Transactions of A. Razmadze
Mathematical Institute
Vol. **174** (2020), issue 3, 277–302

BOUNDARY VALUE PROBLEMS OF THERMOELASTIC DIFFUSION THEORY WITH MICROTEMPERATURES AND MICROCONCENTRATIONS

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Abstract. The paper deals with the linear theory of thermoelastic diffusion for elastic isotropic and homogeneous materials with microtempeatures and microconcetrations. For the system of the corresponding differential equations of pseudo-oscillations the fundamental matrix is constructed explicitly in terms of elementary functions. With the help of Green's identities the general integral representation formula of solutions is derived by means of generalized layer and Newtonian potentials. The basic Dirichlet and Neumann type boundary value problems are formulated in appropriate function spaces and the uniqueness theorems are proved. The existence theorems for classical solutions are established by using the potential method.

1. Introduction

Construction of a refined mathematical model of continuum mechanics with regard for different physical fields and their investigation is a very important problem from the theoretical and practical points of view, due to the rapidly increasing use of composite materials in modern technological processes, as well as in geology, biology, medicine, etc.

One such refined model, a thermoelastic diffusion theory with microtemperatures and microconcentrations, is proposed by M. Aouadi, M. Ciarletta, and V, Tibullo [1]. In this paper, the dynamical problems for a thermoelastic material with diffusion, whose microelements are assumed to possess microtemperatures and microconcentrations, are considered. The constitutive and field equations of the thermodynamic for the homogeneous and isotropic bodies are derived. Using the semigroup theory for linear operators, they show that a wide class of mixed problems with appropriate initial and boundary conditions are well posed, and the asymptotic behavior of solutions is established for a sufficiently large time parameter.

Recently, in [2], a linear dynamical problem involving a thermoelastic material with diffusion, whose microelements are assumed to possess microtemperatures and microconcentrations, has been analyzed. The problem is studied from the numerical point of view, introducing a fully discrete approximation by using the finite element method and the implicit Euler scheme. A discrete stability property is established and some a priori error estimates are obtained.

The system of differential equations of thermodynamic diffusion linear theory for isotropic homogeneous elastic materials with microtemperatures and microconcentrations with respect to the displacement vector, microconcentration vector, microtemperature vector, chemical potential function and temperature function, represents a fully coupled complex system of second order partial differential equations (see [1]).

If the physical characteristics involved in the dynamical system of differential equations are time harmonic dependent (i.e., they are represented as the product of the time dependent exponential function $exp(-i\sigma t)$ with a complex parameter $\sigma = \sigma_1 + i\sigma_2$, $\sigma_1 \in \mathbb{R}$, $\sigma_2 > 0$, and a function of the spatial variable $x \in \mathbb{R}^3$), then we have the so- called *system of pseudo-oscillation equations.* The corresponding matrix differential operator is strongly elliptic, formally non-self-adjoint operator with constant coefficients.

2020 *Mathematics Subject Classification.* 31B10, 47G10, 47G30, 74F05, 74A15.

Key words and phrases. Microtemperatures; Microconcentrations; Thermoelastic diffusion; Potential theory; Integral equations.

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The present paper is devoted to the investigation of the basic boundary value problems for the system of pseudo-oscillation equations for homogeneous isotropic materials by using the potential method.

To this end, we construct the matrix of fundamental solutions explicitly in terms of elementary functions for the pseudo-oscillation equations and investigate mapping properties of the corresponding volume and layer potential operators.

Using the approaches developed in [5,6,8,12,15], with the help of the potential method we reduce the Dirichlet and Neumann type boundary value problems to the corresponding system of singular integral equations and prove the existence theorems in the space of regular vector functions.

2. Constitutive Relations and Basic Differential Equations

Denote by $\mathbb{R}^3$ the three-dimensional Euclidean space and let $\Omega^+\subset\mathbb{R}^3$ be a bounded domain with boundary $S:=\partial\Omega^+$, $\overline{\Omega^+}=\Omega^+\cup S$. Further, let $\Omega^-=\mathbb{R}^3\backslash\overline{\Omega^+}$. We assume that $\overline{\Omega}\in\{\overline{\Omega^+},\overline{\Omega^-}\}$ is filled with a thermoelastic diffusion isotropic and homogeneous material with microtemperatures and microconcentration. Denote by $u=(u_1,u_2,u_3)^\top$, $C=(C_1,C_2,C_3)^\top$, and $T=(T_1,T_2,T_3)^\top$ the displacement vector, the microconcentration vector and the microtemperatures vector, respectively. By P we denote the chemical potential of material and by ϑ the temperature, measured from fixed absolute temperature T_0. We assume that T_0 is a given positive constant. The symbol $(\cdot)^\top$ denotes transposition.

Denote by t_{ij}, η_{ij}, q_{ij}, η_j, and q_j the stress tensor, the first mass diffusion flux moment tensor, the first heat flux moment tensor, the flux vector of mass diffusion, and the heat flux vector, respectively. By C^*, S^*, σ_i^*, ζ_i^*, Ω_i^*, and ϵ_i^* we denote the concentration of the diffusive material, the microentropy, the micromass, the microheat flux average, the first moment of mass diffusion, and the first moment of energy vector, respectively.

In the case of an isotropic and homogeneous thermoelastic diffusion material, with microtemperatures and microconcentration, the constitutive equations read as follows [1]

$$t_{ij}=t_{ij}(U):=\mu\,(\partial_j\,u_i+\partial_i\,u_j)+\delta_{ij}\,(\lambda_0\ \mathrm{div}\,u-\gamma_2\,P-\gamma_1\,\vartheta), \tag{2.1}$$
$$\eta_{ij}=\eta_{ij}(U):=-h_4\,\delta_{ij}\ \mathrm{div}\,C-h_5\,\partial_j\,C_i-h_6\,\partial_i\,C_j, \tag{2.2}$$
$$q_{ij}=q_{ij}(U):=-k_4\,\delta_{ij}\ \mathrm{div}\,T-k_5\,\partial_j\,T_i-k_6\,\partial_i\,T_j, \tag{2.3}$$
$$\eta_i=\eta_i(U):=h_1\,C_i+h\,\partial_i\,P, \tag{2.4}$$
$$q_i=q_i(U):=k_1\,T_i+k\,\partial_i\,\vartheta, \tag{2.5}$$
$$\rho\,S^*(U):=\gamma_1\ \mathrm{div}\,u+c\vartheta+\varkappa\,P,$$
$$C^*(U):=\gamma_2\ \mathrm{div}\,u+\varkappa\,\vartheta+m\,P,$$
$$\sigma_i^*(U):=(h-h_3)\,\partial_i\,P+(h_1-h_2)\,C_i\,,$$
$$\zeta_i^*(U):=(k-k_3)\,\partial_i\,\vartheta+(k_1-k_2)\,T_i\,,$$
$$\rho\Omega_i^*(U):=-m_1\,C_i-\varkappa_1\,T_i\,,$$
$$\rho\,\epsilon_i^*(U):=-\varkappa_1\,C_i-c_1\,T_i\,,$$

where $U=(u,C,T,P,\vartheta)^\top$, δ_{ij} is the Kronecker delta, $\partial=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$, $j=1,2,3$;

$$\lambda_0=\lambda-\frac{\beta_2^2}{\varrho},\ \ \gamma_1=\beta_1+\frac{\overline{\omega}\beta_2}{\varrho},\ \ \gamma_2=\frac{\beta_2}{\varrho},$$

λ and μ are Lame's constants, $\beta_1=(3\lambda+2\mu)\alpha_t$, $\beta_2=(3\lambda+2\mu)\alpha_c$, where α_t is the coefficient of linear thermal expansion and α_c is the coefficient of linear diffusion expansion; $\overline{\omega}$ and ϱ are the measures of thermodiffusion and diffusive effects, respectively; ρ is the mass density and h, k, h_j, k_j, $j=1,2,\dots,6$, are the thermoelastic material constants;

$$c=\frac{\rho c_E}{T_0}+\frac{\overline{\omega}^2}{\varrho},\quad \varkappa=\frac{\overline{\omega}}{\varrho},\quad m=\frac{1}{\varrho},$$

where c_E is the specific heat at constant strain; c_1 and m_1 are the constants of microthermal and microdiffusion conductivity, respectively; $\varkappa_1$ is measure of microthermodiffusion.

In the sequel, we assume that the above constitutive coefficients satisfy the following assumptions [1]

$$\begin{aligned}&\rho>0,\ \ \mu>0,\ \ 3\lambda_0+2\mu>0,\ \ c>0,\ \ c_1>0,\ \ cm-\varkappa^2>0,\ \ c_1m_1-\varkappa_1^2>0,\\&h>0,\ \ 3h_4+h_5+h_6\geq 0,\ \ h_6\pm h_5\geq 0,\ \ 4h\,h_2-(h_1+h_3)^2\geq 0,\\&k>0,\ \ 3k_4+k_5+k_6\geq 0,\ \ k_6\pm k_5\geq 0,\ \ 4\,T_0\,k\,k_2-(\,k_1+T_0\,k_3)^2\geq 0.\end{aligned}\tag{2.6}$$

The linear field equations of dynamics of the thermoelasticity diffusion theory with microtemperatures and microconcentrations of homogeneous and isotropic bodies have the form [1]

$$\begin{aligned}&\mu\Delta u(x,t)+(\lambda_0+\mu)\,\mathrm{grad\ div}\,u(x,t)-\gamma_2\ \mathrm{grad}\,P(x,t)-\gamma_1\ \mathrm{grad}\,\vartheta(x,t)+\rho\,F(x,t)=\rho\frac{\partial^2u(x,t)}{\partial t^2},\\&h_6\Delta C(x,t)+(h_4+h_5)\,\mathrm{grad\ div}\,C(x,t)-h_2C(x,t)-h_3\,\mathrm{grad}\,P(x,t)=m_1\frac{\partial C(x,t)}{\partial t}+\varkappa_1\frac{\partial T(x,t)}{\partial t},\\&k_6\Delta T(x,t)+(k_4+k_5)\,\mathrm{grad\ div}\,T(x,t)-k_2T(x,t)-k_3\,\mathrm{grad}\ \vartheta(x,t)-\rho\,G(x,t)\\&\qquad\qquad\qquad=\varkappa_1\frac{\partial C(x,t)}{\partial t}+c_1\frac{\partial T(x,t)}{\partial t},\\&-\gamma_2\,\frac{\partial}{\partial t}\ \mathrm{div}\,u(x,t)+h_1\ \mathrm{div}\,C(x,t)+h\,\Delta P(x,t)=m\frac{\partial P(x,t)}{\partial t}+\varkappa\frac{\partial\vartheta(x,t)}{\partial t},\\&-\gamma_1\,\frac{\partial}{\partial t}\ \mathrm{div}\,u(x,t)+\frac{k_1}{T_0}\ \mathrm{div}\,T(x,t)+\frac{k}{T_0}\,\Delta\,\vartheta(x,t)+\frac{\rho}{T_0}\,s(x,t)=\varkappa\frac{\partial P(x,t)}{\partial t}+c\frac{\partial\vartheta(x,t)}{\partial t},\end{aligned}\tag{2.7}$$

where Δ is the Laplace operator, t is the time variable, $F=(F_1,F_2,F_3)^\top$ is the body force vector per unit mass, $G=(G_1,G_2,G_3)^\top$ is the first moment of the heat source vector, s is the heat source per unit mass.

If all the vector and scalar functions in (2.7) are harmonic time dependent, i.e.,

$$\begin{aligned}u(x,t)=&u(x)\ \exp\{-i\,t\sigma\},\ \ C(x,t)=C(x)\ \exp\{-i\,t\sigma\},\ \ T(x,t)=T(x)\ \exp\{-i\,t\sigma\},\\P(x,t)=&P(x)\ \exp\{-i\,t\sigma\},\ \ \vartheta(x,t)=\vartheta(x)\ \exp\{-i\,t\sigma\},\\F(x,t)=&F(x)\ \exp\{-i\,t\sigma\},\ \ G(x,t)=G(x)\ \exp\{-i\,t\sigma\},\ \ s(x,t)=s(x)\ \exp\{-i\,t\sigma\},\end{aligned}$$

with $\sigma\in\mathbb{R}$ and $i=\sqrt{-1}$, we obtain the system of steady state oscillation equations of the thermoelastic diffusion linear theory with microtemperatures and microconcentrations:

$$\mu\Delta u(x)+(\lambda_0+\mu)\,\mathrm{grad\,div}\,u(x)+\rho\sigma^2u(x)-\gamma_2\ \mathrm{grad}\,P(x)-\gamma_1\ \mathrm{grad}\,\vartheta(x)=-\rho\,F(x),\tag{2.8}$$

$$h_6\Delta C(x)+(h_4+h_5)\,\mathrm{grad\,div}\,C(x)+\delta\,C(x)+i\sigma\varkappa_1T(x)-h_3\ \mathrm{grad}\,P(x)=0,\tag{2.9}$$

$$k_6\Delta T(x)+(k_4+k_5)\,\mathrm{grad\,div}\,T(x)+\varkappa_0\,T(x)+i\sigma\varkappa_1C(x)-k_3\ \mathrm{grad}\,\vartheta(x)=\rho G(x),\tag{2.10}$$

$$i\sigma\gamma_2\,\mathrm{div}\,u(x)+h_1\ \mathrm{div}\,C(x)+h\,\Delta\,P(x)+i\sigma m\,P(x)+i\sigma\varkappa\,\vartheta(x)=0,\tag{2.11}$$

$$i\sigma\gamma_1\,T_0\ \mathrm{div}\,u(x)+k_1\ \mathrm{div}\,T(x)+i\sigma\varkappa\,T_0\,P(x)+k\,\Delta\,\vartheta(x)+i\sigma c\,T_0\,\vartheta(x)=-\rho\,s(x),\tag{2.12}$$

where

$$\delta=i\sigma m_1-h_2,\quad \varkappa_0=i\sigma c_1-k_2;$$

u, C, T, F, and G are complex-valued vector functions, while P, ϑ, and s are complex-valued scalar functions, and σ is a frequency parameter. If $\sigma=\sigma_1+i\,\sigma_2$ is a complex parameter with $\sigma_2\neq 0$, then the above equations are called the *pseudo–oscillation equations*, while for $\sigma=0$, they represent the *equilibrium equations of statics*. Note that the pseudo–oscillation equations are obtained from the equations of dynamical system (2.7) by the Laplace transform with the complex parameter σ.

Throughout the paper, we assume that σ is a complex parameter,

$$\sigma=\sigma_1+i\,\sigma_2,\quad \sigma_1\in\mathbb{R},\quad \sigma_2>0.\tag{2.13}$$

Let us introduce the matrix differential operator

$$L(\partial,\sigma) := \begin{bmatrix} L^{(1)}(\partial,\sigma) & L^{(6)}(\partial,\sigma) & L^{(11)}(\partial,\sigma) & L^{(16)}(\partial,\sigma) & L^{(21)}(\partial,\sigma) \\ L^{(2)}(\partial,\sigma) & L^{(7)}(\partial,\sigma) & L^{(12)}(\partial,\sigma) & L^{(17)}(\partial,\sigma) & L^{(22)}(\partial,\sigma) \\ L^{(3)}(\partial,\sigma) & L^{(8)}(\partial,\sigma) & L^{(13)}(\partial,\sigma) & L^{(18)}(\partial,\sigma) & L^{(23)}(\partial,\sigma) \\ L^{(4)}(\partial,\sigma) & L^{(9)}(\partial,\sigma) & L^{(14)}(\partial,\sigma) & L^{(19)}(\partial,\sigma) & L^{(24)}(\partial,\sigma) \\ L^{(5)}(\partial,\sigma) & L^{(10)}(\partial,\sigma) & L^{(15)}(\partial,\sigma) & L^{(20)}(\partial,\sigma) & L^{(25)}(\partial,\sigma) \end{bmatrix}_{11\times 11}, \tag{2.14}$$

where

$$\begin{aligned} L^{(1)}(\partial,\sigma) :=& (\mu\,\Delta + \rho\sigma^2) I_3 + (\lambda_0+\mu) Q(\partial), \quad L^{(2)}(\partial,\sigma) := [0]_{3\times 3}, \\ L^{(3)}(\partial,\sigma) :=& [0]_{3\times 3}, \quad L^{(4)}(\partial,\sigma) := i\sigma\gamma_2\,\nabla, \quad L^{(5)}(\partial,\sigma) := i\sigma\gamma_1\,T_0\,\nabla, \\ L^{(6)}(\partial,\sigma) :=& [0]_{3\times 3}, \quad L^{(7)}(\partial,\sigma) := (h_6\,\Delta + \delta) I_3 + (h_4+h_5) Q(\partial), \\ L^{(8)}(\partial,\sigma) :=& i\sigma\varkappa_1\, I_3, \quad L^{(9)}(\partial,\sigma) := h_1 \nabla, \quad L^{(10)}(\partial,\sigma) := [0]_{1\times 3}, \\ L^{(11)}(\partial,\sigma) :=& [0]_{3\times 3}, \quad L^{(12)}(\partial,\sigma) := i\sigma\varkappa_1\, I_3, \\ L^{(13)}(\partial,\sigma) :=& (k_6\Delta + \varkappa_0) I_3 + (k_4+k_5) Q(\partial), \quad L^{(14)}(\partial,\sigma) := [0]_{1\times 3}, \\ L^{(15)}(\partial,\sigma) :=& k_1\,\nabla, \quad L^{(16)}(\partial,\sigma) := -\gamma_2\,\nabla^\top, \quad L^{(17)}(\partial,\sigma) := -h_3\,\nabla^\top, \\ L^{(18)}(\partial,\sigma) :=& [0]_{3\times 1}, \quad L^{(19)}(\partial,\sigma) := h\Delta + i\sigma m, \quad L^{(20)}(\partial,\sigma) := i\sigma\varkappa T_0, \\ L^{(21)}(\partial,\sigma) :=& -\gamma_1 \nabla^\top, \quad L^{(22)}(\partial,\sigma) := [0]_{3\times 1}, \quad L^{(23)}(\partial,\sigma) := -k_3 \nabla^\top, \\ L^{(24)}(\partial,\sigma) :=& i\sigma\varkappa, \quad L^{(25)}(\partial,\sigma) := k\,\Delta + i\sigma c\, T_0. \end{aligned} \tag{2.15}$$

Here and in the sequel, I_k stands for the $k\times k$ unit matrix and

$$Q(\partial) := [\partial_k\partial_j]_{3\times 3}, \ \nabla := [\partial_1,\partial_2,\partial_3], \ \partial_k = \partial/\partial_{x_k}.$$

It is easy to show that for $V = (V_1,V_2,V_3)^\top$,

$$Q(\partial)V = \operatorname{grad}\operatorname{div} V, \quad Q(\partial) = [Q(\partial)]^\top, \quad [Q(\partial)]^2 = \Delta Q(\partial). \tag{2.16}$$

Due to the above notation, system (2.8)–(2.12) can be rewritten in a matrix form as

$$L(\partial,\sigma)U(x) = \Phi(x),$$

where $U = (u, C, T, P, \vartheta)^\top$, $\Phi(x) = \big(-\rho\, F(x),\ 0,\ \rho\, G(x),\ 0,\ -\rho\, s(x)\big)^\top$. The operator $L(\partial,\sigma)$ is not formally self-adjoint differential operator.

