Transactions of A. Razmadze
Mathematical Institute
Vol. **174** (2020), issue 3, 271–275

A SYMMETRIZATION IN π-REGULAR RINGS

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Abstract. We introduce and study the so-called (m,n)-*regularly nil clean rings* by showing that these rings are, in fact, a non-trivial generalization of the classical π-regular rings. Our results somewhat supply a recent publication of the author in Turk. J. Math. (2019) and some recent assertions from an own draft (2020).

1. Introduction and Background

Throughout this paper, all rings are assumed to be associative and unital. Our standard terminology and notations are in the most part in agreement with those in [8,9]. Specifically, we let $U(R)$ denote the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R, $J(R)$ the Jacobson radical of R, and $C(R)$ the center of R. About some of the more specific notions, we shall state them in detail below.

Let us recollect that a ring R is called *von Neumann regular* or just *regular* for short if, for every $a \in R$, there is $b \in R$ such that $a = aba$. If, in addition, $b \in U(R)$, the ring R is said to be *unit-regular*. If, however, $b \in Id(R)$, we surprisingly arrive at the so-called *Boolean* rings in which every element is an idempotent. Indeed, it is pretty easy to see that $U(R) = \{1\}$ whence R is reduced (that is, it does not possess any non-trivial nilpotent) and thus abelian (that is, each its idempotent is central). Therefore, as both $b, ab \in Id(R)$, it must be $a = a^2b = ab$, and hence $a = a.a = a^2$, as required.

In that direction, we recall also that a ring R is called π-*regular* if, for each $a \in R$, there are $n \in \mathbb{N}$ and $b \in R$, both depending on a, such that $a^n = a^nba^n$; if $b \in U(R)$, we say that R is *unit* π-*regular*. In case $b = d^n$ for some $d \in R$ and, possibly, $n \geq 2$, R is then called *perfectly regular*, as well as if $d \in U(R)$, we call R *perfectly unit-regular*. In the same vein, we recall that a ring R is said to be *strongly* π-*regular* if, for each $a \in R$, there are $n \in \mathbb{N}$ and $c \in R$, both depending on a, with the property that $a^n = a^{n+1}c = ca^{n+1}$ or, equivalently, $a^n = a^{2n}c^n$. This leads to the fact that any strongly π-regular ring is perfectly unit-regular and the latter one is obviously unit π-regular. It was established in [1] that strongly π-regular rings are always π-regular, whereas the converse is not generally true; however, it holds for abelian rings and for rings with a bounded index of nilpotence. Another interesting class of rings is the class of the so-termed π-*boolean* rings that are rings R for which, for every $a \in R$, there is $i \in \mathbb{N}$ with $a^i = a^{i+1}$. These are, certainly, strongly π-regular by taking $c = 1$. Likewise, it is a principal fact that strongly π-regular rings are unit-regular, provided they are regular.

It is worthwhile noticing that π-regularity was successfully generalized to some non-elementary ways in [6], [7] and [3], [4], respectively. In this connection, as a non-trivial extension of the aforementioned π-regular rings, it was recently defined in [4] the class of the so-called *regularly nil clean rings* as those rings R having the property that for any $r \in R$, there exists $e \in Rr \cap Id(R)$ with $r(1-e) \in Nil(R)$ (or, equivalently, $(1-e)r \in Nil(R)$. In [4, Proposition 1.3] is given the following left-right symmetric property, namely that there exists $f \in rR$ with the property $r(1-f) \in Nil(R)$ (or, in an equivalent form, $(1-f)r \in Nil(R)$. Likewise, it is proved in [4, Proposition 2.1] that π-regular rings are by themselves regularly nil clean.

2020 *Mathematics Subject Classification.* 16U99, 16E50, 16W10, 13B99.

Key words and phrases. π-Regular rings; Regularly nil clean rings; D-Regularly nil clean rings; (m,n)-Regularly nil clean rings.

