \to H^0(S_1) \end{aligned} \tag{A.20}$$

are continuous.

Lemma A.6. *There are positive constants $C_1' > 0$ and $C_2' > 0$ such that*

$$\mathrm{Re}\,\langle -\mathcal{K}^{-1}\mathcal{M}\psi\,,\,\overline{\psi}\rangle_{S_1} \geq C_1'\,\|\psi\|^2_{H^{\frac{1}{2}}(S_1)} - C_2'\,\|\psi\|^2_{H^0(S_1)} \ \text{ for all } \ \psi \in H^{\frac{1}{2}}(S_1).$$

Proof. In view of (A.19), (A.20) and the Schwartz inequality for all $\psi \in H^{\frac{1}{2}}(S_1)$, we have

$$\begin{aligned} \mathrm{Re}\big\langle -\mathcal{K}^{-1}\mathcal{M}\psi\,,\,\overline{\psi}\big\rangle_{S_1} &= \mathrm{Re}\big\langle -[\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}} + \mathcal{G}]\psi\,,\,\overline{\psi}\big\rangle_{S_1} \\ &\geqslant \mathrm{Re}\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\psi\,,\,\overline{\psi}\big\rangle_{S_1} - |\langle\mathcal{G}\psi\,,\,\overline{\psi}\rangle_{S_1}| \\ &= \mathrm{Re}\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\psi\,,\,\overline{\psi}\big\rangle_{S_1} - \Big|\int\limits_{S_1}\overline{\psi}\,\mathcal{G}\psi\,dS\Big| \\ &\geqslant \mathrm{Re}\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\psi\,,\,\overline{\psi}\big\rangle_{S_1} - \|\mathcal{G}\psi\|_{H^0(S_1)}\,\|\overline{\psi}\|_{H^0(S_1)} \\ &\geqslant \mathrm{Re}\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\psi\,,\,\overline{\psi}\big\rangle_{S_1} - c_1\,\|\psi\|_{H^{\frac{1}{2}}(S_1)}\,\|\psi\|_{H^0(S_1)} \end{aligned} \tag{A.21}$$

with some positive constant c_1.

To estimate the first summand from below, we proceed as follows. The general integral representation formula for an exponentially decaying solution to the homogeneous equation $A_1(\partial_x, i)w = 0$ in Ω_1 reads as

$$w + \widetilde{V}(T_1^- w) - \widetilde{W}(\gamma_{S_1}^- w) = 0 \quad \text{in} \quad \Omega_1.$$

Substituting here $w = \widetilde{W}\varphi$ with arbitrary $\varphi \in H^{\frac{1}{2}}(S_1)$ and taking the generalized trace of the conormal derivative on S_1, we obtain

$$\widetilde{\mathcal{L}}\Big(\frac{1}{2}I + \widetilde{\mathcal{W}}\Big)\varphi = \Big(\frac{1}{2}I + \widetilde{\mathcal{W}}'\Big)\,\widetilde{\mathcal{L}}\varphi \ \text{ on } \ S_1.$$

This implies the following operator relation with the domain of definition $H^{\frac{1}{2}}(S_1)$ and the range $H^{-\frac{1}{2}}(S_1)$,

$$\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}} = \Big(\frac{1}{2}I + \widetilde{\mathcal{W}}'\Big)^{-1}\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}\Big(\frac{1}{2}I + \widetilde{\mathcal{W}}\Big)^{-1}. \tag{A.22}$$

Further, substituting $u = \widetilde{W}g$ with $g = \big(\frac{1}{2}I + \widetilde{\mathcal{W}}\big)^{-1}\varphi$ and $\varphi \in H^{\frac{1}{2}}(S_1)$ in (A.10) for $\Omega^- = \Omega_1$ and taking into consideration the equalities $T_1^- u = \widetilde{\mathcal{L}}\Big(\frac{1}{2}I + \widetilde{\mathcal{W}}\Big)^{-1}\varphi$ and (A.22), we get