Here, the central dot denotes the real scalar product $a\cdot b = \sum_{k=1}^N a_k b_k$ for $a, b\in\mathbb{C}^N$, and $[c\times d]$ denotes the cross product of two vectors $c, d\in\mathbb{C}^3$.

In view of the constitutive equations (2.1)–(2.3), the components of the stress vector $t^{(n)}(U)$, the first mass diffusion flux moment vector $\eta^{(n)}(U)$, and the first heat flux moment vector $q^{(n)}(U)$, acting on a surface element with a unit outward normal vector $n = (n_1,n_2,n_3)^\top$, read as

$$t_j^{(n)}(U) = \sum_{p=1}^3 t_{pj}(U)\, n_p, \quad \eta_j^{(n)}(U) = \sum_{p=1}^3 \eta_{pj}(U)\, n_p, \quad q_j^{(n)}(U) = \sum_{p=1}^3 q_{pj}(U)\, n_p, \quad j=1,2,3. \tag{2.17}$$

It is easy to see that (2.17) can be rewritten as

$$\begin{aligned} t^{(n)}(U) &= 2\mu\,\partial_n\, u + \lambda_0\, n\ \operatorname{div} u + \mu[n\times \operatorname{curl}\ u] - \gamma_2\, n\, P - \gamma_1\, n\,\vartheta, \\ \eta^{(n)}(U) &= -(h_5+h_6)\,\partial_n\, C - h_4\, n\ \operatorname{div} C - h_5[n\times \operatorname{curl}\ C], \\ q^{(n)}(U) &= -(k_5+k_6)\,\partial_n\, T - k_4\, n\ \operatorname{div} T - k_5[n\times \operatorname{curl}\ T], \end{aligned}$$

where $\partial_n = \partial/\partial n$ stands for the normal derivative.

Due to the constitutive equation (2.4) and (2.5), the normal components of the flux vector of mass diffusion and the heat flux vector across a surface element with a unit outward normal vector $n=(n_1,n_2,n_3)^\top$, are expressed as follows:

$$\eta_n(U)=\sum_{j=1}^{3}\eta_j(U)\,n_j=h_1\,n\cdot C+h\;\partial_n P,\qquad q_n(U)=\sum_{j=1}^{3}q_j(U)\,n_j=k_1\,n\cdot T+k\;\partial_n\vartheta.$$

Throughout the paper, we will refer the eleventh vector $(t^{(n)},\,\eta^{(n)},\,q^{(n)},\,\eta_n,\,q_n)^\top$ as the *generalized stress vector*. Further, let us introduce the generalized stress operator

$$\mathcal{P}(\partial,n):=\begin{bmatrix}\mathcal{P}^{(1)}(\partial,n) & [0]_{3\times3} & [0]_{3\times3} & -\gamma_2 n & -\gamma_1 n\\ [0]_{3\times3} & \mathcal{P}^{(2)}(\partial,n) & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & [0]_{3\times3} & \mathcal{P}^{(3)}(\partial,n) & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{1\times3} & h_1\,n^\top & [0]_{1\times3} & h\,\partial_n & 0\\ [0]_{1\times3} & [0]_{1\times3} & k_1\,n^\top & 0 & k\,\partial_n\end{bmatrix}_{11\times11}, \tag{2.18}$$

where

$$\begin{aligned}&\mathcal{P}^{(l)}(\partial,n)=\Big[\mathcal{P}^{(l)}_{kj}(\partial,n)\Big]_{3\times3},\ l=1,2,3,\\ &\mathcal{P}^{(1)}_{kj}(\partial,n)=\mu\,\delta_{kj}\,\partial_n+\lambda_0\,n_k\,\partial_j+\mu\,n_j\,\partial_k,\\ &\mathcal{P}^{(2)}_{kj}(\partial,n)=h_6\,\delta_{kj}\,\partial_n+h_4\,n_k\,\partial_j+h_5\,n_j\,\partial_k,\\ &\mathcal{P}^{(3)}_{kj}(\partial,n)=k_6\,\delta_{kj}\,\partial_n+k_4\,n_k\,\partial_j+k_5\,n_j\,\partial_k.\end{aligned} \tag{2.19}$$

Note that for an arbitrary vector $U=(u,C,T,P,\vartheta)^\top$, the eleventh vector $\mathcal{P}(\partial,n)\,U$ is related to the components of the generalized stress vector as follows:

$$\mathcal{P}(\partial,n)\,U=(t^{(n)},\,-\eta^{(n)},\,-q^{(n)},\,\eta_n,\,q_n)^\top.$$

Let us introduce the associated boundary operator which is related to the adjoint differential operator $L^*(\partial,\sigma):=L^\top(-\partial,\sigma)$,

$$\mathcal{P}^*(\partial,n):=\begin{bmatrix}\mathcal{P}^{(1)}(\partial,n) & [0]_{3\times3} & [0]_{3\times3} & -i\sigma\gamma_2 n & -i\sigma\gamma_1 T_0 n\\ [0]_{3\times3} & \mathcal{P}^{(2)}(\partial,n) & [0]_{3\times1} & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & [0]_{3\times1} & \mathcal{P}^{(3)}(\partial,n) & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{1\times3} & h_3\,n^\top & [0]_{1\times3} & h\,\partial_n & [0]_{1\times3}\\ [0]_{1\times3} & [0]_{1\times3} & k_3\,n^\top & [0]_{1\times3} & k\,\partial_n\end{bmatrix}_{11\times11}, \tag{2.20}$$

where $\mathcal{P}^{(j)}(\partial,n)$, $j=1,2,3$, are given by (2.19).

3. Green's Formulas

Here we assume that the boundary $\partial\Omega^+$ of Ω^+ is a Lyapunov surface and n stands for the outward unit normal vector to $\partial\Omega^+$.

Definition 3.1. A vector function $U=(u,C,T,P,\vartheta)^\top$ is said to be regular in the domain Ω^+ if $U\in C^2(\Omega^+)\cap C^1(\overline{\Omega^+})$.

For regular vector functions $U=(u,C,T,P,\vartheta)^\top$ and $U'=(u',C',T',P',\vartheta')^\top$ in the domain Ω^+, we have the following Green's formulas:

$$\int\limits_{\Omega^+}U'\cdot L(\partial,\sigma)U\,dx=\int\limits_{\partial\Omega^+}\{U'\}^+\cdot\{\mathcal{P}(\partial,n)U\}^+dS-\int\limits_{\Omega^+}E(U',U)dx, \tag{3.1}$$

$$\int\limits_{\Omega^+}U\cdot L^*(\partial,\sigma)U'\,dx=\int\limits_{\partial\Omega^+}\{U\}^+\cdot\{\mathcal{P}^*(\partial,n)U'\}^+dS-\int\limits_{\Omega^+}E(U',U)dx, \tag{3.2}$$

where the differential operator $L(\partial,\sigma)$ is given by (2.14), $L^*(\partial,\sigma)=L^\top(-\partial,\sigma)$ is the formally adjoint operator to $L(\partial,\sigma)$, the boundary operators $\mathcal{P}(\partial,n)$ and $\mathcal{P}^*(\partial,n)$ are defined by (2.18) and (2.20), respectively; the symbols $\{\cdot\}^\pm$ denote one-sided limiting values on $\partial\Omega^+$ from $\Omega^\pm$, respectively; $E(\cdot,\cdot)$ is the so-called energy bilinear form

$$\begin{aligned} E(U',U)=\ & E^{(1)}(u',u)+E^{(2)}(C',C)+E^{(3)}(T',T)-\rho\sigma^2\,u'\cdot u-(\gamma_2\,P+\gamma_1\,\vartheta)\operatorname{div}u'-\delta\,C'\cdot C\\ & -i\sigma\,\varkappa_1\,C'\cdot T+h_3\,C'\cdot\operatorname{grad}P-\varkappa_0\,T'\cdot\,T-i\sigma\,\varkappa_1\,T'\cdot C+k_3\,T'\cdot\operatorname{grad}\vartheta\\ & -i\sigma\,m\,P'\,P-i\sigma\,\gamma_2\,P'\,\operatorname{div}u-i\sigma\,\varkappa\,P'\,\vartheta+h_1\,C\cdot\operatorname{grad}\,P'+h\operatorname{grad}P'\cdot\operatorname{grad}P\\ & +k\,\operatorname{grad}\,\vartheta'\cdot\operatorname{grad}\,\vartheta-i\sigma\,c\,T_0\,\vartheta'\,\vartheta-i\sigma\,\gamma_1\,T_0\,\vartheta'\,\operatorname{div}u+k_1\,T\cdot\operatorname{grad}\,\vartheta'-i\sigma\,\varkappa\,T_0\,P\,\vartheta', \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} E^{(1)}(u',u)=&\frac{3\lambda_0+2\mu}{3}\operatorname{div}u'\operatorname{div}u\\ &+\frac{\mu}{3}\sum_{k,j=1}^{3}\left(\frac{\partial u'_k}{\partial x_k}-\frac{\partial u'_j}{\partial x_j}\right)\left(\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}\right)\\ &+\frac{\mu}{2}\sum_{k,j=1,\,k\neq j}^{3}\left(\frac{\partial u'_k}{\partial x_j}+\frac{\partial u'_j}{\partial x_k}\right)\left(\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\right), \end{aligned} \tag{3.4}$$

$$\begin{aligned} E^{(2)}(C',C)=&\frac{3h_4+h_5+h_6}{3}\operatorname{div}C'\operatorname{div}C+\frac{h_6-h_5}{2}\operatorname{curl}C'\cdot\operatorname{curl}C\\ &+\frac{h_5+h_6}{4}\sum_{k,j=1,\,k\neq j}^{3}\left(\frac{\partial C'_k}{\partial x_j}+\frac{\partial C'_j}{\partial x_k}\right)\left(\frac{\partial C_k}{\partial x_j}+\frac{\partial C_j}{\partial x_k}\right)\\ &+\frac{h_5+h_6}{6}\sum_{k,j=1}^{3}\left(\frac{\partial C'_k}{\partial x_k}-\frac{\partial C'_j}{\partial x_j}\right)\left(\frac{\partial C_k}{\partial x_k}-\frac{\partial C_j}{\partial x_j}\right), \end{aligned} \tag{3.5}$$

$$\begin{aligned} E^{(3)}(T',T)=&\frac{3k_4+k_5+k_6}{3}\operatorname{div}T'\operatorname{div}T+\frac{k_6-k_5}{2}\operatorname{curl}T'\cdot\operatorname{curl}T\\ &+\frac{k_5+k_6}{4}\sum_{k,j=1,\,k\neq j}^{3}\left(\frac{\partial T'_k}{\partial x_j}+\frac{\partial T'_j}{\partial x_k}\right)\left(\frac{\partial T_k}{\partial x_j}+\frac{\partial T_j}{\partial x_k}\right)\\ &+\frac{k_5+k_6}{6}\sum_{k,j=1}^{3}\left(\frac{\partial T'_k}{\partial x_k}-\frac{\partial T'_j}{\partial x_j}\right)\left(\frac{\partial T_k}{\partial x_k}-\frac{\partial T_j}{\partial x_j}\right). \end{aligned} \tag{3.6}$$

With the help of relations (3.1) and (3.2) we can show that the following second Green's identity

$$\begin{aligned} &\int\limits_{\Omega^+}\left[U'\cdot L(\partial,\sigma)U-U\cdot L^*(\partial,\sigma)U'\right]dx\\ &=\int\limits_{\partial\Omega^+}\Big[\{U'\}^+\cdot\{\mathcal{P}(\partial,n)U\}^+-\{U\}^+\cdot\{\mathcal{P}^*(\partial,n)U'\}^+\Big]\,dS \end{aligned} \tag{3.7}$$

holds.

Let us note that the differential operator

$$L(\partial):=L(\partial,0) \tag{3.8}$$

corresponds to the static equilibrium case, while the formally self-adjoint differential operator

$$L_0(\partial):=\begin{bmatrix} L_0^{(1)}(\partial) & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & L_0^{(7)}(\partial) & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{3\times3} & [0]_{3\times3} & L_0^{(13)}(\partial) & [0]_{3\times1} & [0]_{3\times1}\\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & h\,\Delta & 0\\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & k\Delta \end{bmatrix}_{11\times11} \tag{3.9}$$

with

$$\begin{aligned} L_0^{(1)}(\partial) &:= \mu\Delta I_3 + (\lambda_0+\mu)Q(\partial),\\ L_0^{(7)}(\partial) &:= h_6\Delta I_3 + (h_4+h_5)Q(\partial),\\ L_0^{(13)}(\partial) &:= k_6\Delta I_3 + (k_4+k_5)Q(\partial), \end{aligned} \tag{3.10}$$

represents the principal homogeneous part of operators (2.14) and (3.8). With the help of inequalities (2.6), one can show that the differential operators $L_0(\partial)$ and $L(\partial,\sigma)$ are strongly elliptic and the following inequality

$$D_2|\xi|^2|\zeta|^2 \geq L_0(\xi)\zeta\cdot\zeta = \sum_{k,j=1}^{11} L_0(\xi)_{kj}\zeta_j\overline{\zeta_k} \geq D_1|\xi|^2|\zeta|^2$$

holds with some constants $D_k>0$ $(k=1,2)$ for an arbitrary $\xi\in\mathbb{R}^3$ and arbitrary complex vector $\zeta\in\mathbb{C}^{11}$.

4. The Matrix of Fundamental Solutions

Note that the construction of the fundamental matrix is carried out by the same method as indicated in [6, 7, 14]. Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse distributional Fourier transform in the space of tempered distributions (Schwartz space $\mathcal{S}'(\mathbb{R}^3)$), which for regular summable functions f and $\widehat{f}$ reads as follows:

$$\mathcal{F}_{x\to\xi}[f] = \int\limits_{\mathbb{R}^3} f(x)\,e^{i\,x\cdot\xi}dx = \widehat{f}(\xi), \quad \mathcal{F}^{-1}_{\xi\to x}[\widehat{f}\,] = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\widehat{f}(\xi)\,e^{-i\,x\cdot\xi}d\xi = f(x),$$

where $x=(x_1,x_2,x_3)$ and $\xi=(\xi_1,\xi_2,\xi_3)$. Note that for an arbitrary multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $f\in\mathcal{S}'(\mathbb{R}^3)$

$$\mathcal{F}[\partial^\alpha f] = (-i\,\xi)^\alpha\mathcal{F}[f], \quad \mathcal{F}^{-1}[\xi^\alpha\widehat{f}\,] = (i\,\partial)^\alpha\mathcal{F}^{-1}[\widehat{f}\,], \tag{4.1}$$

where $|\alpha|=\alpha_1+\alpha_2+\alpha_3$ and $\xi^\alpha=\xi_1^{\alpha_1}\,\xi_2^{\alpha_2}\,\xi_3^{\alpha_3}$. Denote by $\Gamma(x,\sigma)=[\Gamma_{kj}(x,\sigma)]_{11\times 11}$ the matrix of fundamental solutions of the operator $L(\partial,\sigma)$ (see (2.14), (2.15))

$$L(\partial,\sigma)\,\Gamma(x,\sigma) = \delta(x)\,I_{11}, \tag{4.2}$$

where $\delta(\,\cdot\,)$ is Dirac's distribution.