On the other hand, referring to [5], a ring R is said to be *double regularly nil clean* or just *D-regularly nil clean* for short if, for each $a \in R$, there exists $e \in (aRa) \cap Id(R)$ such that $a(1-e) \in Nil(R)$ (and hence, $(1-e)a \in Nil(R)$).

Certainly, the requirement $e \in (aRa) \cap Id(R)$ is obviously equivalent to the relation $e \in (aR) \cap (Ra) \cap Id(R)$ as the idempotent $e \in aR \cap Ra$ makes sense that $e = e.e \in aRa$.

Apparently, D-regularly nil clean rings are always regularly nil clean. Reformulating [4, Problem 3.1], an intriguing question is of whether or not the properties of being regularly nil clean and D-regularly nil clean are independent of each other, i.e., does there exist a regularly nil clean ring that is *not* D-regularly nil clean?

By what we have discussed so far, our further work is mainly motivated by the following new and more general concept:

Definition 1.1. A ring R is called (m,n)*-regularly nil clean* if, for any $a \in R$, there exist two nonnegative integers m,n and an idempotent $e \in a^mRa^n$ such that $a^m(1-e)a^n \in Nil(R)$.

It is clear that the condition $a^m(1-e)a^n \in Nil(R)$ is equivalent to $a^{m+n}(1-e) \in Nil(R)$, that is, $(1-e)a^{m+n} \in Nil(R)$ as for any two elements x,y of a ring R it follows that $xy \in Nil(R) \iff yx \in Nil(R)$.

Moreover, if $m = 0$ and $n \geq 1$, we just obtain the R-version (marked for right), while if $m \geq 1$ and $n = 0$, we obtain the L-version (marked for left), which both versions were discussed above.

2. Preliminary and Main Results

We begin here with the following technicality, which could be useful for further applications.

Lemma 2.1. *Suppose that R is a ring and m,n are nonnegative integers. Then R is (m,n)-regularly nil clean, if and only if $R/J(R)$ is (m,n)-regularly nil clean and $J(R)$ is nil.*

Proof. Before we proceed to proving this assertion, we need the following folklore fact:

If P is a ring with a nil-ideal I and if $d \in P$ with $d + I \in Id(P/I)$, then $d + I = e + I$ for some $e \in Id(P) \cap dPd$ such that $de = ed$.

The left-to-right implication, being valid in the same manner as in [4, Theorem 2.9], we will deal with the right-to-left one. So, given an arbitrary element a of R, there exists $b+J(R) \in Id(R/J(R)) \cap (a+J(R))(R/J(R))(a+J(R))$ with $(a+J(R))(1+J(R)-(b+J(R)) \in Nil(R/J(R))$. Consequently, bearing in mind the above folklore fact, there is $r \in R$ such that $b+J(R) = (a+J(R))(r+J(R))(a+J(R)) = ara + J(R) = e + J(R)$ for some $e \in Id(R) \cap (ara)R(ara) \subseteq Id(R) \cap aRa$. Furthermore,

$$\begin{aligned}(a + J(R))(1 + J(R) - (e + J(R))) &= (a + J(R))(1 - e + J(R)) \\ &= a(1-e) + J(R) \in Nil(R/J(R))\end{aligned}$$

and, therefore, there exists $m \in \mathbb{N}$ having the property that $[a(1-e)]^m \in J(R) \subseteq Nil(R)$. This means that $a(1-e) \in Nil(R)$, as required. □

Although it has been long ago known that the center of an exchange ring need not to be again exchange, the following statement is somewhat curious even in the light of [4, Proposition 2.7].

Proposition 2.2. *The center of an (m,n)-regularly nil clean ring is again an (m,n)-regularly nil clean ring.*

Proof. Letting R be such a ring and given $c \in C(R)$, we can write that $(c(1-e))^m = c^m(1-e) = 0$ for some $e \in Id(R) \cap cRc = c^2R$. What suffices to prove is that $e \in C(R)$. To do that, for all $r \in R$, it must be that $er(1-e) \in c^mR(1-e) = Rc^m(1-e) = 0$ as $e \in c^2R$ implies at once that $e = e^m \in c^mR$. Thus $er = ere$ and, by a reason of similarity, we also have $re = ere$. Hence, it now immediately follows that $er = re$, proving the claim about the centrality of e.