$$-\big\langle T_1^- u\,,\,\overline{\gamma_{S_1}^- u}\big\rangle_{S_1} = \big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\varphi\,,\,\overline{\varphi}\big\rangle_{S_1} = \int\limits_{\Omega_1}\big[\,E_1(u,\overline{u}) + \kappa_1|u(x)|^2\,\big]\,dx, \tag{A.23}$$

where E_1 is defined in (2.9). Since the matrix $\mathbf{a}_1 = \big[\,a_{kj}^{(1)}\,\big]_{k,j=1}^3$ is positive definite, $\kappa_1 > 0$ and $\gamma_{S_1}^- u = \varphi$, with the help of the trace theorem, from (A.23), we deduce

$$\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\varphi\,,\,\overline{\varphi}\big\rangle_{S_1} \geqslant c_2\,\|u\|^2_{H^1(\Omega_1)} \geqslant c_3\,\|\gamma_{S_1}^- u\|^2_{H^{\frac{1}{2}}(S_1)} = c_3\,\|\varphi\|^2_{H^{\frac{1}{2}}(S_1)}, \tag{A.24}$$

where c_2 and c_3 are some positive constants.

Now, using the inequalities (A.24) and

$$\|\psi\|_{H^{\frac{1}{2}}(S_1)}\,\|\psi\|_{H^0(S_1)} \leqslant \varepsilon\,\|\psi\|^2_{H^{\frac{1}{2}}(S_1)} + \frac{1}{4\varepsilon}\,\|\psi\|^2_{H^0(S_1)},$$

from (A.21), we finally obtain

$$\begin{aligned}\operatorname{Re}\big\langle -\mathcal{K}^{-1}\mathcal{M}\psi\,,\,\overline{\psi}\big\rangle_{S_1} &\geqslant \operatorname{Re}\big\langle -\widetilde{\mathcal{T}}^{-1}\widetilde{\mathcal{L}}\psi\,,\,\overline{\psi}\big\rangle_{S_1} - c_1\,\|\psi\|_{H^{\frac{1}{2}}(S_1)}\,\|\psi\|_{H^0(S_1)}\\ &\geqslant c_3\,\|\psi\|^2_{H^{\frac{1}{2}}(S_1)} - c_1\,\|\psi\|_{H^{\frac{1}{2}}(S_1)}\,\|\psi\|_{H^0(S_1)}\\ &\geqslant (c_3 - \varepsilon c_1)\,\|\psi\|^2_{H^{\frac{1}{2}}(S_1)} - (4\varepsilon)^{-1}c_1\,\|\psi\|^2_{H^0(S_1)},\end{aligned}$$

where ε is an arbitrarily small positive number. This completes the proof. □

Remark A.7. In many papers, the one-sided boundary traces of layer potentials and their conormal derivaives are understood in the nontangential limit sense (for details see, e.g., [17, 32, 43]). Note that in the case of a bounded Lipschitz domain Ω, a single layer potential $V(h)$ with a density $h \in H^{-\frac{1}{2}}(\partial\Omega)$, as well as a double layer potential $W(g)$ with a density $g \in H^{\frac{1}{2}}(\partial\Omega)$, belong to the space $H^1(\Omega)$ and possess the Sobolev boundary traces belonging to the space $H^{\frac{1}{2}}(\partial\Omega)$ (see Lemma A.1). Therefore, for these potentials the nontangential boundary values exist almost everywhere on $\partial\Omega$ and the corresponding nontangential maximal functions are square integrable (see [32, 43]). Consequently, for these potentials the Sobolev traces and the nontangential traces on $\partial\Omega$ coincide (see, e.g., [2, Remark 6.7]).

Acknowledgement

This work was supported by the Shota Rustaveli National Science Foundation of Georgia (SRNSF) (Grant number FR-18-126).