We represent the matrix $\Gamma(x,\sigma)$ in the blockwise form

$$\Gamma(x,\sigma) = \begin{bmatrix} \Gamma^{(1)}(x,\sigma) & \Gamma^{(2)}(x,\sigma) & \Gamma^{(3)}(x,\sigma) & \Gamma^{(4)}(x,\sigma) & \Gamma^{(5)}(x,\sigma)\\ \Gamma^{(6)}(x,\sigma) & \Gamma^{(7)}(x,\sigma) & \Gamma^{(8)}(x,\sigma) & \Gamma^{(9)}(x,\sigma) & \Gamma^{(10)}(x,\sigma)\\ \Gamma^{(11)}(x,\sigma) & \Gamma^{(12)}(x,\sigma) & \Gamma^{(13)}(x,\sigma) & \Gamma^{(14)}(x,\sigma) & \Gamma^{(15)}(x,\sigma)\\ \Gamma^{(16)}(x,\sigma) & \Gamma^{(17)}(x,\sigma) & \Gamma^{(18)}(x,\sigma) & \Gamma^{(19)}(x,\sigma) & \Gamma^{(20)}(x,\sigma)\\ \Gamma^{(21)}(x,\sigma) & \Gamma^{(22)}(x,\sigma) & \Gamma^{(23)}(x,\sigma) & \Gamma^{(24)}(x,\sigma) & \Gamma^{(25)}(x,\sigma) \end{bmatrix}_{11\times 11},$$

where

$$\begin{aligned} \Gamma^{(j)}(x,\sigma) &= \Big[\Gamma^{(j)}_{pq}(x,\sigma)\Big]_{3\times 3}, & j&=1,2,3,6,7,8,11,12,13,\\ \Gamma^{(j)}(x,\sigma) &= \Big[\Gamma^{(j)}_{pq}(x,\sigma)\Big]_{3\times 1}, & j&=4,5,9,10,14,15,\\ \Gamma^{(j)}(x,\sigma) &= \Big[\Gamma^{(j)}_{pq}(x,\sigma)\Big]_{1\times 3}, & j&=16,17,18,21,22,23, \end{aligned}$$

and $\Gamma^{(19)}(x,\sigma)$, $\Gamma^{(20)}(x,\sigma)$, $\Gamma^{(24)}(x,\sigma)$, and $\Gamma^{(25)}(x,\sigma)$ are scalar functions. By $\widehat{\Gamma}(\xi,\sigma)$ and $\widehat{\Gamma}^{(k)}(\xi,\sigma)$ we denote the Fourier transforms of the matrices $\Gamma(x,\sigma)$ and $\Gamma^{(k)}(x,\sigma)$, $k=1,2,\dots,25$. Applying the Fourier transform to equation (4.2) and taking into consideration (4.1) and the equality $\mathcal{F}[\delta(\cdot)]=1$, we get

$$L(-i\,\xi,\sigma)\,\widehat{\Gamma}(\xi,\sigma) = I_{11}. \tag{4.3}$$

We have to find $\widehat{\Gamma}(\xi,\sigma)$ from (4.3) and afterwards with the help of the inverse Fourier transform construct the fundamental matrix $\Gamma(x,\sigma)$ explicitly in terms of the standard elementary functions.

First of all, we have to represent the matrix $\widehat{\Gamma}(\xi,\sigma)=[L(-i\,\xi,\sigma)]^{-1}$ in such a form which is convenient for calculation of the inverse Fourier transform. To this end, we proceed as follows. We set $r:=|\xi|=\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}$ and introduce the notation

$$\begin{aligned} A(\xi) :=& L^{(1)}(-i\,\xi,\sigma)=(\rho\,\sigma^2-\mu\,r^2)\,I_3-(\lambda_0+\mu)\,Q(\xi),\\ B(\xi) :=& L^{(7)}(-i\,\xi,\sigma)=(\delta-h_6\,r^2)\,I_3\,-(h_4+h_5)Q(\xi),\\ D(\xi) :=& L^{(13)}(-i\,\xi,\sigma)=(\varkappa_0-k_6r^2)\,I_3-(k_4+k_5)\,Q(\xi), \end{aligned} \tag{4.4}$$

where $Q(\cdot)$ is defined by (2.16). Applying the relations (2.16) and (4.4) we can easily show that

$$A(\xi)=A(-\xi)=A^\top(\xi),\quad B(\xi)=B(-\xi)=B^\top(\xi),$$
$$D(\xi)=D(-\xi)=D^\top(\xi),\quad Q(\xi)=Q^\top(\xi),\quad [Q(\xi)]^2=r^2Q(\xi),$$

and the matrices A,B, and D commute to each other.

In view of (2.14)–(2.16) from (4.3) we derive

$$\begin{aligned} & A(\xi)\,\widehat{\Gamma}^{(j)}(\xi,\sigma)+i\gamma_2\xi^\top\,\widehat{\Gamma}^{(j+15)}(\xi,\sigma)+i\gamma_1\xi^\top\,\widehat{\Gamma}^{(j+20)}(\xi,\sigma)=\delta_{1j}I_3\,,\\ & B(\xi)\,\widehat{\Gamma}^{(j+5)}(\xi,\sigma)+i\sigma\,\varkappa_1\,\widehat{\Gamma}^{(j+10)}(\xi,\sigma)+i\,h_3\,\xi^\top\,\widehat{\Gamma}^{(j+15)}(\xi,\sigma)=\delta_{2j}I_3\,,\\ & i\sigma\,\varkappa_1\,\widehat{\Gamma}^{(j+5)}(\xi,\sigma)+D(\xi)\,\widehat{\Gamma}^{(j+10)}(\xi,\sigma)+i\,k_3\,\xi^\top\,\widehat{\Gamma}^{(j+20)}(\xi,\sigma)=\delta_{3j}I_3\,,\\ & \sigma\,\gamma_2\xi\,\widehat{\Gamma}^{(j)}(\xi,\sigma)-i\,h_1\xi\,\widehat{\Gamma}^{(j+5)}(\xi,\sigma)+(i\sigma\,m-h\,r^2)\,\widehat{\Gamma}^{(j+15)}(\xi,\sigma)+i\sigma\,\varkappa\,\widehat{\Gamma}^{(j+20)}(\xi,\sigma)=\delta_{4j},\\ & \sigma\,\gamma_1\,T_0\,\xi\,\widehat{\Gamma}^{(j)}(\xi,\sigma)-i\,k_1\,\xi\,\widehat{\Gamma}^{(j+10)}(\xi,\sigma)+i\sigma\,\varkappa\,T_0\,\widehat{\Gamma}^{(j+15)}(\xi,\sigma)+(i\sigma\,c\,T_0\,-k\,r^2)\,\widehat{\Gamma}^{(j+20)}(\xi,\sigma)=\delta_{5j}\,,\\ & \qquad j=1,2,\dots,5. \end{aligned} \tag{4.5}$$

From the system (4.5), by direct calculations, we can show that the elements of the matrix $\widehat{\Gamma}(\xi,\sigma)$ have the form

$$\begin{aligned} \widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}[a_j(\xi)I_3+b_j(\xi)Q(\xi)],\quad j=1,7,8,12,13,\\ \widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}b_j(\xi)Q(\xi),\quad j=2,3,6,11,\\ \widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}c_j(\xi)\xi^\top,\quad j=4,5,9,10,14,15,\\ \widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}c_j(\xi)\xi,\quad j=16,17,18,21,22,23,\\ \widehat{\Gamma}^{(j)}(\xi,\sigma)&=\frac{1}{\Lambda(\xi)}a_j(\xi),\quad j=19,20,24,25. \end{aligned}$$

Here,

$$\begin{aligned} &\Lambda(\xi)=\det L(-i\xi,\sigma)=a'(\xi)\left(a'(\xi)+b'(\xi)r^2\right)a(\xi)\left(a(\xi)+b(\xi)r^2\right)\Lambda_0(\xi)=d_1\prod_{j=1}^{11}(r^2-\lambda_j^2),\\ &\quad a'(\xi)=\rho\,\sigma^2-\mu\,r^2=-\mu(r^2-\lambda_1^2),\quad \lambda_1^2=\rho\,\sigma^2\,\mu^{-1},\quad b'(\xi)=-(\lambda_0+\mu),\\ &\quad a'(\xi)+b'(\xi)\,r^2=\rho\,\sigma^2-(\lambda_0+2\mu)\,r^2=-(\lambda_0+2\mu)(r^2-\lambda_2^2),\quad \lambda_2^2=\rho\,\sigma^2(\lambda_0+2\mu)^{-1},\\ &\quad d_1=-\mu\,(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\,d,\ \ d=(\lambda_0+2\mu)h_0\,k_0\,h\,k,\ \ h_0=h_4+h_5+h_6,\quad k_0=k_4+k_5+k_6; \end{aligned} \tag{4.6}$$

$\pm\lambda_3$, $\pm\lambda_4$ and $\pm\lambda_5$, $\pm\lambda_6$ are the roots, with respect to $r=|\xi|$, of the equations $a(\xi)=0$ and $a(\xi)+b(\xi)\,r^2=0$, respectively;

$$\begin{aligned} a(\xi)&=(h_6\,r^2-\delta)(k_6\,r^2-\varkappa_0)+\sigma^2\varkappa_1^2=h_6\,k_6\,(r^2-\lambda_3^2)(r^2-\lambda_4^2),\\ b(\xi)&=(h_4+h_5)(k_4+k_5)r^2+(k_4+k_5)(h_6\,r^2-\delta)+(h_4+h_5)(k_6\,r^2-\varkappa_0),\\ a(\xi)+b(\xi)\,r^2&=(h_0\,r^2-\delta)(k_0\,r^2-\varkappa_0)+\sigma^2\,\varkappa_1^2=\,h_0\,k_0\,(r^2-\lambda_5^2)(r^2-\lambda_6), \end{aligned} \tag{4.7}$$

$\pm\lambda_j,\ j = 7, 8, \dots, 11$, are the roots of the equation $\Lambda_0(\xi) = 0$ with respect to $r = |\xi|$, where

$$\begin{aligned}\Lambda_0(\xi) = {}&\big[h_1\,h_3(\varkappa_0 - k_0\,r^2)\,r^2 - (i\sigma\,m - h\,r^2)(a(\xi) + b(\xi)\,r^2)\big]\,\big[i\,\sigma\,T_0\,\gamma_1^2\,r^2 - (i\,\sigma\,c\,T_0 - k\,r^2)(\rho\,\sigma^2 -\\ &- (\lambda_0 + 2\mu)\,r^2)\big] + \big[\varkappa\,T_0\,(a(\xi) + b(\xi)\,r^2) + \varkappa_1\,(h_1\,k_3\,T_0 + k_1\,h_3)r^2\big]\big[\sigma^2\,\varkappa\,(\,\rho\,\sigma^2 - (\lambda_0 + 2\mu)\,r^2) -\\ &- \sigma^2\,\gamma_1\,\gamma_2\,r^2\big] + k_1\,k_3(\,\rho\,\sigma^2 - (\lambda_0 + 2\mu)\,r^2)\big[h_1\,h_3\,r^2 - (i\sigma\,m - h\,r^2)(\delta - h_0\,r^2)\big]r^2 -\\ &- \sigma^2\,\gamma_1\,\gamma_2\,\varkappa\,T_0\,(a(\xi) + b(\xi)\,r^2) + i\sigma\,\gamma_2^2\,r^2\big[k_1\,k_3\,(\delta - h_0\,r^2)r^2 -\\ &- (i\,\sigma\,c\,T_0 - k\,r^2)\,(a(\xi) + b(\xi)\,r^2)\big] = -d\prod_{j=7}^{11}(r^2 - \lambda_j^2);\end{aligned} \tag{4.8}$$

$$\begin{aligned}a_1(\xi) &= (a'(\xi) + b'(\xi)r^2)\,a(\xi)\,(a(\xi) + b(\xi)r^2)\Lambda_0(\xi),\\ a_7(\xi) &= a'(\xi)\,(a'(\xi) + b'(\xi)r^2)\,(a(\xi) + b(\xi)r^2)(\varkappa_0 - k_6\,r^2)\Lambda_0(\xi),\\ a_8(\xi) &= -i\sigma\,\varkappa_1\,a'(\xi)\,(a'(\xi) + b'(\xi)r^2)\,(a(\xi) + b(\xi)r^2)\Lambda_0(\xi),\\ a_{12}(\xi) &= -i\sigma\,\varkappa_1\,a'(\xi)\,(a'(\xi) + b'(\xi)r^2)\,(a(\xi) + b(\xi)r^2)\Lambda_0(\xi),\\ a_{13}(\xi) &= a'(\xi)\,(a'(\xi) + b'(\xi)r^2)\,(a(\xi) + b(\xi)r^2)(\delta - h_6\,r^2)\Lambda_0(\xi),\end{aligned} \tag{4.9}$$

$$\begin{aligned}a_{19}(\xi) &= a'(\xi)\,(a'(\xi) + b'(\xi)r^2)\,a(\xi)\,(a(\xi) + b(\xi)r^2)\,\gamma_{44}(\xi),\\ a_{20}(\xi) &= a'(\xi)\,(a'(\xi) + b'(\xi)r^2)\,a(\xi)\,(a(\xi) + b(\xi)r^2)\,\gamma_{45}(\xi),\\ a_{24}(\xi) &= a'(\xi)\,(a'(\xi) + b'(\xi)r^2)\,a(\xi)\,(a(\xi) + b(\xi)r^2)\,\gamma_{54}(\xi),\\ a_{25}(\xi) &= a'(\xi)\,(a'(\xi) + b'(\xi)r^2)\,a(\xi)\,(a(\xi) + b(\xi)r^2)\,\gamma_{55}(\xi),\end{aligned} \tag{4.10}$$

$$\begin{aligned}\gamma_{44}(\xi) ={}&(a(\xi) + b(\xi)r^2)\big[(a'(\xi) + b'(\xi)r^2)(i\sigma\,c\,T_0 - k\,r^2) - i\sigma\,\gamma_1^2\,T_0\,r^2\big]\\ &- k_1\,k_3(a'(\xi) + b'(\xi)r^2)(\delta - h_0\,r^2)\,r^2,\\ \gamma_{45}(\xi) ={}&i\sigma\,(a(\xi) + b(\xi)r^2)\big[\gamma_1\,\gamma_2\,r^2 - \varkappa\,(a'(\xi) + b'(\xi)r^2)\big] - i\sigma\,\varkappa_1\,h_1\,k_3\,(a'(\xi) + b'(\xi)r^2)\,r^2,\\ \gamma_{54}(\xi) ={}&i\sigma\,T_0(a(\xi) + b(\xi)r^2)\big[\gamma_1\,\gamma_2\,r^2 - \varkappa\,(a'(\xi) + b'(\xi)r^2)\big] - i\sigma\,\varkappa_1\,k_1\,h_3\,(a'(\xi) + b'(\xi)r^2)\,r^2,\\ \gamma_{55}(\xi) ={}&(a(\xi) + b(\xi)r^2)\big[(a'(\xi) + b'(\xi)r^2)(i\sigma\,m - h\,r^2) - i\sigma\,\gamma_2^2\,r^2\big]\\ &- h_1\,h_3\,(a'(\xi) + b'(\xi)r^2)(\varkappa_0 - k_0\,r^2)\,r^2.\end{aligned} \tag{4.11}$$

$$\begin{aligned}b_1(\xi) &= -a(\xi)(a(\xi) + b(\xi)r^2)\,\big\{b'(\xi)\,\Lambda_0(\xi) + i\,a'(\xi)\big[\gamma_2\,\gamma_{41}(\xi) + \gamma_1\,\gamma_{51}(\xi)\big]\big\},\\ b_2(\xi) &= -ia'(\xi)a(\xi)(a(\xi) + b(\xi)r^2)\,\big[\gamma_2\,\gamma_{42}(\xi) + \gamma_1\,\gamma_{52}(\xi)\big],\\ b_3(\xi) &= -ia'(\xi)a(\xi)(a(\xi) + b(\xi)r^2)\,\big[\gamma_2\,\gamma_{43}(\xi) + \gamma_1\,\gamma_{53}(\xi)\big],\\ b_6(\xi) &= a'(\xi)(a'(\xi) + b'(\xi)r^2)a(\xi)\,\big[i\,h_3\,(k_0\,r^2 - \varkappa_0)\,\gamma_{41}(\xi) - \sigma\,\varkappa_1\,k_3\,\gamma_{51}(\xi)\big],\\ b_7(\xi) &= a'(\xi)(a'(\xi) + b'(\xi)r^2)\,\big\{\Lambda_0(\xi)\,\big[(k_6\,r^2 - \varkappa_0)\,b(\xi) - (k_4 + k_5)\,a(\xi)\big]\\ &\quad + a(\xi)\big[ih_3\,(k_0\,r^2 - \varkappa_0)\gamma_{42}(\xi) - \sigma\,\varkappa_1\,k_3\,\gamma_{52}(\xi)\big]\big\},\\ b_8(\xi) &= a'(\xi)(a'(\xi) + b'(\xi)r^2)\,\big\{i\sigma\,\varkappa_1\,b(\xi)\Lambda_0(\xi)\\ &\quad + a(\xi)\big[ih_3\,(k_0\,r^2 - \varkappa_0)\gamma_{43}(\xi) - \sigma\,\varkappa_1\,k_3\,\gamma_{53}(\xi)\big]\big\},\end{aligned} \tag{4.12}$$

