Transactions of A. Razmadze
Mathematical Institute
Vol. **174** (2020), issue 3, 265–269

# ON AN APPLICATION OF POWER INCREASING SEQUENCES

HÜSEYİN BOR

**Abstract.** In this paper we prove a general new summability factor theorem for infinite series involving quasi-power increasing sequences. Some new results are also deduced.

## 1. Introduction

A positive sequence $(X_n)$ is said to be a quasi-$\sigma$-power increasing sequence if there exists a constant $K = K(\sigma, X) \geq 1$ such that $Kn^\sigma X_n \geq m^\sigma X_m$ for all $n \geq m \geq 1$ (see [18]). For any sequence $(\lambda_n)$ we write that $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. We denote by $u_n^\alpha$ and $t_n^\alpha$ the $n$th Cesàro means of order $\alpha$, $\alpha > -1$, of the sequences $(s_n)$ and $(na_n)$, respectively, that is (see [13]),

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \ \ ({t_n}^1 = t_n),$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.$$

A series $\sum a_n$ is said to be summable $|C, \alpha; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [15])

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1}|u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} n^{\delta k - 1}|t_n^\alpha|^k < \infty.$$

If we set $\delta$=0, then we get the $|C, \alpha|_k$ summability (see [14]). Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \ i \geq 1).$$

The sequence-to-sequence transformation

$$v_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v$$

defines the sequence $(v_n)$ of the Riesz mean, or simply, the $(\bar{N}, p_n)$ mean of the sequence $(s_n)$, generated by the sequence of coefficients $(p_n)$ (see [16]). The series $\sum a_n$ is said to be the $|\bar{N}, p_n; \delta|_k$ summable, $k \geq 1$ and $\delta \geq 0$, if (see [7])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1}|v_n - v_{n-1}|^k < \infty.$$

If we set $\delta = 0$, then we obtain the $|\bar{N}, p_n|_k$ summability (see [1]). If we take $p_n = 1$ for all n, then we get the $|C, 1; \delta|_k$ summability. Finally, if we set $\delta = 0$ and $k = 1$, then we get the $|\bar{N}, p_n|$ summability (see [20]).

---

2020 *Mathematics Subject Classification.* 26D15, 40D15, 40F05.

*Key words and phrases.* Absolute summability; Summability factors; Riesz mean; Power increasing sequences; Hölder's inequality; Minkowski's inequality.

## 2. Known Result

Several theorems have been proved dealing with the $|\bar{N}, p_n; \delta|_k$ summability factors of infinite series (see [3, 5–12, 17]). Among them, in [10], the following theorem has been proved.

**Theorem A.** *Let $(X_n)$ be a quasi-$\sigma$-power increasing sequence for some $\sigma$ $(0 < \sigma < 1)$. Suppose that there exist the sequences $(\beta_n)$ and $(\lambda_n)$ such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{2}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{3}$$

$$|\lambda_n|X_n = O(1). \tag{4}$$

*If*

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{|s_v|^k}{vX_v^{k-1}} = O(X_n) \quad as \quad n \to \infty \tag{5}$$

*and $(p_n)$ is a sequence such that*

$$P_n = O(np_n), \tag{6}$$

$$P_n\Delta p_n = O(p_n p_{n+1}), \tag{7}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\Big(\Big(\frac{P_v}{p_v}\Big)^{\delta k} \frac{1}{P_v}\Big) \quad as \quad m \to \infty, \tag{8}$$

*hold, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is the $|\bar{N}, p_n; \delta|_k$ summable, $k \geq 1$ and $0 \leq \delta < 1/k$.*

## 3. The Main Result

The aim of this paper is to prove Theorem A under weaker conditions. Now, we prove the following

**Theorem.** *Let $(X_n)$ be a quasi-$\sigma$-power increasing sequence for some $\sigma$ $(0 < \sigma < 1)$. If the conditions* (1)*,* (2)*,* (3)*,* (4)*,* (6)*,* (7)*,* (8) *and*

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{|t_v|^k}{vX_v^{k-1}} = O(X_n) \quad as \quad n \to \infty \tag{9}$$

*hold, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is the $|\bar{N}, p_n; \delta|_k$ summable, $k \geq 1$ and $0 \leq \delta < 1/k$.*

**Remark.** It should be noted that the condition (5) implies the condition (9) but the converse is need not be true (see [4, 19]).