References

1. M. S. Agranovich, Elliptic singular integro-differential operators. (Russian) *Uspehi Mat. Nauk* **20** (1965), no. 5 (125), 3–120.
2. M. S. Agranovich, Spectral properties of potential-type operators for a class of strongly elliptic systems on smooth and Lipschitz surfaces. (Russian) *translated from Tr. Mosk. Mat. Obs.* **62** (2001), 3–53 *Trans. Moscow Math. Soc.* 2001, 1–47.
3. M. S. Agranovich, Spectral problems for second-order strongly elliptic systems in domains with smooth and non-smooth boundaries. (Russian) *translated from Uspekhi Mat. Nauk* **57** (2002), no. 5 (347), 3–78 *Russian Math. Surveys* **57** (2002), no. 5, 847–920.
4. M. S. Agranovich, *Sobolev Spaces, their Generalizations and Elliptic Problems in Smooth and Lipschitz Domains.* Revised translation of the 2013 Russian original. Springer Monographs in Mathematics. Springer, Cham, 2015.

5. M. S. Agranovich, R. Mennicken, Spectral problems for the Helmholtz equation with a spectral parameter in the boundary conditions on a nonsmooth surface. (Russian) *translated from Mat. Sb.* **190** (1999), no. 1, 29–68 *Sb. Math.* **190** (1999), no. 1-2, 29–69.
6. H. Brakhage, P. Werner, Über das Dirichletsche Aussenraumproblem für die Helmholtzsche Schwingungsgleichung. (German) *Arch. Math.* **16** (1965), 325–329.
7. A. Buffa, R. Hiptmair, Regularized combined field integral equations. *Numer. Math.* **100** (2005), no. 1, 1–19.
8. A. J. Burton, G. F. Miller, The application of integral equation methods to the numerical solution of some exterior boundary-value problems. *Proc. Roy. Soc. London Ser. A* **323** (1971), 201–210.
9. G. Chen, J. Zhou, *Boundary Element Methods.* Computational Mathematics and Applications. Academic Press, Ltd., London, 1992.
10. O. Chkadua, S. E. Mikhailov, D. Natroshvili, Singular localised boundary-domain integral equations of acoustic scattering by inhomogeneous anisotropic obstacle. *Math. Methods Appl. Sci.* **41** (2018), no. 17, 8033–8058.
11. D. Colton, R. Kress, *Integral Equation Methods in Scattering Theory.* Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1983.
12. D. Colton, R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory.* Second edition. Applied Mathematical Sciences, 93. Springer-Verlag, Berlin, 1998.
13. M. Costabel, Boundary integral operators on Lipschitz domains: Elementary results. *SIAM J. Math. Anal.* **19** (1988), 613–626.
14. M. Costabel, E. Stephan, A direct boundary integral equation method for transmission problems. *J. Math. Anal. Appl.* **106** (1985), no. 2, 367–413.
15. Z. Ding, A proof of the trace theorem of Sobolev spaces on Lipschitz domains. *Proc. Amer. Math. Soc.* **124** (1996), no. 2, 591–600.
16. E. Fabes, O. Mendez, M. Mitrea, Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains. *J. Funct. Anal.* **159** (1998), no. 2, 323–368.
17. W. Gao, Layer potentials and boundary value problems for elliptic systems in Lipschitz domains. *J. Funct. Anal.* **95** (1991), no. 2, 377–399.
18. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.
19. L. Jentsch, D. Natroshvili, Interaction between thermoelastic and scalar oscillation fields. *Integral Equations Operator Theory* **28** (1997), no. 3, 261–288.
20. L. Jentsch, D. Natroshvili, Non-local approach in mathematical problems of fluid-structure interaction. *Math. Methods Appl. Sci.* **22** (1999), no. 1, 13–42.
21. D. Jerison, C. Kenig, The Dirichlet problem in nonsmooth domains. *Ann. of Math. (2)* **113** (1981), no. 2, 367–382.
22. D. Jerison, C. Kenig, The Neumann problem on Lipschitz domains. *Bull. Amer. Math. Soc. (N.S.)* **4** (1981), no. 2, 203–207.
23. D. Jerison, C. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.* **130** (1995), no. 1, 161–219.
24. R. Kittappa, R. Kleinman, Acoustic scattering by penetrable homogeneous objects. *J. Mathematical Phys.* **16** (1975), 421–432.
25. R. Kleinman, P. Martin, On single integral equations for the transmission problem of acoustics. *SIAM J. Appl. Math.* **48** (1988), no. 2, 307–325.
26. R. Kress, G. Roach, Transmission problems for the Helmholtz equation. *J. Mathematical Phys.* **19** (1978), no. 6, 1433–1437.
27. R. Leis, Zur Dirichletschen Randwertaufgabe des Aussenraumes der Schwingungsgleichung. (German) *Math. Z.* **90** (1965), 205–211.
28. P. A. Martin, Acoustic scattering by inhomogeneous obstacles. *SIAM J. Appl. Math.* **64** (2003), no. 1, 297–308.
29. W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations.* Cambridge University Press, Cambridge, 2000.
30. S. E. Mikhailov, Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains. *J. Math. Anal. Appl.* **378** (2011), no. 1, 324–342.
31. S. G. Mikhlin, S. Prössdorf, *Singular Integral Operators.* translated from the German by Albrecht Böttcher and Reinhard Lehmann. Springer-Verlag, Berlin, 1986.
32. M. Mitrea, Boundary value problems and Hardy spaces associated to the Helmholtz equation in Lipschitz domains. *J. Math. Anal. Appl.* **202** (1996), no. 3, 819–842.
33. M. Mitrea, M. Wright, Boundary value problems for the Stokes system in arbitrary Lipschitz domains. *Astérisque* no. **344** (2012), viii+241 pp.
34. D. Natroshvili, S. Kharibegashvili, Z. Tediashvili, Direct and inverse fluid-structure interaction problems. Dedicated to the memory of Gaetano Fichera (Italian). *Rend. Mat. Appl. (7)* **20** (2000), 57–92.
35. J.-C. Nédélec, *Acoustic and Electromagnetic Equations.* Integral representations for harmonic problems. Applied Mathematical Sciences, 144. Springer-Verlag, New York, 2001.
36. O. I. Panič, On the solubility of exterior boundary-value problems for the wave equation and for a system of Maxwell's equations. (Russian) *Uspehi Mat. Nauk* **20** (1965), no. 1 (121), 221–226.