What remains to be shown is just that $e \in cC(R)c = c^2C(R)$. Indeed, write $e = c^2b$ for some $b \in R$. This forces that $e = c^2be = c^2y$, where $y = be = eb$ as e is central. We claim that $y \in C(R)$, as needed. In fact, for any $z \in R$, one derives that $yz(1-e) = (1-e)yz = (1-e)ebz = 0$ and that $(1-e)zy = zy(1-e) = zbe(1-e) = 0$, because $1-e \in C(R)$, which tells us that $yz = yze$ and $zy = ezy$. Further, $yz = yzc^2y = c^2yzy$ and $zy = c^2yzy$ and, finally, $yz = zy$, as claimed. □

It was proved in [4, Proposition 2.6] that the corner subring of any regularly nil clean ring is again regularly nil clean as well as in [5] the same claim was proved for D-regularly nil clean ring. The next assertion parallels to these two statements.

Proposition 2.3. *Given two integers $m, n \geq 0$, if R is an (m, n)-regularly nil clean ring, then so is the corner ring eRe for any $e \in Id(R)$. In particular, if $\mathbb{M}_n(R)$ is (m, n)-regularly nil clean, then so does R.*

Proof. Choose an arbitrary element $ere \in eRe$ for some $r \in R$. Since $ere \in R$, it follows that there is an idempotent f in R with $f \in (ere)R(ere)$ such that $(1-f)ere \in Nil(R)$. But this could be written as $ere - fere = (e-f)ere = (e-fe)ere = q \in Nil(R)$. Thus $(e-efe)ere = eq = eqe \in Nil(eRe)$ with $efe \in Id(eRe) \cap (ere)(eRe)(ere) = Id(eRe) \cap (ere)R(ere)$, because $efe = f$ and $qe = q$ so that $eq \in Nil(R)$, as expected.

The second part-half appears to be a direct consequence of the first part-half as R is always isomorphic to a corner subring of $\mathbb{M}_n(R)$. □

An important, but seemingly rather difficult problem is the reciprocal implication of the last assertion, namely, if both eRe and $(1-e)R(1-e)$ are (m, n)-regularly nil clean rings, does the same hold for R, too?

Let us now denote by $\mathbb{T}_n(R)$ the upper triangular matrix ring over a ring R, where n runs over $\mathbb{N}$. The next result sheds some more light on the structure of this ring.

Proposition 2.4. *The ring $\mathbb{T}_n(R)$ is (m, n)-regularly nil clean, if and only if the ring R is (m, n)-regularly nil clean.*

Proof. It is well known that

$$\mathbb{T}_n(R)/I \cong \underbrace{R \times \cdots \times R}_{n-times}$$

for a proper nil-ideal I of $\mathbb{T}_n(R)$. So, the claim follows at once by using the standard arguments, leaving the check to the interested readers. □

The next two tricky technicalities are pivotal.

Lemma 2.5. *If R is a ring and $x, y \in R$ with $x = xyx$, then for the element $y' := yxy$ the following two relations*

$(*)$ $x = xy'x$;

$(**)$ $y' = y'xy'$

are fulfilled.

Proof. About the first relationship, $xy'x = x(yxy)x = (xyx)yx = xyx$. As for the second one, $y'xy' = (yxy)x(yxy) = y(xyx)yxy = y(xyx)y = yxy = y'$, as promised. □

It is worthwhile noticing that in [4] it was showed that if a is a π-regular element, that is, a^n is regular for some $n \in \mathbb{N}$, then a is regularly nil clean, too. Nevertheless, this pleasant implication perhaps cannot be happen in the situation of D-regular nil cleanness. Specifically, the following critical assertion is valid:

Proposition 2.6. *If R is a ring having an element a such that a^n is regular for some $n \geq 2$, then a is $(1, 1)$-regularly nil clean of index not greater than n.*