$$\begin{aligned}b_{11}(\xi) &= a'(\xi)(a'(\xi) + b'(\xi)r^2)\,a(\xi)\big[i\ k_3\,(h_0\,r^2 - \delta)\gamma_{51}(\xi) - \sigma\,\varkappa_1\,h_3\,\gamma_{41}(\xi)\big],\\ b_{12}(\xi) &= a'(\xi)(a'(\xi) + b'(\xi)r^2)\big\{a(\xi)\big[ik_3\,(h_0\,r^2 - \delta)\gamma_{52}(\xi)\\ &\quad - \sigma\,\varkappa_1\,h_3\,\gamma_{42}(\xi)\big] - i\sigma\,\varkappa_1\,b(\xi)\,\Lambda_0(\xi)\big\},\\ b_{13}(\xi) &= a'(\xi)(a'(\xi) + b'(\xi)r^2)\big\{a(\xi)\big[ik_3\,(h_0\,r^2 - \delta)\gamma_{53}(\xi) - \sigma\,\varkappa_1\,h_3\,\gamma_{43}(\xi)\big]\\ &\quad - \big[(h_4 + h_5)\,a(\xi) + (\delta - h_6\,r^2)\,b(\xi)\big]\,\Lambda_0(\xi)\big\},\end{aligned} \tag{4.13}$$

$$
\begin{aligned}
c_4(\xi) &= -ia'(\xi)a(\xi)(a(\xi)+b(\xi)r^2)\big[\gamma_2\,\gamma_{44}(\xi)+\gamma_1\,\gamma_{54}(\xi)\big],\\
c_5(\xi) &= -ia'(\xi)a(\xi)(a(\xi)+b(\xi)r^2)\big[\gamma_2\,\gamma_{45}(\xi)+\gamma_1\,\gamma_{55}(\xi)\big],\\
c_9(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)\big[i\,h_3\,(k_0\,r^2-\varkappa_0)\,\gamma_{44}(\xi)-\sigma\,\varkappa_1\,k_3\,\gamma_{54}(\xi)\big],\\
c_{10}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)\big[i\,h_3\,(k_0\,r^2-\varkappa_0)\,\gamma_{45}(\xi)-\sigma\,\varkappa_1\,k_3\,\gamma_{55}(\xi)\big],\\
c_{14}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)\big[i\,k_3\,(h_0\,r^2-\delta)\,\gamma_{54}(\xi)-\sigma\,\varkappa_1\,h_3\,\gamma_{44}(\xi)\big],\\
c_{15}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)\big[i\,k_3\,(h_0\,r^2-\delta)\,\gamma_{55}(\xi)-\sigma\,\varkappa_1\,h_3\,\gamma_{45}(\xi)\big],\\
c_{16}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{41}(\xi),\\
c_{17}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{42}(\xi),\\
c_{18}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{43}(\xi),\\
c_{21}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{51}(\xi),\\
c_{22}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{52}(\xi),\\
c_{23}(\xi) &= a'(\xi)(a'(\xi)+b'(\xi)r^2)a(\xi)(a(\xi)+b(\xi)r^2)\gamma_{53}(\xi),
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
\gamma_{41}(\xi) =& \big(a(\xi)+b(\xi)r^2\big)\big[i\sigma^2\,\varkappa\,\gamma_1\,T_0-\sigma\,\gamma_2(i\sigma\,c\,T_0-k\,r^2)\big]\\
&+\big[i\sigma^2\,\varkappa_1\,\gamma_1\,h_1\,k_3\,T_0+\sigma k_1\,k_3\,\gamma_2(\delta-h_0\,r^2)\big]r^2,\\
\gamma_{42}(\xi) =& \sigma\,\gamma_1^2\,h_1\,T_0\,r^2(\varkappa_0-k_0\,r^2)+i\sigma^2\,\varkappa_1\,k_1\,\gamma_1\,\gamma_2\,r^2\\
&-(a'(\xi)+b'(\xi)r^2)\big[i\sigma^2\,\varkappa\,\varkappa_1\,k_1+i\,h_1\,k_1\,k_3\,r^2-i\,h_1\,(\varkappa_0-k_0\,r^2)(i\sigma\,c\,T_0-k\,r^2)\big],\\
\gamma_{43}(\xi) =& \sigma\,k_1\,\gamma_1\,\gamma_2\,(h_0\,r^2-\delta)r^2-i\sigma^2\,\varkappa_1\,h_1\,\gamma_1^2\,T_0\,r^2\\
&+(a'(\xi)+b'(\xi)r^2)\big[\sigma\,\varkappa\,k_1\,(\delta-h_0\,r^2)+\sigma\,\varkappa_1\,h_1\,\big(i\sigma\,c\,T_0-k\,r^2\big)\big],\\
\gamma_{51}(\xi) =& T_0\,(a(\xi)+b(\xi)r^2)\big[i\sigma^2\,\varkappa\,\gamma_2-\sigma\,\gamma_1(i\sigma\,m-h\,r^2)\big]\\
&+\sigma\,\gamma_1\,h_1\,h_3\,T_0(\varkappa_0-k_0\,r^2)r^2+i\sigma^2\,\varkappa_1\,k_1\,h_3\,\gamma_2\,r^2,\\
\gamma_{52}(\xi) =& \sigma\,h_1\,\gamma_1\,\gamma_2\,T_0\,(k_0\,r^2-\varkappa_0)\,r^2-i\,\sigma^2\,\varkappa_1\,k_1\,\gamma_2^2\,r^2\\
&-(a'(\xi)+b'(\xi)r^2)\big[\sigma\,\varkappa_1\,k_1\,(h\,r^2-i\sigma\,m)-\sigma\,\varkappa\,h_1\,T_0(k_0\,r^2-\varkappa_0)\big],\\
\gamma_{53}(\xi) =& \sigma\,k_1\,\gamma_2^2\,(\delta-h_0\,r^2)\,r^2+i\sigma^2\,\varkappa_1\,h_1\,\gamma_1\,\gamma_2\,T_0\,r^2+\\
&+(a'(\xi)+b'(\xi)r^2)\big[i\,k_1(\delta-h_0\,r^2)(i\sigma\,m-h\,r^2)-i\,k_1\,h_1\,h_3\,r^2-i\sigma^2\,\varkappa\,\varkappa_1\,h_1\,T_0\big],
\end{aligned}
\tag{4.15}
$$

Now, we can represent the matrix $\widehat{\Gamma}(\xi,\sigma)$ in the form

$$
\widehat{\Gamma}(\xi,\sigma)=[L(-i\,\xi,\sigma)]^{-1}=\frac{1}{\Lambda(\xi)}\,\mathcal{M}(\xi,\sigma),
\tag{4.16}
$$

where

$$
\mathcal{M}(\xi,\sigma):=\begin{bmatrix}
a_1(\xi)\,I_3 & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\
[0]_{3\times3} & a_7(\xi)\,I_3 & a_8(\xi)\,I_3 & [0]_{3\times1} & [0]_{3\times1}\\
[0]_{3\times3} & a_{12}(\xi)\,I_3 & a_{13}(\xi)\,I_3 & [0]_{3\times1} & [0]_{3\times1}\\
[0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & a_{19}(\xi) & a_{20}(\xi)\\
[0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & a_{24}(\xi) & a_{25}(\xi)
\end{bmatrix}
$$

$$
+\begin{bmatrix}
b_1(\xi)\,Q(\xi) & b_2(\xi)\,Q(\xi) & b_3(\xi)\,Q(\xi) & [0]_{3\times1} & [0]_{3\times1}\\
b_6(\xi)\,Q(\xi) & b_7(\xi)\,Q(\xi) & b_8(\xi)\,Q(\xi) & [0]_{3\times1} & [0]_{3\times1}\\
b_{11}(\xi)\,Q(\xi) & b_{12}(\xi)\,Q(\xi) & b_{13}(\xi)\,Q(\xi) & [0]_{3\times1} & [0]_{3\times1}\\
[0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & 0\\
[0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & 0
\end{bmatrix}
\tag{4.17}
$$

$$
+ \begin{bmatrix} [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & c_4(\xi)\,\xi^\top & c_5(\xi)\,\xi^\top \\ [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & c_9(\xi)\,\xi^\top & c_{10}(\xi)\,\xi^\top \\ [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & c_{14}(\xi)\,\xi^\top & c_{15}(\xi)\,\xi^\top \\ c_{16}(\xi)\,\xi & c_{17}(\xi)\,\xi & c_{18}(\xi)\,\xi & 0 & 0 \\ c_{21}(\xi)\,\xi & c_{22}(\xi)\,\xi & c_{23}(\xi)\,\xi & 0 & 0 \end{bmatrix}.
$$

Note that the entries of the matrix $\mathcal{M}(\xi,\sigma)$ are polynomials in ξ. Therefore, to invert the Fourier transform and find an explicit form for the fundamental matrix $\Gamma(x,\sigma)$ we need the roots with respect to $r=|\xi|$ of the equation

$$
\Lambda(\xi) = \det L(-i\,\xi,\sigma) = 0. \tag{4.18}
$$

Due to the evenness of the function $\Lambda(\xi)$ with respect to $r=|\xi|$, it is clear that if $r=r_0$ is a root of the equation $\Lambda(\xi)=0$, then so is $r=-r_0$. In view of (4.6) the roots of the equation $\Lambda(\xi)=0$ are $\pm\lambda_j$, $j=1,2,\dots,11$. For the sake of simplicity, we assume that $\lambda_j\neq\lambda_k$, for $j\neq k$, $\operatorname{Im}\lambda_j>0$, and if $\operatorname{Im}\lambda_j=0$, then $\lambda_j>0$, (see Appendix A). Therefore, in view of (4.16) we can represent the fundamental solution as

$$
\Gamma(x,\sigma) = \mathcal{F}^{-1}_{\xi\to x}\Big[\widehat{\Gamma}(\xi,\sigma)\Big] = \frac{1}{d_1}\,\mathcal{F}^{-1}_{\xi\to x}\Big[\mathcal{M}(\xi,\sigma)\,\frac{1}{\Phi(r)}\Big] = \frac{1}{d_1}\,\mathcal{M}(i\,\partial,\sigma)\,\mathcal{F}^{-1}_{\xi\to x}\Big[\,\frac{1}{\Phi(r)}\,\Big], \tag{4.19}
$$

where

$$
\Phi(r) = \prod_{j=1}^{11}(r^2-\lambda_j^2), \qquad d_1 = -\mu\,(\lambda_0+2\mu)\,h_0\,k_0\,h_6\,k_6\,d.
$$

Note that

$$
\frac{1}{\Phi(r)} = \sum_{j=1}^{11}\frac{p_j}{r^2-\lambda_j^2},
$$

where the parameters $p_1,p_2,\dots,p_{11}$ solve the system of linear algebraic equations

$$
\lambda_1^{2m}\,p_1+\lambda_2^{2m}\,p_2+\cdots+\lambda_{11}^{2m}\,p_{11}=0, \quad m=0,1,\dots,9,
$$

$$
\lambda_1^{20}\,p_1+\lambda_2^{20}\,p_2+\cdots+\lambda_{11}^{20}\,p_{11}=1.
$$

They can be represented as follows:

$$
p_j = \left[\prod_{l=1,\,l\neq j}^{11}(\lambda_l^2-\lambda_j^2)\right]^{-1}.
$$

Note that if $\operatorname{Im}\lambda_j\geq 0$, then

$$
\mathcal{F}^{-1}_{\xi\to x}\left[\frac{1}{r^2-\lambda_j^2}\right] = \frac{e^{i\,\lambda_j|x|}}{4\pi|x|}.
$$

Therefore,

$$
\mathcal{F}^{-1}_{\xi\to x}\Big[\,\frac{1}{\Phi(r)}\,\Big] = \frac{1}{4\pi}\sum_{j=1}^{11}p_j\,\frac{e^{i\,\lambda_j|x|}}{|x|}.
$$

Now, from (4.19), we deduce

$$
\Gamma(x,\sigma) = \frac{1}{4\,\pi\,d_1}\,\mathcal{M}(i\,\partial,\sigma)\,\sum_{j=1}^{11}p_j\,\frac{e^{i\,\lambda_j|x|}}{|x|}, \tag{4.20}
$$

or

$$
\Gamma(x,\sigma) = \frac{1}{4\,\pi\,d_1}\mathcal{M}(i\,\partial,\sigma)\,\Psi(x,\sigma)\,,
$$

where the differential operator $\mathcal{M}(i\,\partial,\sigma)$ is given by (4.17) with $i\,\partial$ for ξ and

$$\Psi(x,\sigma)=\sum_{j=1}^{11}p_j\,\frac{e^{i\,\lambda_j|x|}}{|x|}\,.$$

We can simplify $\mathcal{M}(i\,\partial,\sigma)\Psi(x,\sigma)$ and rewrite the fundamental solution in a more explicit form. To this end, let us note that

$$\Delta\,\frac{e^{i\,\lambda_j|x|}}{|x|}=-\lambda_j^2\,\frac{e^{i\,\lambda_j|x|}}{|x|},\quad |x|\neq 0,$$

and apply formulas (4.7)–(4.15) to obtain

$$a(i\,\partial)\Psi(x,\sigma)=\sum_{j=1}^{11}p_j\,a^{(j)}\,\frac{e^{i\,\lambda_j|x|}}{|x|},\qquad b(i\,\partial)\Psi(x,\sigma)=\sum_{j=1}^{11}p_j\,b^{(j)}\,\frac{e^{i\,\lambda_j|x|}}{|x|},$$

$$a_l(i\,\partial)\Psi(x,\sigma)=\sum_{j=1}^{11}p_ja_l^{(j)}\,\frac{e^{i\,\lambda_j|x|}}{|x|}\,,\quad l=1,7,8,12,13,19,20,24,25,$$

$$b_l(i\,\partial)\Psi(x,\sigma)=\sum_{j=1}^{11}p_jb_l^{(j)}\,\frac{e^{i\,\lambda_j|x|}}{|x|}\,,\quad l=1,2,3,6,7,8,11,12,13,$$

$$c_l(i\,\partial)\Psi(x,\sigma)=\sum_{j=1}^{11}p_jc_l^{(j)}\,\frac{e^{i\,\lambda_j|x|}}{|x|}\,,\quad l=4,5,9,10,14,15,16,17,18,21,22,23,$$

where

$$a^{(j)}=h_6\,k_6(\lambda_j^2-\lambda_3^2)(\lambda_j^2-\lambda_4^2),$$

$$b^{(j)}=(h_4+h_5)(k_4+k_5)\lambda_j^2+(k_4+k_5)(h_6\,\lambda_j^2-\delta)+(h_4+h_5)(k_6\,\lambda_j^2-\varkappa_0),$$

$$a_1^{(j)}=(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\,d\prod_{l=2}^{11}(\lambda_j^2-\lambda_l^2),$$

$$a_7^{(j)}=-\mu\,(\lambda_0+2\mu)h_0\,k_0\,d\,(\varkappa_0-k_6\,\lambda_j^2)\prod_{l=1}^{2}(\lambda_j^2-\lambda_l^2)\prod_{l=5}^{11}(\lambda_j^2-\lambda_l^2),$$

$$a_8^{(j)}=i\sigma\,\varkappa_1\,\mu\,(\lambda_0+2\mu)h_0\,k_0\,d\prod_{l=1}^{2}(\lambda_j^2-\lambda_l^2)\prod_{l=5}^{11}(\lambda_j^2-\lambda_l^2),$$

$$a_{12}^{(j)}=a_8^{(j)},$$

$$a_{13}^{(j)}=-\mu\,(\lambda_0+2\mu)h_0\,k_0\,d\,(\delta-h_6\,\lambda_j^2)\prod_{l=1}^{2}(\lambda_j^2-\lambda_l^2)\prod_{l=5}^{11}(\lambda_j^2-\lambda_l^2),$$

$$a_{19}^{(j)}=\mu\,(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{44}^{(j)},$$

$$a_{20}^{(j)}=\mu\,(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{45}^{(j)},$$

$$a_{24}^{(j)}=\mu\,(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{54}^{(j)},$$

$$a_{25}^{(j)}=\mu\,(\lambda_0+2\mu)h_0\,k_0\,h_6\,k_6\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{55}^{(j)},$$

$$\gamma_{44}^{(j)}=h_0\,k_0\,(\lambda_j^2-\lambda_5^2)(\lambda_j^2-\lambda_6^2)\big[(\lambda_0+2\mu)\,(\lambda_j^2-\lambda_2^2)\,(k\,\lambda_j^2-i\sigma\,c\,T_0)-i\sigma\,\gamma_1^2\lambda_j^2\big]$$

$$+ k_1\, k_3\, (\lambda_0 + 2\mu)\, (\lambda_j^2 - \lambda_2^2)(\delta - h_0\, \lambda_j^2)\lambda_j^2$$

$$\gamma_{45}^{(j)} = i\sigma\, h_0\, k_0\, (\lambda_j^2 - \lambda_5^2)(\lambda_j^2 - \lambda_6^2)\big[\gamma_1\, \gamma_2 \lambda_j^2 + \varkappa\, (\lambda_0 + 2\mu)\, (\lambda_j^2 - \lambda_2^2)\big] + i\sigma\, \varkappa_1\, h_1\, k_3 (\lambda_0 + 2\mu)\, (\lambda_j^2 - \lambda_2^2)\lambda_j^2,$$