To prove our theorem, we need the following lemmas.

**Lemma 1** ([18])**.** *Under the conditions on $(X_n)$, $(\beta_n)$ and $(\lambda_n)$ as as expressed in the statement of the theorem, we have the following:*

$$nX_n\beta_n = O(1),$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{10}$$

**Lemma 2** ([2])**.** *If the conditions* (6) *and* (7) *are satisfied, then*

$$\Delta\left(\frac{P_n}{n^2p_n}\right) = O\left(\frac{1}{n^2}\right)$$

## 4. Proof of the theorem

Let $(T_n)$ be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n p_n}$. Then, by the definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{r p_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{v p_v},$$

and hence

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{v p_v}, \quad n \geq 1, \quad (P_{-1} = 0).$$

Using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} =& \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta \left( \frac{P_{v-1} P_v \lambda_v}{v^2 p_v} \right) \sum_{r=1}^{v} r a_r + \frac{\lambda_n}{n^2} \sum_{v=1}^{n} v a_v \\ =& \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{p_v} (v+1) t_v p_v \frac{\lambda_v}{v^2} \\ &+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v (v+1) \frac{t_v}{v^2 p_v} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} (v+1) t_v \Delta(P_v / v^2 p_v) \\ &+ \lambda_n t_n (n+1)/n^2 = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,r}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$

Now, applying Hölder's inequality, we have

$$\begin{aligned} \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |T_{n,1}|^k =& O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v |t_v| |\lambda_v| \frac{1}{v} \right\}^k \\ =& O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left( \frac{P_v}{p_v} \right)^k p_v |t_v|^k |\lambda_v|^k \frac{1}{v^k} \\ & \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\ =& O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k p_v |t_v|^k |\lambda_v|^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \\ =& O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^k |\lambda_v|^{k-1} |\lambda_v| p_v |t_v|^k \frac{1}{v^k} \frac{1}{P_v} \left( \frac{P_v}{p_v} \right)^{\delta k} \\ =& O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{k-1} |\lambda_v| \left( \frac{1}{X_v} \right)^{k-1} |t_v|^k \frac{1}{v^k} \left( \frac{P_v}{p_v} \right)^{\delta k} \\ =& O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k} v^{k-1} \frac{1}{v^k} \left( \frac{1}{X_v} \right)^{k-1} |\lambda_v| |t_v|^k \\ =& O(1) \sum_{v=1}^{m} |\lambda_v| \left( \frac{P_v}{p_v} \right)^{\delta k} \frac{|t_v|^k}{v {X_v}^{k-1}} \\ =& O(1) \sum_{v=1}^{m-1} \Delta |\lambda_v| \sum_{r=1}^{v} \left( \frac{P_r}{p_r} \right)^{\delta k} \frac{|t_r|^k}{r {X_r}^{k-1}} \end{aligned}$$

$$
\begin{aligned}
&+O(1)|\lambda_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{v{X_v}^{k-1}}\\
&=O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&=O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)|\lambda_m|X_m=O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}
$$

by the hypotheses of the theorem and Lemma 1. Now, using (6), we obtain

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,2}|^k &=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v}p_v|\Delta\lambda_v||t_v|\right\}^k\\
&=O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k p_v|\Delta\lambda_v|^k|t_v|^k\\
&\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}\\
&=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k p_v(\beta_v)^k|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\\
&=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\left(\frac{P_v}{p_v}\right)^{k-1}(\beta_v)^k|t_v|^k\\
&=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}v^{k-1}(\beta_v)^{k-1}(\beta_v)|t_v|^k\\
&=O(1)\sum_{v=1}^{m}v\beta_v\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{v{X_v}^{k-1}}\\
&=O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{|t_r|^k}{r{X_r}^{k-1}}+O(1)m\beta_m\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{v{X_v}^{k-1}}\\
&=O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)m\beta_mX_m\\
&=O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}
$$

by the hypotheses of the theorem and Lemma 1. Again, using Lemma 1 and Lemma 2, as in $T_{n,1}$, we have

$$
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,3}|^k=O(1)\quad\text{as}\quad m\to\infty.
$$