Proof. Writing $a^n = a^n b a^n$ for some existing $b \in R$, then with Lemma 2.5 at hand, we can also write that $b = ba^n b$. Indeed, setting $b' = ba^n b$, by consulting with the cited lemma we will have $a^n = a^n b' a^n$ and $b' = b' a^n b'$, so that without loss of generality, we could replace b' via b. Furthermore, letting $e := aba^{n-1}$, we easily check that $e \in Id(R) \cap (aRa)$ – by a way of similarity we may also consider the idempotent $f = a^{n-1}ba$. By a direct inspection, one verifies that $[a(1-e)a]^n = 0$. In fact, first of all, one finds that $a(1-e)a = a^2 - a^2 b a^n$ and that $[a(1-e)a]^2 = a^4 - a^4 b a^n$. So, by induction, $[a(1-e)a]^n = a^{2n} - a^{2n} b a^n = 0$, as expected. □

We are now ready to proceed by proving with the following

Theorem 2.7. *All π-regular rings are $(1,1)$-regularly nil clean.*

Proof. For such a ring R, letting $r \in R$, if r^2 is a regular element, we are set applying Proposition 2.6. However, if not, since $r^2 \in R$, there is an integer $k > 1$ such that $(r^2)^k = r^{2k}$ is a regular element. As $2k > 2$, again Proposition 2.6 is applicable to conclude the claim. □

Now, the same idea as that in [5] can be adopted to get the following statement which could be of independent interest, as well.

Lemma 2.8. *Let V be a vector space over an arbitrary field K, let $R = End_K(V)$ whose elements are being written to the left of elements of V, and let $a \in R$. Then there exists an idempotent $e \in aRa$ such that $(a(1-e)a)^2 = 0$.*

We are now in a position to show the existence of a concrete construction of an $(1,1)$-regularly nil clean ring that is surely not π-regular.

Example 2.9. For any pair of natural numbers (m,n), there exists an (m,n)-regularly nil clean ring which is *not* π-regular.

Proof. We will restrict our attention to the pair $(1,1)$ as the argumentation follows by analogy with the situation in [4] bearing in mind Lemma 2.8. The general construction can be exhibited by induction, but it is a rather technical matter and so we leave all details to the interested reader. □

In regard to our considerations alluded to above, we state the following

Problem 2.10. What is the relationship between D-regularly nil clean rings and (m,n)-regularly nil clean rings? Are they independent to each other or not?

In the case where the elements a and e from Definition 1.1 commute, these two notions are deducible one of other, i.e., they coincide.

Another question of interest could be the following: It is long ago known that a ring R is *exchange* if, for any $r \in R$, there exists $e \in Id(R) \cap rR$ such that $1-e \in (1-r)R$. This, curiously, is equivalent to the existence of an integer $k > 1$ with the properties that $e \in r^kRr^k$ and $1-e \in (1-r)^kR(1-r)^k$. This can be directly proved observing that both elements r^k and $(1-r)^k$ are commuting and co-maximal.

In that aspect, we define a ring R to be *strongly π-exchange* if, for every $x \in R$, there exists an idempotent $e \in x^nR^n$ for some natural $n > 1$ such that $1-e \in (1-x)^nR^n$, and we define R to be *π-exchange* if, for every $x \in R$, there exists an idempotent $e \in ex^nR^n$ for some natural $n > 1$ such that $1-e \in (1-e)(1-x)^nR^n$, where $R^n = \{r^n \mid r \in R\}$ is a subset of R consisting of all n-th powers of elements from R.

So, we come to our final problem.

Problem 2.11. Determine the structure of strongly π-exchange and π-exchange rings.

In view of [2], there are rather unexpected and non-trivial examples of strongly π-exchange rings, so that the posed question seems to be hard. Indeed, in virtue of our discussion in the introductional section, are perfectly regular rings always π-exchange?

Acknowledgement

The work of the author is partially supported by the Bulgarian National Science Fund under Grant KP-06 N 32/1 of December 07, 2019.

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(Received 09.04.2020)

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