$$\gamma_{54}^{(j)} = i\sigma\, h_0\, k_0\, T_0\, (\lambda_j^2 - \lambda_5^2)(\lambda_j^2 - \lambda_6^2)\big[\gamma_1\, \gamma_2 \lambda_j^2 + \varkappa\, (\lambda_0 + 2\mu)\, (\lambda_j^2 - \lambda_2^2)\big] + i\, \sigma\, \varkappa_1\, k_1\, h_3 (\lambda_0 + 2\mu)\, (\lambda_j^2 - \lambda_2^2)\lambda_j^2,$$

$$\gamma_{55}^{(j)} = h_0\, k_0\, (\lambda_j^2 - \lambda_5^2)(\lambda_j^2 - \lambda_6^2)\big[(\lambda_0 + 2\mu)\, (\lambda_j^2 - \lambda_2^2)(h\, \lambda_j^2 - i\, \sigma\, m) - i\sigma\, \gamma_2^2\, \lambda_j^2\big] + h_1\, h_3 (\lambda_0 + 2\mu)\, (\lambda_j^2 - \lambda_2^2)(\varkappa_0 - k_0\, \lambda_j^2)\lambda_j^2,$$

$$b_1^{(j)} = -h_0\, k_0\, h_6\, k_6\, \big\{(\lambda_0 + \mu)\, d \prod_{l=3}^{11} (\lambda_j^2 - \lambda_l^2) - i\mu(\lambda_j^2 - \lambda_1^2) \prod_{l=3}^{6} (\lambda_j^2 - \lambda_l^2)\big[\gamma_2\, \gamma_{41}^{(j)} + \gamma_1\, \gamma_{51}^{(j)}\big]\big\},$$

$$b_2^{(j)} = i\, \mu\, h_0\, k_0\, h_6\, k_6\, (\lambda_j^2 - \lambda_1^2) \prod_{l=3}^{6} (\lambda_j^2 - \lambda_l^2)\big[\gamma_2\, \gamma_{42}^{(j)} + \gamma_1\, \gamma_{52}^{(j)}\big],$$

$$b_3^{(j)} = i\, \mu\, h_0\, k_0\, h_6\, k_6\, (\lambda_j^2 - \lambda_1^2) \prod_{l=3}^{6} (\lambda_j^2 - \lambda_l^2)\big[\gamma_2\, \gamma_{43}^{(j)} + \gamma_1\, \gamma_{53}^{(j)}\big],$$

$$b_6^{(j)} = \mu\, (\lambda_0 + 2\mu)\, h_6\, k_6 \prod_{l=1}^{4} (\lambda_j^2 - \lambda_l^2)\big[i h_3 (k_0 \lambda_j^2 - \varkappa_0)\gamma_{41}^{(j)} - \sigma\, \varkappa_1\, k_3\, \gamma_{51}^{(j)}\big],$$

$$b_7^{(j)} = \mu\, (\lambda_0 + 2\mu)(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\Big\{ - d \prod_{l=7}^{11} (\lambda_j^2 - \lambda_l^2)\big[(k_6 \lambda_j^2 - \varkappa_0)\, b^{(j)} - (k_4 + k_5) a^{(j)}\big] + a^{(j)}\big[i h_3 (k_0 \lambda_j^2 - \varkappa_0)\gamma_{42}^{(j)} - \sigma\, \varkappa_1\, k_3\, \gamma_{52}^{(j)}\big]\Big\},$$

$$b_8^{(j)} = \mu\, (\lambda_0 + 2\mu)(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\big\{ - i\sigma\, \varkappa_1\, d\, b^{(j)} \prod_{l=7}^{11} (\lambda_j^2 - \lambda_l^2) + a^{(j)}\big[i h_3 (k_0 \lambda_j^2 - \varkappa_0)\gamma_{43}^{(j)} - \sigma\, \varkappa_1\, k_3\, \gamma_{53}^{(j)}\big]\big\},$$

$$b_{11}^{(j)} = \mu\, (\lambda_0 + 2\mu) h_6\, k_6 \prod_{l=1}^{4} (\lambda_j^2 - \lambda_l^2)\big[i k_3 (h_0 \lambda_j^2 - \delta)\gamma_{51}^{(j)} - \sigma\, \varkappa_1\, h_3\, \gamma_{41}^{(j)}\big],$$

$$b_{12}^{(j)} = \mu\, (\lambda_0 + 2\mu)(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\Big\{a^{(j)}\big[i k_3 (h_0 \lambda_j^2 - \delta)\gamma_{52}^{(j)} - \sigma\, \varkappa_1\, h_3\, \gamma_{42}^{(j)}\big] + i\sigma \varkappa_1 d\, b^{(j)} \prod_{l=7}^{11} (\lambda_j^2 - \lambda_l^2)\Big\},$$

$$b_{13}^{(j)} = \mu\, (\lambda_0 + 2\mu)(\lambda_j^2 - \lambda_1^2)(\lambda_j^2 - \lambda_2^2)\big\{a^{(j)}\big[i k_3 (h_0 \lambda_j^2 - \delta)\gamma_{53}^{(j)} - \sigma\, \varkappa_1\, h_3\, \gamma_{43}^{(j)}\big] + d \prod_{l=7}^{11} (\lambda_j^2 - \lambda_l^2)\big[(h_4 + h_5) a^{(j)} + (\delta - h_6 \lambda_j^2) b^{(j)}\big]\big\},$$

$$c_4^{(j)} = i\, \mu\, h_0\, k_0\, h_6\, k_6\, (\lambda_j^2 - \lambda_1^2) \prod_{l=3}^{6} (\lambda_j^2 - \lambda_l^2)\big[\gamma_2\, \gamma_{44}^{(j)} + \gamma_1\, \gamma_{54}^{(j)}\big],$$

$$c_5^{(j)} = i\, \mu\, h_0\, k_0\, h_6\, k_6\, (\lambda_j^2 - \lambda_1^2) \prod_{l=3}^{6} (\lambda_j^2 - \lambda_l^2)\big[\gamma_2\, \gamma_{45}^{(j)} + \gamma_1\, \gamma_{55}^{(j)}\big],$$

$$c_9^{(j)} = \mu\, (\lambda_0 + 2\mu)\, h_6\, k_6 \prod_{l=1}^{4} (\lambda_j^2 - \lambda_l^2)\big[i h_3\, (k_0 \lambda_j^2 - \varkappa_0)\, \gamma_{44}^{(j)} - \sigma\, \varkappa_1\, k_3\, \gamma_{54}^{(j)}\big],$$

$$c_{10}^{(j)} = \mu\, (\lambda_0 + 2\mu)\, h_6\, k_6 \prod_{l=1}^{4} (\lambda_j^2 - \lambda_l^2)\big[i h_3\, (k_0 \lambda_j^2 - \varkappa_0)\, \gamma_{45}^{(j)} - \sigma\, \varkappa_1\, k_3\, \gamma_{55}^{(j)}\big],$$

$$c_{14}^{(j)} = \mu\,(\lambda_0+2\mu)\,h_6\,k_6\,\prod_{l=1}^{4}(\lambda_j^2-\lambda_l^2)\big[ik_3\,(h_0\lambda_j^2-\delta)\,\gamma_{54}^{(j)}-\sigma\,\varkappa_1\,h_3\,\gamma_{44}^{(j)}\big],$$

$$c_{15}^{(j)} = \mu\,(\lambda_0+2\mu)\,h_6\,k_6\,\prod_{l=1}^{4}(\lambda_j^2-\lambda_l^2)\big[ik_3\,(h_0\lambda_j^2-\delta)\,\gamma_{55}^{(j)}-\sigma\,\varkappa_1\,h_3\,\gamma_{45}^{(j)}\big],$$

$$c_{16}^{(j)} = \mu\,(\lambda_0+2\mu)\,h_6\,k_6\,h_0\,k_0\,\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{41}^{(j)},$$

$$c_{17}^{(j)} = \mu\,(\lambda_0+2\mu)\,h_6\,k_6\,h_0\,k_0\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{42}^{(j)},$$

$$c_{18}^{(j)} = \mu\,(\lambda_0+2\mu)\,h_6\,k_6\;h_0\,k_0\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{43}^{(j)},$$

$$c_{21}^{(j)} = \mu\,(\lambda_0+2\mu)\,h_6\,k_6\;h_0\,k_0\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{51}^{(j)},$$

$$c_{22}^{(j)} = \mu\,(\lambda_0+2\mu)\,h_6\,k_6\;h_0\,k_0\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{52}^{(j)},$$

$$c_{23}^{(j)} = \mu\,(\lambda_0+2\mu)\,h_6\,k_6\;h_0\,k_0\prod_{l=1}^{6}(\lambda_j^2-\lambda_l^2)\,\gamma_{53}^{(j)},$$

$$\begin{aligned}
\gamma_{41}^{(j)} &= h_0\,k_0\,(\lambda_j^2-\lambda_5^2)(\lambda_j^2-\lambda_6^2)\big[i\sigma^2\,\varkappa\,\gamma_1\,T_0-\sigma\gamma_2(i\sigma\,c\,T_0-k\lambda_j^2)\big]\\
&\quad+\big[i\sigma^2\,\varkappa_1\,\gamma_1\,h_1\,k_3T_0+\sigma\,k_1\,k_3\,\gamma_2(\delta-h_0\,\lambda_j^2)\big]\lambda_j^2,\\
\gamma_{42}^{(j)} &= \sigma\,h_1\,\gamma_1^2\,T_0\,\lambda_j^2(\varkappa_0-k_0\,\lambda_j^2)+i\sigma^2\,\varkappa_1\,k_1\,\gamma_1\gamma_2\,\lambda_j^2\\
&\quad+(\lambda_0+2\mu)(\lambda_j^2-\lambda_2^2)\big[i\sigma^2\,\varkappa\,\varkappa_1\,k_1+i\,h_1\,k_1\,k_3\lambda_j^2-ih_1(\varkappa_0-k_0\lambda_j^2)(i\sigma\,c\,T_0-k\,\lambda_j^2)\big],\\
\gamma_{43}^{(j)} &= \sigma\,k_1\,\gamma_1\,\gamma_2\,(h_0\,\lambda_j^2-\delta)\lambda_j^2-i\sigma^2\,\varkappa_1\,h_1\,\gamma_1^2\,T_0\,\lambda_j^2\\
&\quad-(\lambda_0+2\mu)(\lambda_j^2-\lambda_2^2)\,\big[\sigma\,\varkappa\,k_1\,(\delta-h_0\,\lambda_j^2)+\sigma\,\varkappa_1\,h_1\,\big(i\sigma\,c\,T_0-k\,\lambda_j^2\big)\big],\\
\gamma_{51}^{(j)} &= h_0\,k_0\,T_0\,(\lambda_j^2-\lambda_5^2)(\lambda_j^2-\lambda_6^2)\big[i\sigma^2\,\varkappa\,\gamma_2-\sigma\,\gamma_1(i\sigma\,m-h\,\lambda_j^2)\big]\\
&\quad+\sigma\,\gamma_1\,h_1\,h_3\,T_0(\varkappa_0-k_0\,\lambda_j^2)\lambda_j^2+i\sigma^2\,\varkappa_1\,k_1\,h_3\,\gamma_2\,\lambda_j^2,\\
\gamma_{52}^{(j)} &= \sigma\,h_1\,\gamma_1\,\gamma_2\,T_0\,(k_0\,\lambda_j^2-\varkappa_0)\,\lambda_j^2-i\,\sigma^2\,\varkappa_1\,k_1\,\gamma_2^2\,\lambda_j^2\\
&\quad+(\lambda_0+2\mu)(\lambda_j^2-\lambda_2^2)\,\big[\sigma\,\varkappa_1\,k_1\,(h\,\lambda_j^2-i\sigma\,m)-\sigma\,\varkappa\,h_1\,T_0(k_0\,\lambda_j^2-\varkappa_0)\big],\\
\gamma_{53}^{(j)} &= \sigma\,k_1\,\gamma_2^2\,(\delta-h_0\,\lambda_j^2)\,\lambda_j^2+i\sigma^2\,\varkappa_1\,h_1\,\gamma_1\,\gamma_2\,T_0\,\lambda_j^2\\
&\quad-(\lambda_0+2\mu)(\lambda_j^2-\lambda_2^2)\,\big[i\,k_1(\delta-h_0\,\lambda_j^2)(i\sigma\,m-h\,\lambda_j^2)-i\,k_1\,h_1\,h_3\,\lambda_j^2-i\sigma^2\,\varkappa\,\varkappa_1\,h_1\,T_0\big],
\end{aligned}$$

From (4.17) and (4.19), for the fundamental matrix, we get the following representation:

$$\Gamma(x,\sigma)=\frac{1}{4\,\pi\,d_1}\left\{\begin{bmatrix}
\Psi_1(x,\sigma)\,I_3 & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1}\\
[0]_{3\times3} & \Psi_7(x,\sigma)\,I_3 & \Psi_8(x,\sigma)\,I_3 & [0]_{3\times1} & [0]_{3\times1}\\
[0]_{3\times3} & \Psi_{12}(x,\sigma)\,I_3 & \Psi_{13}(x,\sigma)I_3 & [0]_{3\times1} & [0]_{3\times1}\\
[0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & \Psi_{19}(x,\sigma) & \Psi_{20}(x,\sigma)\\
[0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & \Psi_{24}(x,\sigma) & \Psi_{25}(x,\sigma)
\end{bmatrix}\right.$$

$$+\begin{bmatrix} Q(\partial)\widetilde{\Psi}_1(x,\sigma) & Q(\partial)\widetilde{\Psi}_2(x,\sigma) & Q(\partial)\widetilde{\Psi}_3(x,\sigma) & [0]_{3\times1} & [0]_{3\times1} \\ Q(\partial)\widetilde{\Psi}_6(x,\sigma) & Q(\partial)\widetilde{\Psi}_7(x,\sigma) & Q(\partial)\widetilde{\Psi}_8(x,\sigma) & [0]_{3\times1} & [0]_{3\times1} \\ Q(\partial)\widetilde{\Psi}_{11}(x,\sigma) & Q(\partial)\widetilde{\Psi}_{12}(x,\sigma) & Q(\partial)\widetilde{\Psi}_{13}(x,\sigma) & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & 0 \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & 0 \end{bmatrix} \tag{4.21}$$

$$\left.+\begin{bmatrix} [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & \nabla^\top\Psi'_4(x,\sigma) & \nabla^\top\Psi'_5(x,\sigma) \\ [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & \nabla^\top\Psi'_9(x,\sigma) & \nabla^\top\Psi'_{10}(x,\sigma) \\ [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & \nabla^\top\Psi'_{14}(x,\sigma) & \nabla^\top\Psi'_{15}(x,\sigma) \\ \nabla\Psi'_{16}(x,\sigma) & \nabla\Psi'_{17}(x,\sigma) & \nabla\Psi'_{18}(x,\sigma) & 0 & 0 \\ \nabla\Psi'_{21}(x,\sigma) & \nabla\Psi'_{22}(x,\sigma) & \nabla\Psi'_{23}(x,\sigma) & 0 & 0 \end{bmatrix}\right\},$$

where

$$\Psi_l(x,\sigma) = \sum_{j=1}^{11} p_j\, a_l^{(j)} \frac{e^{i\,\lambda_j|x|}}{|x|}, \qquad l = 1,7,8,12,13,19,20,24,25,$$

$$\widetilde{\Psi}_l(x,\sigma) = -\sum_{j=1}^{11} p_j\, b_l^{(j)} \frac{e^{i\,\lambda_j|x|}}{|x|}, \qquad l = 1,2,3,6,7,8,11,12,13,$$

$$\Psi'_l(x,\sigma) = i\sum_{j=1}^{11} p_j\, c_l^{(j)} \frac{e^{i\,\lambda_j|x|}}{|x|}, \quad l = 4,5,9,10,14,15,16,17,18,21,22,23.$$

Remark 4.1. Note that (4.20) can be rewritten in the form

$$\Gamma(x,\sigma) = \sum_{j=1}^{11} \Phi^{(j)}(x,\sigma), \tag{4.22}$$

where

$$\Phi^{(j)}(x,\sigma) = \frac{p_j}{4\pi\, d_1}\, \mathcal{M}(i\,\partial,\sigma)\, \frac{e^{i\lambda_j|x|}}{|x|}, \tag{4.23}$$

and $\mathcal{M}(i\,\partial,\sigma)$ is defined by (4.17). Since $\mathcal{M}(i\,\partial,\sigma)$ is a matrix differential operator with constant coefficients, from the representation (4.23) it follows that the entries of the matrix $\Phi^{(j)}(x,\sigma) = \Big[\Phi^{(j)}_{pq}(x,\sigma)\Big]_{11\times11}$ are metaharmonic functions corresponding to the wave number λ_j, i.e., they are solutions of the Helmholtz equation

$$(\Delta + \lambda_j^2)\, \Phi^{(j)}_{pq}(x,\sigma) = 0, \quad |x| \neq 0,$$

and decay exponentially at infinity:

$$\frac{\partial}{\partial|x|}\, \Phi^{(j)}_{pq}(x,\sigma) - i\,\lambda_j\, \Phi^{(j)}_{pq}(x,\sigma) = \exp\{-\mathrm{Im}\,\lambda_j\,|x|\}\, O(|x|^{-2}), \quad p,q = \overline{1,11},$$

as $|x| \to +\infty$. The entries of the matrix $\Phi^{(j)}(x,\sigma)$ and its derivatives likewise satisfy at infinity the following decay conditions [16]:

$$\Phi^{(j)}_{pq}(x,\sigma) = \exp\{-\mathrm{Im}\,\lambda_j\,|x|\}\, O(|x|^{-1}),$$

$$\frac{\partial}{\partial x_l}\, \Phi^{(j)}_{pq}(x,\sigma) - i\,\lambda_j\, \frac{x_l}{|x|}\, \Phi^{(j)}_{pq}(x,\sigma) = \exp\{-\mathrm{Im}\,\lambda_j\,|x|\}\, O(|x|^{-2}), \quad l = 1,2,3.$$

These asymptotic relations can be differentiated any times with respect to the variable x.