Finally, as in $T_{n,1}$, we have

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,4}|^k &=O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\frac{n+1}{n}\right)^k\frac{1}{n^k}|\lambda_n|^k|t_n|^k\\
&=O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}n^{k-1}\frac{1}{n^k}|\lambda_n|^{k-1}|\lambda_n||t_n|^k\\
&=O(1)\sum_{n=1}^{m}|\lambda_n|\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{|t_n|^k}{n{X_n}^{k-1}}=O(1)\quad\text{as}\quad m\to\infty.
\end{aligned}
$$

This completes the proof of the theorem. If we set $\delta$=0, then we have a result dealing with $|\bar{N},p_n|_k$ summability factors of infinite series. Also, if we take $p_n=1$ for all n, then we obtain a new result concerning the $|C,1;\delta|_k$ summability factors of infinite series. Finally, if we set $\delta=0$ and $k=1$, then we get a result related to the $|\bar{N},p_n|$ summability factors of infinite series.

## References


1. H. Bor, On two summability methods. *Math. Proc. Cambridge Philos. Soc.* **97** (1985), no. 1, 147–149.
2. H. Bor, Absolute summability factors for infinite series. *Indian J. Pure Appl. Math.* **19** (1988), no. 7, 664–671.
3. H. Bor, A relation between two summability methods. *Riv. Mat. Univ. Parma (4)* **14** (1988), 107–112 (1989).
4. H. Bor, On the absolute Cesàro summability factors. *Anal. Numér. Théor. Approx.* 20 (1991), no. 1-2, 11–14.
5. H. Bor, Factors for $|\overline{N},p_n;\delta|_k$ summability of infinite series. *Bull. Calcutta Math. Soc.* **84** (1992), no. 4, 371–374.
6. H. Bor, Multipliers for $|\overline{N},p_n;\delta|_k$ summability of infinite series. *Proc. Indian Acad. Sci. Math. Sci.* **103** (1993), no. 2, 153–157.
7. H. Bor, On local property of $|\overline{N},p_n;\delta|_k$ summability of factored Fourier series. *J. Math. Anal. Appl.* **179** (1993), no. 2, 646–649.
8. H. Bor, On absolute summability factors. *Z. Anal. Anwendungen* **15** (1996), no. 2, 545–549.
9. H. Bor, On $|\overline{N},p_n;\delta|_k$ summability factors of infinite series. *Taiwanese J. Math.* **1** (1997), no. 3, 327–332.
10. H. Bor, A further application of power increasing sequences. *Adv. Pure Appl. Math.* **4** (2013), no. 2, 179–188.
11. H. Bor, Some equivalence theorems on absolute summability methods. *Acta Math. Hungar.* **149** (2016), no. 1, 208–214.
12. H. Bor, On some new results for non-decreasing sequences. *Tbilisi Math. J.* **10** (2017), no. 2, 57–64.
13. E. Cesàro, Sur la multiplication des séries. *Bull. Sci. Math.* **14** (1890), 114–120.
14. T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley. *Proc. London Math. Soc. (3)* **7** (1957), 113–141.
15. T. M. Flett, Some more theorems concerning the absolute summability of Fourier series and power series. *Proc. London Math. Soc. (3)* **8** (1958), 357–387.
16. G. H. Hardy, *Divergent Series.* Oxford, at the Clarendon Press, 1949.
17. J. O. Lee, On the summability of infinite series and Hüseyin Bor. *Journal for History of Mathematics* **30** (2017), no. 6, 353–365.
18. L. Leindler, A new application of quasi power increasing sequences. *Publ. Math. Debrecen* **58** (2001), no. 4, 791–796.
19. S. M. Mazhar, On $|C,1|_k$ summability factors of infinite series. *Indian J. Math.* **14** (1972), 45–48.
20. G. Sunouchi, Notes on Fourier analysis. XVIII. Absolute summability of series with constant terms. *Tohoku Math. J. (2)* **1** (1949), 57–65.


(Received 20.02.2020)

P.O. Box 121, TR-06502 Bahçelievler, Ankara, Turkey
*E-mail address*: hbor33@gmail.com