37. F. Rellich, Über das asymptotische Verhalten der Lösungen von $(\Delta + k^2)u = 0$ in anendlichen Gebieten. *Jber. Deutsch. Math. Verein* **53**(1943), 57–65.
38. M. Renardy, R. C. Rogers, *An Introduction to Partial Differential Equations*. Second edition. Texts in Applied Mathematics, 13. Springer-Verlag, New York, 2004.
39. X. X. Tao, S. Y. Zhang, On the unique continuation properties for elliptic operators with singular potentials. *Acta Math. Sin. (Engl. Ser.)* **23** (2007), no. 2, 297–308.
40. R. H. Torres, G. V. Welland, The Helmholtz equation and transmission problems with Lipschitz interfaces. *Indiana Univ. Math. J.* **42** (1993), no. 4, 1457–1485.
41. B. R. Vainberg, Principles of radiation, limiting absorption and limiting amplitude in the general theory of partial differential equations. (Russian) *Uspehi Mat. Nauk* **21** (1966), no. 3 (129), 115–194.
42. I. N. Vekua, On metaharmonic functions. (Russian) *Trav. Inst. Math. Tbilissi* **12** (1943), 105–174.
43. G. Verchota, Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. Funct. Anal.* **59** (1984), no. 3, 572–611.
44. P. Werner, Zur mathematischer Theorie akustischer Wellenfelder. (German) *Arch. Rational Mech. Anal.* **6** (1960), 231–260 (1960).
45. P. Werner, Beugungsprobleme der mathematischen Akustik. (German) *Arch. Rational Mech. Anal.* **12** (1963), 155–184.

(Received 26.07.2020)

[1]Departament of Mathematics, Georgian Technical University, 77 Kostava Str., Tbilisi 0175, Georgia

[2]I. Javakhishvili Tbilisi State University, I. Vekua Institute of Applied Mathematics, 2 University Str., Tbilisi 0186, Georgia

E-mail address: 18barbare@gmail.com
E-mail address: m_mrevlishvili@yahoo.com
E-mail address: natrosh@hotmail.com