In accordance with formulas (4.22), (4.23) and Corollary A.2 (see Appendix A) we see that for $\operatorname{Im}\sigma = \sigma_2 > 0$ the entries of the matrix $\Gamma(x,\sigma)$ decay exponentially as $|x| \to \infty$, since $\operatorname{Im}\lambda_j > 0$, $j = \overline{1,11}$.

Remark 4.2. Note that the matrix $\Gamma^*(x,\sigma) := [\Gamma(-x,\sigma)]^\top$ represents a fundamental solution to the formally adjoint differential operator $L^*(\partial,\sigma) \equiv \big[L(-\partial,\sigma)\big]^\top$,

$$L^*(\partial,\sigma)\big[\Gamma(-x,\sigma)\big]^\top = I_{11}\,\delta(x).$$

In the case of repeated roots the fundamental solution can be obtained from (4.20) by the standard limiting procedure.

5. Principal Singular Part of the Fundamental Matrix

The principal singular part $\Gamma_0(x)$ of the fundamental matrix (4.21) represents an 11×11 fundamental matrix of the operator $L_0(\partial)$ defined by (3.9), (3.10) and solves the equation

$$L_0(\partial)\Gamma_0(x) = \delta(x) I_{11}.$$

It is easy to show that

$$\Gamma_0(x) = \begin{bmatrix} \Gamma_0^{(1)}(x) & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{3\times3} & \Gamma_0^{(7)}(x) & [0]_{3\times3} & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{3\times3} & [0]_{3\times3} & \Gamma_0^{(13)}(x) & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & \Gamma_0^{(19)}(x) & 0 \\ [0]_{1\times3} & [0]_{1\times3} & [0]_{1\times3} & 0 & \Gamma_0^{(25)}(x) \end{bmatrix}_{11\times11},$$

where

$$\Gamma_0^{(1)}(x) = -\frac{1}{8\pi\,\mu}\Big\{\frac{2}{|x|}\,I_3 - \frac{\lambda_0+\mu}{\lambda_0+2\mu}Q(\partial)|x|\Big\},$$

$$\Gamma_0^{(7)}(x) = -\frac{1}{8\pi\,h_6}\Big\{\frac{2}{|x|}\,I_3 - \frac{h_4+h_5}{h_0}Q(\partial)|x|\Big\},$$

$$\Gamma_0^{(13)}(x) = -\frac{1}{8\pi\,k_6}\Big\{\frac{2}{|x|}\,I_3 - \frac{k_4+k_5}{k_0}Q(\partial)|x|\Big\},$$

$$\Gamma_0^{(19)}(x) = -\frac{1}{4\pi\,h\,|x|},$$

$$\Gamma_0^{(25)}(x) = -\frac{1}{4\pi\,k\,|x|}.$$

Note that $\Gamma_0(x) = \Gamma_0^\top(x) = \Gamma_0(-x)$ and the entries of the matrix $\Gamma_0(x)$ are homogeneous functions of order -1. For an arbitrary multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and an arbitrary complex number σ it can easily be shown that in a neighbourhood of the origin (i.e., for small $|x|$)

$$\partial^\alpha[\Gamma(x,\sigma) - \Gamma_0(x)] = \mathcal{O}(|x|^{-\alpha}),\quad |\alpha| = \alpha_1+\alpha_2+\alpha_3,$$

which shows that $\Gamma_0(x)$ is a principal singular part of the matrix $\Gamma(x,\sigma)$.

6. Potentials and Their Properties

Let us introduce the generalized single and double-layer potentials, and the Newton type volume potential,

$$V(\varphi)(x) = \int\limits_S \Gamma(x-y,\sigma)\,\varphi(y)\,dS_y,\quad x\in\mathbb{R}^3\setminus S, \tag{6.1}$$

$$W(\varphi)(x) = \int\limits_S \left[\mathcal{P}^*(\partial_y, n(y))\Gamma^\top(x-y,\sigma)\right]^\top \varphi(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{6.2}$$

$$N_{\Omega^\pm}(\psi)(x) = \int\limits_{\Omega^\pm} \Gamma(x-y,\sigma)\,\psi(y)\,dy, \quad x \in \mathbb{R}^3, \tag{6.3}$$

where $\Gamma(\cdot,\sigma)$ is the fundamental matrix given by (4.20) or (4.21), $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_{11})^\top$ is a density vector-function defined on S, while a density vector-function $\psi = (\psi_1, \dots, \psi_{11})^\top$ is defined on $\Omega^\pm$ and we assume that in the case of Ω^- the support of the density vector-function ψ of the Newtonian potential (6.3) is a compact set, $\mathcal{P}^*(\partial_y, n(y))$ is the boundary differential operator defined by (2.20). It can be checked that the potentials defined by (6.1) and (6.2) are C^∞-smooth in $\mathbb{R}^3 \setminus S$ and solve the homogeneous equation $L(\partial,\sigma)U = 0$ in $\mathbb{R}^3 \setminus S$ for an arbitrary continuous vector function φ. The volume potential solves the nonhomogeneous equation

$$L(\partial,\sigma)N_{\Omega^\pm}(\psi) = \psi \ \text{ in } \ \Omega^\pm \ \text{ for } \ \psi \in C^{0,\alpha}(\overline{\Omega^\pm}). \tag{6.4}$$

Theorem 6.1. *Let $S = \partial\Omega^+$ be $C^{1,\gamma'}$smooth with $0 < \gamma' \le 1$, $\sigma = \sigma_1 + i\,\sigma_2$ with $\sigma_2 > 0$, and let U be a regular vector function of the class $C^2(\overline{\Omega^+})$. Then the integral representation formula*

$$W(\{U\}^+)(x) - V(\{\mathcal{P}U\}^+)(x) + N_{\Omega^+}(L(\partial,\sigma)U)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \in \Omega^-. \end{cases}$$

holds.

This follows from Green's formula (3.7) (see [4, Appendix D]).

Similar representation formula holds in the exterior domain Ω^- if the vector U and its derivatives possess some asymptotic properties at infinity. In particular, the following assertion holds.

Theorem 6.2. *Let $S = \partial\Omega^-$ be $C^{1,\gamma'}$smooth with $0 < \gamma' \le 1$ and let U be a regular vector of the class $C^2(\overline{\Omega^-})$ such that for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $0 \le |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \le 2$, the function $\partial^\alpha U_j$ is polynomially bounded at infinity, i.e., for sufficiently large $|x|$,*

$$|\partial^\alpha U_j(x)| \le C_0\,|x|^m, \ \ j = 1, 2, \dots, 11,$$

with some constants m and $C_0 > 0$. Then the integral representation formula

$$-W(\{U\}^-)(x) + V(\{\mathcal{P}U\}^-)(x) + N_{\Omega^-}(L(\partial,\sigma)U)(x) = \begin{cases} 0 & \text{for } x \in \Omega^+, \\ U(x) & \text{for } x \in \Omega^-, \end{cases}$$

where $\sigma = \sigma_1 + i\,\sigma_2$ with $\sigma_2 > 0$, holds.

The proof immediately follows from Theorem 6.1 and Remark 4.1.

From Theorem 6.2, it follows immediately that if $U \in C^2(\overline{\Omega^-})$ grows at infinity polynomially, and $L(\partial,\sigma)U$ possesses a compact support, then actually U and its all partial derivatives decay exponentially at infinity and the following Green's formula

$$\int\limits_{\Omega^-} U' \cdot L(\partial,\sigma)U dx = -\int\limits_{\partial\Omega^-} \{U'\}^- \cdot \{\mathcal{P}(\partial,n)U\}^- dS - \int\limits_{\Omega^-} E(U',U)dx \tag{6.5}$$

holds for all polynomially bounded vector functions $U' \in C^1(\overline{\Omega^-})$.

Now let us consider the mapping and regularity properties of the single and double-layer potentials and the boundary pseudodifferential operators generated by them in the Hölder $C^{m,\gamma'}$ spaces. They can be established by standard methods. We remark only that the layer potentials corresponding to the fundamental matrices with different values of the parameter σ have the same smoothness properties and possess the same jump relations. Therefore, using the word for word arguments given in [3,4,8,9,11–14], we can prove the following theorems concerning the above-introduced layer potentials. Unless otherwise stated, for simplicity, we assume that

$$\begin{aligned} &S = \partial\Omega^\pm \in C^{m,\gamma'} \text{ with integer } m \ge 2 \ \text{ and } \ 0 < \gamma' \le 1; \\ &\sigma = \sigma_1 + i\,\sigma_2, \ \ \sigma_1 \in \mathbb{R}, \ \ \operatorname{Im}\sigma = \sigma_2 > 0. \end{aligned} \tag{6.6}$$

Theorem 6.3. *Let S, m, and γ' be as in* (6.6), $0<\delta'<\gamma'$, *and let $k\le m-1$ be an integer. Then the operators*

$$V\ :\ C^{k,\,\delta'}(S)\to C^{k+1,\,\delta'}(\overline{\Omega^{\pm}}),\quad W\ :\ C^{k,\,\delta'}(S)\to C^{k,\,\delta'}(\overline{\Omega^{\pm}}) \tag{6.7}$$

are continuous. For any $g\in C^{0,\,\delta'}(S)$, $h\in C^{1,\,\delta'}(S)$, and any $x\in S$,

$$[\,V(g)(x)\,]^{\pm}=V(g)(x)=\mathcal{H}\,g(x), \tag{6.8}$$

$$\{\,\mathcal{P}(\partial_x,n(x))\,V(g)(x)\,\}^{\pm}=\big[\,\mp 2^{-1}I_{11}+\mathcal{K}\,\big]\,g(x), \tag{6.9}$$

$$\{\,W(g)(x)\,\}^{\pm}=\big[\,\pm 2^{-1}I_{11}+\mathcal{N}\,\big]\,g(x), \tag{6.10}$$

$$\{\mathcal{P}(\partial_x,n(x))\,W(h)(x)\,\}^{+}=\{\,\mathcal{P}(\partial_x,n(x))\,W(h)(x)\,\}^{-}=\mathcal{L}\,h(x), \tag{6.11}$$

where

$$\mathcal{H}\,g(x):=\int\limits_{S}\Gamma(x-y,\sigma)\,g(y)\,dS_y\,, \tag{6.12}$$

$$\mathcal{K}\,g(x):=\int\limits_{S}\big[\,\mathcal{P}(\partial_x,n(x))\,\Gamma(x-y,\sigma)\,\big]\;g(y)\,dS_y\,, \tag{6.13}$$

$$\mathcal{N}\,g(x):=\int\limits_{S}\big[\,\mathcal{P}^{*}(\partial_y,n(y))\,\Gamma^{\top}(x-y,\sigma)\,\big]^{\top}\,g(y)\,dS_y\,, \tag{6.14}$$

$$\mathcal{L}\,h(x):=\lim_{\Omega^{\pm}\ni z\to x\in S}\mathcal{P}(\partial_z,n(x))\int\limits_{S}\big[\,\mathcal{P}^{*}(\partial_y,n(y))\,\Gamma^{\top}(z-y,\sigma)\,\big]^{\top}\,h(y)\,dS_y\,. \tag{6.15}$$

The proof of the relations (6.7)–(6.11) can be performed by standard arguments (see, e.g., [6,8,12]). The relation (6.11) is called the *Liapunov-Tauber type theorem.*

With the help of the explicit form of the fundamental matrix $\Gamma(x-y,\sigma)$ it can be shown that the operators $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators, $\mathcal{H}$ is a smoothing (weakly singular) integral operator, while $\mathcal{L}$ is a singular integro-differential operator.

Theorem 6.4. *Let S, m, γ', δ' and k be as in Theorem* 6.3. *Then the operators*

$$\mathcal{H}:C^{k,\,\delta'}(S)\to C^{k+1,\,\delta'}(S)\,, \tag{6.16}$$

$$\mathcal{K}:C^{k,\,\delta'}(S)\to C^{k,\,\delta'}(S)\,, \tag{6.17}$$

$$\mathcal{N}:C^{k,\,\delta'}(S)\to C^{k,\,\delta'}(S)\,, \tag{6.18}$$

$$\mathcal{L}:C^{k,\,\delta'}(S)\to C^{k-1,\,\delta'}(S)\,, \tag{6.19}$$

are continuous. Moreover, 1) *the principal homogeneous symbol matrices of the operators $\pm 2^{-1}I_{11}+\mathcal{K}$ and $\pm 2^{-1}I_{11}+\mathcal{N}$ are non-degenerate, while the principal homogeneous symbol matrices of the operators $-\mathcal{H}$ and $\mathcal{L}$ are positive definite;* 2) *the operators $\mathcal{H}$, $\pm 2^{-1}I_{11}+\mathcal{K}$, $\pm 2^{-1}I_{11}+\mathcal{N}$, and $\mathcal{L}$ are elliptic pseudodifferential operators* (*of order* -1, 0, 0, *and* 1, *respectively*) *with zero index; in appropriate function spaces, the following equalities* 3)

$$\begin{aligned} &\mathcal{N}\,\mathcal{H}=\mathcal{H}\,\mathcal{K}, &&\mathcal{L}\,\mathcal{N}=\mathcal{K}\,\mathcal{L},\\ &\mathcal{H}\,\mathcal{L}=-4^{-1}\,I_{11}+\mathcal{N}^{\,2}, &&\mathcal{L}\,\mathcal{H}=-4^{-1}\,I_{11}+\mathcal{K}^{2}. \end{aligned} \tag{6.20}$$

hold.

The mapping properties (6.16)–(6.19) are standard and can be proved as their counterparts in [8,11,13,14]. Items 1) and 2) are based on the positive definiteness of the potential energy functional and positive definiteness of the symbol matrix $L_0(\xi)$ for $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3\setminus\{0\}$ (see (3.9), (3.10)), (cf. [3,4,10,11,14] and [6]). Item 3) follows from the jump relations for the layer potentials and the general integral representation formulas of solutions to the homogeneous equation $L(\partial,\sigma)U=0$.

7. Formulation of Boundary Value Problems and Uniqueness Theorems

Let us formulate the basic interior and exterior boundary value problems for the domains Ω^+ and Ω^-. We assume that $S=\partial\Omega^+\in C^{1,\gamma'}$, $0<\gamma'\le 1$.

Problem $(I^{(\sigma)})^\pm$ **(The Dirichlet problem).** Find a regular solution vector function $U=(u,C,T,P,\vartheta)^\top$ to the system of differential equations

$$L(\partial,\sigma)\,U(x)=\Phi^\pm(x),\ \ x\in\Omega^\pm, \tag{7.1}$$

satisfying the boundary condition

$$\{U(z)\}^\pm=f(z),\quad z\in S. \tag{7.2}$$

Problem $(II^{(\sigma)})^\pm$ **(The Neumann problem).** Find a regular solution vector function $U=(u,C,T,P,\vartheta)^\top$ to system (7.1), satisfying the boundary condition

$$\{\mathcal{P}(\partial,n)\,U(z)\}^\pm=F(z),\ \ z\in S. \tag{7.3}$$

We assume that the data of the boundary value problems belong to the appropriate classes,

$$\Phi^\pm\in C^{0,\alpha'}(\overline{\Omega})^\pm,\quad f\in C^{1,\alpha'}(S),\quad F\in C^{0,\alpha'}(S),\quad 0<\alpha'<\gamma'\le 1.$$

In addition, in the case of exterior problems we assume that the vector function Φ^- is compactly supported in Ω^-. Now we prove the following uniqueness theorem.

Theorem 7.1. *Let $\sigma=\sigma_1+i\sigma_2$, with $\sigma_1\in\mathbb{R}$ and $\sigma_2>0$. Then the homogeneous boundary value problems $(I^{(\sigma)})^\pm$ and $(II^{(\sigma)})^\pm$ have only the trivial solution in the class of regular vector functions.*

Proof. Let $U=(u,C,T,P,\vartheta)^\top$ be a regular solution of the homogeneous boundary value problem $(I^{(\sigma)})^\pm$ or $(II^{(\sigma)})^\pm$. Apply Green's formula (3.1) or (6.5) for the vector functions U and U', where

$$U'=\big(i\,\overline{\sigma}\,\overline{u},\,\overline{C},\,\overline{T},\,\overline{P},\,\frac{1}{T_0}\,\overline{\vartheta}\big)^\top.$$

Keeping in mind (3.3)–(3.6), we get the following relation:

$$\pm\int\limits_{\partial\Omega^\pm}\{U'\}^\pm\cdot\{\mathcal{P}(\partial,n)U\}^\pm dS-\int\limits_{\Omega^\pm}E(U',U)dx=0, \tag{7.4}$$

where

$$\begin{aligned}E(U',U)=&i\,\overline{\sigma}E^{(1)}(\overline{u},u)+E^{(2)}(\overline{C},C)+E^{(3)}(\overline{T},T)-i\,\overline{\sigma}\,(\gamma_2\,P+\gamma_1\vartheta)\operatorname{div}\overline{u}-i\,\rho\,\sigma\,|\sigma|^2\,|u|^2\\&-\delta\,|C|^2-i\,\sigma\,\varkappa_1\overline{C}\cdot T+h_3\overline{C}\cdot\operatorname{grad}P-\varkappa_0\,|T|^2-i\,\sigma\,\varkappa_1\overline{T}\cdot C+k_3\overline{T}\cdot\operatorname{grad}\vartheta\\&-i\,\sigma\,m|P|^2-i\,\sigma\,\gamma_2\,\overline{P}\operatorname{div}u-i\,\sigma\,\varkappa\,\overline{P}\,\vartheta+h_1\,C\cdot\operatorname{grad}\overline{P}+h\,|\operatorname{grad}P|^2\\&+\frac{k}{T_0}|\operatorname{grad}\vartheta|^2-i\,\sigma\,c\,|\vartheta|^2-i\,\sigma\,\gamma_1\,\overline{\vartheta}\,\operatorname{div}u-i\,\sigma\,\varkappa\,P\,\overline{\vartheta}+\frac{k_1}{T_0}T\cdot\operatorname{grad}\overline{\vartheta};\end{aligned}$$

$$E^{(1)}(\overline{u},u)=\frac{3\lambda_0+2\mu}{3}|\operatorname{div}u|^2+\frac{\mu}{3}\sum_{k,j=1}^{3}\left|\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}\right|^2+\frac{\mu}{2}\sum_{k,j=1,\ k\neq j}^{3}\left|\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\right|^2, \tag{7.5}$$

$$\begin{aligned}E^{(2)}(\overline{C},C)=&\frac{3h_4+h_5+h_6}{3}|\operatorname{div}C|^2+\frac{h_6-h_5}{2}\,|\operatorname{curl}C|^2\\&+\frac{h_5+h_6}{4}\sum_{k,j=1,\ k\neq j}^{3}\left|\frac{\partial C_k}{\partial x_j}+\frac{\partial C_j}{\partial x_k}\right|^2+\frac{h_5+h_6}{6}\sum_{k,j=1}^{3}\left|\frac{\partial C_k}{\partial x_k}-\frac{\partial C_j}{\partial x_j}\right|^2,\end{aligned} \tag{7.6}$$

$$\begin{aligned}E^{(3)}(\overline{T},T)=&\frac{3k_4+k_5+k_6}{3}|\operatorname{div}T|^2+\frac{k_6-k_5}{2}\,|\operatorname{curl}T|^2\\&+\frac{k_5+k_6}{4}\sum_{k,j=1,\ k\neq j}^{3}\left|\frac{\partial T_k}{\partial x_j}+\frac{\partial T_j}{\partial x_k}\right|^2+\frac{k_5+k_6}{6}\sum_{k,j=1}^{3}\left|\frac{\partial T_k}{\partial x_k}-\frac{\partial T_j}{\partial x_j}\right|^2.\end{aligned} \tag{7.7}$$

Since $U=(u,C,T,P,\vartheta)^\top$ solves the homogeneous boundary value problem $(I^{(\sigma)})^{\pm}$, or $(II^{(\sigma)})^{\pm}$, the surface integral in (7.4) vanishes and we arrive at the equation

$$\int\limits_{\Omega^\pm} E(U',U)dx=0.$$

The real part of this equation reads as

$$\begin{aligned}\int\limits_{\Omega^\pm} &\big\{\sigma_2\, E^{(1)}(\overline{u},u)+E^{(2)}(\overline{C},C)+E^{(3)}(\overline{T},T)+\rho\,\sigma_2\,|\sigma|^2|u|^2\\
&+\sigma_2\left[m\,|P|^2+\varkappa\,(P\overline{\vartheta}+\overline{P}\vartheta)+c|\vartheta|^2\right]+\sigma_2\left[m_1|C|^2+\varkappa_1\,(C\cdot\overline{T}+\overline{C}\cdot T)+c_1\,|T|^2\right]\\
&+\frac{1}{2}\left[2\,h_2|C|^2+(h_1+h_3)\,(\overline{C}\cdot\operatorname{grad}P+C\cdot\operatorname{grad}\overline{P})+2h\,|\operatorname{grad}P|^2\,\right]\\
&+\frac{1}{2T_0}\left[2\,k_2\,T_0\,|T|^2+(k_1+T_0\,k_3)\,(\overline{T}\cdot\operatorname{grad}\vartheta+T\cdot\operatorname{grad}\overline{\vartheta})+2\,k\,|\operatorname{grad}\vartheta|^2\right]\big\}dx=0,\end{aligned}\tag{7.8}$$

By means of relations (7.5)–(7.7) we see that $E^{(1)}(\overline{u},u)\geq 0$, $E^{(2)}(\overline{C},C)\geq 0$, and $E^{(3)}(\overline{T},T)\geq 0$. Transforming the integrand and taking into account conditions (2.6), we establish

$$m|P|^2+\varkappa\,(P\overline{\vartheta}+\overline{P}\vartheta)+c|\vartheta|^2=\frac{1}{c}\left[(m\,c-\varkappa^2)\,|P|^2+|\varkappa\,P+c\vartheta|^2\right]\geq 0,$$

$$m_1|C|^2+\varkappa_1\,(C\cdot\overline{T}+\overline{C}\cdot T)+c_1\,|T|^2=\frac{1}{c_1}\left[(m_1\,c_1-\varkappa_1^2)|C|^2+|\varkappa_1\,C+c_1\,T|^2\right]\geq 0,$$

$$\begin{aligned}&\frac{1}{2}\left[2\,h_2|C|^2+(h_1+h_3)(\overline{C}\cdot\operatorname{grad}P+C\cdot\operatorname{grad}\overline{P})+2h\,|\operatorname{grad}P|^2\,\right]\\
&\qquad=\frac{1}{4h}\left\{\left[4h\,h_2-(h_1+h_3)^2\right]|C|^2+|(h_1+h_3)\,C+2h\ \operatorname{grad}P|^2\right\}\geq 0,\end{aligned}$$

$$\begin{aligned}&\frac{1}{2T_0}\left[2\,k_2\,T_0\,|T|^2+(k_1+T_0\,k_3)(\overline{T}\cdot\operatorname{grad}\vartheta+T\cdot\operatorname{grad}\overline{\vartheta})+2\,k\,|\operatorname{grad}\vartheta|^2\right]\\
&\qquad=\frac{1}{4k\,T_0}\left\{\left[4T_0\,k\,k_2-(k_1+T_0\,k_3)^2\right]|T|^2+|(k_1+T_0\,k_3)\,T+2k\ \operatorname{grad}\vartheta|^2\right\}\geq 0.\end{aligned}$$

Consequently, from (7.8) we derive $\operatorname{Re}E(U',U)\geq 0$ in $\Omega^\pm$, implying $U=0$ for $x\in\Omega^\pm$.

□

8. Existence Results

Now, we apply the potential method and prove the existence theorems for the above formulated Dirichlet and Neumann type boundary value problems. We reduce these problems to the equivalent integral equations on the boundary of the elastic body under consideration and investigate their Fredholm properties. We show that the corresponding integral operators are invertible. Without loss of generality, we consider the boundary value problems for the homogeneous differential equation $L(\partial,\sigma)\,U=0$, since a particular solution to the nonhomogeneous equation (7.1) can be written explicitly in the form of the volume potential $N_{\Omega^\pm}(\Phi^\pm)$ (see (6.4)). Moreover, throughout this section we assume that the conditions (6.6) are fulfilled, unless otherwise stated.

8.1. **Investigation of the interior and exterior Dirichlet problems.** We assume that $\Phi^{(\pm)}=0$ and look for solutions in $\Omega^\pm$ in the form of the double-layer potential $U=W(h)$ (see (6.2)). Applying the jump relations for the double-layer potential (see Theorem 6.3) and taking into accoun the boundary conditions (7.2), for the unknown density vector function $h=(h_1,h_2,\dots,h_{11})^\top$ we get the following boundary integral equations:

$$\left[\,2^{-1}I_{11}+\mathcal{N}\,\right]h=f\ \text{ on }\ S,\tag{8.1}$$

in the case of Problem $(I^{(\sigma)})^+$, and

$$\left[\,-2^{-1}I_{11}+\mathcal{N}\,\right]h=f\ \text{ on }\ S,\tag{8.2}$$

in the case of Problem $(I^{(\sigma)})^-$.
Here, the operator $\mathcal{N}$ is given by (6.14). Due to Theorem 6.4, the operators $\pm 2^{-1} I_{11} + \mathcal{N}$ are singular integral operators of normal type with index zero. This leads to the following existence theorems.

Theorem 8.1. *Let $S \in C^{2,\nu}$ and $f \in C^{1,\tau}(S)$ with $0 < \tau < \nu \leq 1$. Then the boundary value problem $(I^{(\sigma)})^+$ is uniquely solvable in the space $C^{1,\tau}(\overline{\Omega^+})$ and the solution can be represented by the double-layer potential $W(h)$ defined by* (6.2)*, where the density $h \in C^{1,\tau}(S)$ is uniquely defined from the integral equation* (8.1)*.*

Proof. The uniqueness follows from Theorem 7.1. Now, let us show that the singular integral operator

$$2^{-1} I_{11} + \mathcal{N} \;:\; C^{1,\tau}(S) \to C^{1,\tau}(S) \tag{8.3}$$

is invertible. Due to Theorem 6.4, we conclude that (8.3) is a Fredholm operator with zero index. Further, we show that ker $\big[\, 2^{-1} I_{11} + \mathcal{N} \,\big]$ is trivial. Indeed, let $h_0 \in C^{1,\tau}(S)$ be a solution of the homogeneous equation

$$\big[\, 2^{-1} I_{11} + \mathcal{N} \,\big]\, h_0 = 0 \;\text{ on }\; S. \tag{8.4}$$

We construct the double-layer potential $W(h_0)$. Evidently, $W(h_0) \in C^{1,\tau}(\overline{\Omega^{\pm}})$ by Theorem 6.3. In view of equation (8.4), we have $\{W(h_0)(x)\}^+ = 0$ for $x \in S$ and by the uniqueness Theorem 7.1, we get $W(h_0)(x) = 0$ for $x \in \Omega^+$. Consequently, $\{\, \mathcal{P}(\partial, n)\, W(h_0)(x)\,\}^+ = 0$ for $x \in S$. By the Liapunov-Tauber theorem (see Theorem 6.3)

$$\{\, \mathcal{P}(\partial, n)\, W(h_0)(x)\}^+ = \{\mathcal{P}(\partial, n)\, W(h_0)(x)\,\}^- = 0, \;\; x \in S,$$

i.e., $W(h_0)$ solves the homogeneous exterior Neumann type boundary value problem $(II^{(\sigma)})^-$ and decays at infinity exponentially. Therefore, $W(h_0)(x) = 0$ in Ω^- by Theorem 7.1. Since

$$\{W(h_0)(x)\}^+ - \{W(h_0)(x)\}^- = 2\, h_0(x), \quad x \in S,$$

we conclude that $h_0 = 0$ on S, which shows that the null space of the operator $2^{-1} I_{11} + \mathcal{N}$ is trivial. Therefore, (8.3) is invertible. □

Quite similarly, with the help of Theorem 7.1, we can show that the operator

$$-2^{-1} I_{11} + \mathcal{N} \;:\; C^{1,\tau}(S) \to C^{1,\tau}(S) \tag{8.5}$$

is invertible, which leads to the existence theorem for the Dirichlet type exterior boundary value problem.

Theorem 8.2. *Let $S \in C^{2,\nu}$ and $f \in C^{1,\nu}(S)$ with $0 < \tau < \nu \leq 1$. Then the boundary value problem $(I^{(\sigma)})^-$ is uniquely solvable in the class of vector functions belonging to the space $C^{1,\tau}(\overline{\Omega^-})$ and decaying at infinity, and the solution is represented by the double-layer potential $W(h)$ defined by* (6.2)*, where $h \in C^{1,\tau}(S)$ is defined by the integral equation* (8.2)*.*

8.2. **Investigation of the interior and exterior Neumann problems.** These problems are formulated in Section 7 as problems $(II^{(\sigma)})^+$ and $(II^{(\sigma)})^-$. As above, we assume that $\Phi^{(\pm)} = 0$ and look for solutions in $\Omega^{\pm}$ in the form of the single-layer potential $U = V(g)$ (see (6.1)). Taking into consideration the boundary conditions (7.3), for the unknown density vector function $g = (g_1, g_2, \ldots, g_{11})^\top$ we get the following boundary integral equations:

$$\big[-\, 2^{-1} I_{11} + \mathcal{K} \,\big]\, g = F \;\text{ on }\; S, \tag{8.6}$$

in the case of Problem $(II^{(\sigma)})^+$, and

$$\big[\, 2^{-1} I_{11} + \mathcal{K} \,\big]\, g = F \;\text{ on }\; S, \tag{8.7}$$

in the case of Problem $(II^{(\sigma)})^-$. Here, the operator $\mathcal{K}$ is given by (6.13). Due to Theorem 6.4, the operators $\pm 2^{-1} I_{11} + \mathcal{K}$ are singular integral operators of normal type with index zero. This leads to the following existence theorems.

Theorem 8.3. *Let $S \in C^{1,\nu}$ and $F \in C^{0,\tau}(S)]$ with $0 < \tau < \nu \leq 1$. Then the boundary value problem $(II^{(\sigma)})^+$ is uniquely solvable in the space $C^{1,\tau}(\overline{\Omega^+})$ and the solution is represented by the single-layer potential $V(g)$ defined by* (6.1)*, where $g \in C^{0,\tau}(S)$ is uniquely defined by the integral equation* (8.6)*.*

Proof. The uniqueness is a consequence of the uniqueness Theorem 7.1. Now, we show that the operator

$$-\,2^{-1}I_{11}+\mathcal{K}\ :\ C^{0,\,\tau}(S)\to C^{0,\,\tau}(S) \tag{8.8}$$

is invertible. Due to Theorem 6.4, the operator (8.8) is a Fredholm operator with zero index. Therefore, it remains to show that the null space of the operator $-2^{-1}I_{11}+\mathcal{K}$ is trivial. Let $g_0\in C^{0,\,\tau}(S)$ solve the homogeneous equation

$$[-\,2^{-1}I_{11}+\mathcal{K}\,]\,g_0=0\ \text{ on }\ S.$$

Construct the single-layer potential $V(g_0)$. Evidently, $V(g_0)\in C^{1,\,\tau}(\overline{\Omega^+})$ due to Theorem 6.3. Moreover, $V(g_0)$ solves the homogeneous Problem $(II^{(\sigma)})^+$ and therefore it vanishes identically in Ω^+, due to Theorem 7.1. Further, by Theorem 6.3, we have $\{\,V(g_0)(x)\,\}^+=\{\,V(g_0)(x)\,\}^-=0$ for $x\in S$, and since it exponentially decays at infinity, by the uniqueness theorem for the Dirichlet exterior boundary value problem, we conclude $V(g_0)(x)\ =\ 0$ for $x\in\Omega^-$. Finally, with the help of the jump relation

$$\{\,\mathcal{P}(\partial,n)\,V(g_0)(x)\,\}^- -\{\,\mathcal{P}(\partial,n)\,V(g_0)(x)\,\}^+=2\,g_0\,(x),\ \ x\in S,$$

we derive $g_0=0$ on S. Thus, the operator (8.8) is invertible. □

By the word for word arguments we can prove that the operator

$$2^{-1}I_{11}+\mathcal{K}\ :\ C^{0,\,\tau}(S)\to C^{0,\,\tau}(S) \tag{8.9}$$

is invertible, which leads to the existence theorem for the Neumann type exterior boundary value problem.

Theorem 8.4. *Let $S\in C^{1,\,\nu}$ and $F\in C^{0,\,\tau}(S)$ with $0<\tau<\nu\le 1$. Then the boundary value problem $(II^{(\sigma)})^-$ is uniquely solvable in the class of vector functions belonging to the space $C^{1,\,\tau}(\overline{\Omega^-})$ and decaying at infinity, and the solution is represented by the single-layer potential $V(g)$ defined by* (6.1)*, where $g\in C^{0,\,\tau}(S)$ is a unique solution of the integral equation* (8.7)*.*

8.3. **Investigation of the basic boundary value problems by the first kind integral equations.** Here we apply an alternative approach and reduce the basic interior and exterior boundary value problems, considered in the previous subsections, to the first kind integral equations (cf. [14]). These results play a crucial role in the study of mixed boundary value problems.

9.3.1. Investigation of the Dirichlet problem with the help of the first kind integral equations. We look for a solution to the problems $(I^{(\sigma)})^+$ and $(I^{(\sigma)})^-$ (see (7.1)–(7.2) with $\Phi^{(\pm)}=0$) in the form of the single-layer potential $U=V(g)$ (see (6.1)). In both cases, for the interior and exterior boundary value problems, we arrive at the equation

$$\mathcal{H}\,g=f\ \text{ on }\ S, \tag{8.10}$$

where $\mathcal{H}$ is defined by (6.12). We have the following existence theorem.

Theorem 8.5. *Let $S\in C^{2,\,\nu}$ and $f\in C^{1,\,\tau}(S)$ with $0<\tau<\nu\le 1$. Then the boundary value problems $(I^{(\sigma)})^\pm$ are uniquely solvable in the class of vector functions belonging to the space $C^{1,\,\tau}(\overline{\Omega^\pm})$ and decaying at infinity, and the solution is represented by the single-layer potential $V(g)$ defined by* (6.1)*, where $g\in C^{0,\,\tau}(S)$ is a unique solution of the integral equation* (8.10)*.*

Proof. The uniqueness follows from Theorem 7.1. Evidently, it remains to show the invertibility of the operator

$$\mathcal{H}\ :\ C^{0,\,\tau}(S)\to C^{1,\,\tau}(S). \tag{8.11}$$

To this end, we apply the operator $\mathcal{L}$ (see (6.15)) to both sides of equation (8.10) and take into consideration the operator equalities (6.20),

$$\mathcal{L}\mathcal{H}g\equiv\big[-\,4^{-1}I_1+\mathcal{K}^2\,\big]\,g=\mathcal{L}f\ \text{ on }\ S. \tag{8.12}$$

Clearly, $\mathcal{L}f\ \in C^{0,\,\tau}(S)$ due to Theorem 6.4. Since the operators (8.8) and (8.9) are invertible, we conclude that the singular integral operator

$$\mathcal{L}\mathcal{H}\ =\ \big[-\,2^{-1}I_{11}+\mathcal{K}\,\big]\,\big[\,2^{-1}I_{11}+\mathcal{K}\,\big]\ :\ C^{0,\,\tau}(S)\to C^{0,\,\tau}(S)$$

is invertible, as well. Therefore, from (8.12), we get the following representation of a solution of equation (8.10),

$$g = \big[-4^{-1} + \mathcal{K}^2 \big]^{-1} \mathcal{L} f \in C^{0,\tau}(S).$$

With the help of the uniqueness Theorem 7.1, one can easily show that the operators

$$\mathcal{H} \ : \ C^{0,\tau}(S) \to C^{1,\tau}(S), \quad \mathcal{L} \ : \ C^{1,\tau}(S) \to C^{0,\tau}(S) \tag{8.13}$$

are injective. Therefore, equations (8.10) and (8.12) are equivalent and the operator (8.11) is invertible, which completes the proof. □

Corollary 8.6. *A solution* $U \in C^{1,\tau}(\overline{\Omega^\pm})$ *of the boundary value problems* $(I^{(\sigma)})^\pm$ *with* $\Phi^{(\pm)} = 0$ *is uniquely representable in the form*

$$U(x) = V(\mathcal{H}^{-1} f)(x), \quad x \in \Omega^\pm,$$

where $f = \{U\}^\pm$ *on* S *and*

$$\mathcal{H}^{-1} \ : \ C^{1,\tau}(S) \to C^{0,\tau}(S)$$

is the inverse to the operator (8.11).

This representation plays a crucial role in the investigation of mixed boundary value problems (cf. [14]).

9.3.2. Investigation of the Neumann problem with the help of the first kind integral equations. We look for a solution to the problems $(II^{(\sigma)})^+$ and $(II^{(\sigma)})^-$ (see (7.1), (7.3) with $\Phi^\pm = 0$) in the form of the double-layer potential $U = W(h)$ (see (6.2)). In both cases, for the interior and exterior boundary value problems, we arrive at the equation

$$\mathcal{L}\, h = F \ \text{ on } \ S, \tag{8.14}$$

where $\mathcal{L}$ is defined by (6.15). We have the following existence theorem.

Theorem 8.7. *Let* $S \in C^{2,\nu}$ *and* $F \in C^{0,\tau}(S)$ *with* $0 < \tau < \nu \le 1$. *Then the boundary value problems* $(II^{(\sigma)})^\pm$ *are uniquely solvable in the class of vector functions belonging to the space* $C^{1,\tau}(\overline{\Omega^\pm})$ *and decaying at infinity, and the solution is represented by the double-layer potential* $W(h)$ *defined by* (6.2), *where* $h \in C^{1,\tau}(S)$ *is a unique solution of the integral equation* (8.14).

Proof. The uniqueness follows from Theorem 7.1. Evidently, it remains to show the invertibility of the operator

$$\mathcal{L} \ : \ C^{1,\tau}(S) \to C^{0,\tau}(S). \tag{8.15}$$

To this end, we apply the operator $\mathcal{H}$ (see (6.12)) to both sides of equation (8.14) and take into consideration the operator equalities (6.20),

$$\mathcal{H}\mathcal{L}h \equiv \big[-4^{-1} I_{11} + \mathcal{N}^2 \big]\, h = \mathcal{H}F \ \text{ on } \ S. \tag{8.16}$$

Clearly, $\mathcal{H}F \in C^{1,\tau}(S)$ due to Theorem 6.4. Since the operators (8.3) and (8.5) are invertible, we conclude that the singular integral operator

$$\mathcal{H}\mathcal{L} = \big[-2^{-1} I_{11} + \mathcal{N} \big] \big[2^{-1} I_{11} + \mathcal{N} \big] \ : \ C^{1,\tau}(S) \to C^{1,\tau}(S)$$

is invertible, as well. Therefore, from (8.16), for a solution of equation (8.14), we get the following representation formula:

$$h = \big[-4^{-1} I_{11} + \mathcal{N}^2 \big]^{-1} \mathcal{H}F \in C^{1,\tau}(S).$$

Since the operators (8.13) are injective, we conclude that equations (8.14) and (8.16) are equivalent and the operator (8.15) is invertible, which completes the proof. □

Corollary 8.8. *A solution* $U \in C^{1,\tau}(\overline{\Omega^\pm})$ *of the boundary value problems* $(II^{(\sigma)})^\pm$ *with* $\Phi^\pm = 0$ *is uniquely representable in the form*

$$U(x) = W(\mathcal{L}^{-1} F)(x), \quad x \in \Omega^\pm,$$

where $F = \{\mathcal{P}(\partial, n)U\}^\pm$ *on* S *and*

$$\mathcal{L}^{-1} \ : \ C^{0,\tau}(S) \to C^{1,\tau}(S)$$

is the inverse to the operator (8.15).

9. Appendix A: Properties of the Characteristic Roots

Here, we investigate the properties of roots of equation (4.6) with respect to r. In particular, we prove the following assertion.

Lemma A.1. *Let us assume that $\sigma=\sigma_1+i\,\sigma_2$ is a complex parameter, where $\sigma_1\in\mathbb{R}$ and $\sigma_2>0$. Then*

$$\det L(-i\,\xi,\sigma)\neq 0$$

for arbitrary $\xi\in\mathbb{R}^3$.

Proof. We prove the lemma by contradiction. Let $\det L(-i\,\xi,\sigma)=0$, $\xi\in\mathbb{R}^3$. Then the system of linear equations $L(-i\,\xi,\sigma)\,X=0$ has a nontrivial solution $X\in\mathbb{C}^{11}\setminus\{0\}$ which can be written as $X=(X^{(1)},X^{(2)},X^{(3)},X^{(4)},X^{(5)})^\top$, where $X^{(j)}=(X_1^{(j)},X_2^{(j)},X_3^{(j)})^\top\in\mathbb{C}^3$, $j=1,2,3$ and $X^{(j)}\in\mathbb{C}$, $j=4,5$, are scalars. Taking into consideration (2.14), the system $L(-i\,\xi,\sigma)\,X=0$ can be rewritten as follows:

$$\begin{gathered}L^{(j)}(-i\,\xi,\sigma)\,X^{(1)}+L^{(j+5)}(-i\,\xi,\sigma)\,X^{(2)}+L^{(j+10)}(-i\,\xi,\sigma)\,X^{(3)}\\ +L^{(j+15)}(-i\,\xi,\sigma)\,X^{(4)}+L^{(j+20)}(-i\,\xi,\sigma)\,X^{(5)}=0,\\ j=1,2,3,4,5,\end{gathered}$$

implying

$$\big[(-\mu\,|\xi|^2+\rho\,\sigma^2)\,I_3-(\lambda_0+\mu)\,Q(\xi)\big]\,X^{(1)}+i\gamma_2\xi^\top X^{(4)}+i\gamma_1\,\xi^\top X^{(5)}=0, \tag{A.1}$$

$$\big[(\delta-h_6\,|\xi|^2)\,I_3-(h_4+h_5)\,Q(\xi)\big]X^{(2)}+i\,\sigma\,\varkappa_1\,X^{(3)}+i\,h_3\,\xi^\top X^{(4)}=0, \tag{A.2}$$

$$i\,\sigma\,\varkappa_1\,X^{(2)}+\big[(\varkappa_0-k_6|\xi|^2)\,I_3-(k_4+k_5)\,Q(\xi)\big]X^{(3)}+i\,k_3\,\xi^\top X^{(5)}=0, \tag{A.3}$$

$$\sigma\,\gamma_2\,\xi\cdot X^{(1)}-i\,h_1\,\xi\cdot X^{(2)}+(i\,\sigma\,m-h\,|\xi|^2)\,X^{(4)}+i\,\sigma\,\varkappa\,X^{(5)}=0, \tag{A.4}$$

$$\sigma\,\gamma_1\,T_0\,\xi\cdot X^{(1)}-ik_1\,\xi\cdot X^{(3)}+i\,\sigma\,T_0\,\varkappa X^{(4)}+\big(i\,\sigma\,c\,T_0-k\,|\xi|^2\big)\,X^{(5)}=0. \tag{A.5}$$

Let us take the dot products of equations (A.1), and (A.2) by the vectors $-i\,\overline{\sigma}\,\overline{X^{(1)}}$ and $-\overline{X^{(2)}}$ respectively, multiply equality (A.3) by the vector $-\overline{X^{(3)}}$, then multiply complex conjugates of equations (A.4) and (A.5) by the functions $-X^{(4)}$ and $-\frac{1}{T_0}\,X^{(5)}$ respectively and sum up the results to obtain

$$\begin{aligned}&i\,\overline{\sigma}\big[\mu\,|\xi|^2-\rho\,\sigma^2\big]\,\big|X^{(1)}\big|^2+i\,\overline{\sigma}\,(\lambda_0+\mu)\,\big|\xi\cdot X^{(1)}\big|^2+\big[h_6\,|\xi|^2-\delta\big]\,\big|X^{(2)}\big|^2+(h_4+h_5)\,\big|\xi\cdot X^{(2)}\big|^2\\ &-i\,\sigma\,\varkappa_1\big[X^{(2)}\cdot\overline{X^{(3)}}+\overline{X^{(2)}}\cdot X^{(3)}\big]-i\,h_3\,\big(\xi\cdot\overline{X^{(2)}}\big)\,X^{(4)}+\big[k_6\,|\xi|^2-\varkappa_0\big]\,\big|X^{(3)}\big|^2\\ &+(k_4+k_5)\,\big|\xi\cdot X^{(3)}\big|^2-i\,k_3\,\big(\xi\cdot\overline{X^{(3)}}\big)X^{(5)}-i\,h_1\,\big(\xi\cdot\overline{X^{(2)}}\big)X^{(4)}+\big[i\,\overline{\sigma}\,m+h\,|\xi|^2\big]\,\big|X^{(4)}\big|^2\\ &+i\,\overline{\sigma}\,\varkappa\,\big[X^{(4)}\,\overline{X^{(5)}}+\overline{X^{(4)}}\,X^{(5)}\big]-\frac{i\,k_1}{T_0}\,\big(\xi\cdot\overline{X^{(3)}}\big)\,X^{(5)}+\Big[i\,\overline{\sigma}\,c+\frac{k}{T_0}\,|\xi|^2\Big]\,\big|X^{(5)}\big|^2=0.\end{aligned}$$

By separating the real part from this equation, we deduce

$$\begin{aligned}&\sigma_2\big[\mu\,|\xi|^2+\rho\,|\sigma|^2\big]\big|X^{(1)}\big|^2+\sigma_2(\lambda_0+\mu)\,\big|\xi\cdot X^{(1)}\big|^2+h_6\,|\xi|^2\,\big|X^{(2)}\big|^2+(h_4+h_5)\,\big|\xi\cdot X^{(2)}\big|^2\\ &+k_6\,|\xi|^2\,\big|X^{(3)}\big|^2+(k_4+k_5)\,\big|\xi\cdot X^{(3)}\big|^2+\sigma_2\big[m_1\,\big|X^{(2)}\big|^2+\varkappa_1\,\big(X^{(2)}\cdot\overline{X^{(3)}}+\overline{X^{(2)}}\cdot X^{(3)}\big)+c_1\,\big|X^{(3)}\big|^2\big]\\ &+\frac{1}{2}\big\{2h_2\,\big|X^{(2)}\big|^2-i\,(h_1+h_3)\,\big[\big(\xi\cdot\overline{X^{(2)}}\big)X^{(4)}-\big(\xi\cdot X^{(2)}\big)\,\overline{X^{(4)}}\big]+2h\,|\xi|^2\,\big|X^{(4)}\big|^2\big\}\\ &+\frac{1}{2T_0}\big\{2T_0\,k_2\,\big|X^{(3)}\big|^2-i\,(k_1+T_0\,k_3)\,\big[\big(\xi\cdot\overline{X^{(3)}}\big)X^{(5)}-\big(\xi\cdot X^{(3)}\big)\,\overline{X^{(5)}}\,\big]+2k\,|\xi|^2\,\big|X^{(5)}\big|^2\big\}\\ &+\sigma_2\big[m\,\big|X^{(4)}\big|^2+\varkappa\,\big(X^{(4)}\cdot\overline{X^{(5)}}+\overline{X^{(4)}}\cdot X^{(5)}\big)+c\,\big|X^{(5)}\big|^2\,\big]=0.\end{aligned} \tag{A.6}$$

With the help of the following relations and inequalities (2.6),

$$|\xi|^2|X^{(j)}|^2 - |\xi \cdot X^{(j)}|^2 = |\,\xi \times X^{(j)}\,|^2, \;\; j = 1,2,3,$$

$$h_6|\xi|^2|X^{(2)}|^2 + (h_4+h_5)|\xi \cdot X^{(2)}|^2 = h_0|\xi \cdot X^{(2)}|^2 + h_6|\,[\xi \times X^{(2)}]\,|^2 \geq 0,$$

$$k_6|\xi|^2|X^{(3)}|^2 + (k_4+k_5)|\xi \cdot X^{(3)}|^2 = k_0\,|\xi \cdot X^{(3)}|^2 + k_6|\,[\xi \times X^{(3)}]\,|^2 \geq 0,$$

$$m_1\,|X^{(2)}|^2 + \varkappa_1\big(X^{(2)} \cdot \overline{X^{(3)}} + \overline{X^{(2)}} \cdot X^{(3)}\big) + c_1\,|X^{(3)}|^2$$
$$= \frac{1}{c_1}\big\{[m_1\,c_1 - \varkappa_1^2]\,|X^{(2)}|^2 + |\varkappa_1\,X^{(2)} + c_1\,X^{(3)}|^2\big\} \geq 0,$$

$$m\,|X^{(4)}|^2 + \varkappa\big(X^{(4)} \cdot \overline{X^{(5)}} + \overline{X^{(4)}} \cdot X^{(5)}\big) + c\,|X^{(5)}|^2$$
$$= \frac{1}{c}\big\{[m\,c - \varkappa^2]\,|X^{(4)}|^2 + |\varkappa\,X^{(4)} + c\,X^{(5)}|^2\big\} \geq 0,$$

$$2h_2\,|X^{(2)}|^2 - i\,(h_1+h_3)\,\big[\big(\xi \cdot \overline{X^{(2)}}\big)X^{(4)} - \big(\xi \cdot X^{(2)}\big)\,\overline{X^{(4)}}\big] + 2h\,|\xi|^2\,|X^{(4)}|^2$$
$$= \frac{1}{2h}\,\big\{[4h\,h_2 - (h_1+h_3)^2]\,|X^{(2)}|^2 + |(h_1+h_3)\,X^{(2)} - 2i\,h\,\xi^\top X^{(4)}|^2\big\} \geq 0,$$

$$2T_0\,k_2\,|X^{(3)}|^2 - i\,(k_1+T_0\,k_3)\,\big[\big(\xi \cdot \overline{X^{(3)}}\big)X^{(5)} - \big(\xi \cdot X^{(3)}\big)\,\overline{X^{(5)}}\big] + 2k\,|\xi|^2\,|X^{(5)}|^2$$
$$= \frac{1}{2k}\,\big\{[4T_0\,k\,k_2 - (k_1+T_0\,k_3)^2]\,|X^{(3)}|^2 + |(k_1+T_0\,k_3)\,X^{(3)} - 2i\,k\,\xi^\top X^{(5)}|^2\big\} \geq 0,$$

from (A.6), we conclude that

$$X^{(j)} = 0, \qquad j = 1,2,3,4,5.$$

Thus, the system $L(-i\,\xi,\sigma)\,X = 0$ possesses only the trivial solution for arbitrary $\xi \in \mathbb{R}^3$. This contradiction proves the lemma. □

Corollary A.2. *Let $\sigma = \sigma_1 + i\,\sigma_2$ be a complex parameter with $\sigma_1 \in \mathbb{R}$ and $\sigma_2 > 0$. Then the equation*

$$\Lambda(\xi) = \det L(-i\,\xi,\sigma) = 0$$

with respect to $|\xi|$ possesses only complex roots $\pm\lambda_j$, $j = \overline{1,11}$ with $\operatorname{Im}\,\lambda_j > 0$, $j = \overline{1,11}$.

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(Received 13.03.2020)

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