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EXTENSION OPERATORS ON SOBOLEV SPACES WITH DECREASING INTEGRABILITY

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Abstract. We study extension operators on Sobolev spaces with decreasing integrability on the base of set functions associated with the operator norms. Sharp necessary conditions are given in terms of the generalized measure density condition and in terms of a weak integral equivalence of the Euclidean metric and an intrinsic metric.

1. Introduction

Let Ω be a domain in the Euclidean space $\mathbb{R}^n$, $n \geq 2$. Recall that the operator

$$E : W^1_p(\Omega) \to W^1_q(\mathbb{R}^n), \quad 1 \leq q \leq p \leq \infty,$$

is called an extension operator on Sobolev spaces (with decreasing integrability in the case $q < p$), if $E(f)\big|_{\Omega} = f$ for any function $f \in W^1_p(\Omega)$ and

$$\|E\| = \sup_{f \in W^1_p(\Omega)\setminus\{0\}} \frac{\|E(f) \mid W^1_q(\mathbb{R}^n)\|}{\|f \mid W^1_p(\Omega)\|} < \infty.$$

Sobolev extension operators arise in the analysis of PDE (see, for example, [15, 21]) and play an important role in the Sobolev spaces theory. In the present article we prove the sharp Ahlfors type necessary generalized (p, q)-measure density condition for extension operators of seminormed Sobolev spaces: *Let there exist a continuous linear extension operator $E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n)$, $n < q \leq p < \infty$, then*

$$\Phi(B(x, r))^{p-q} |B(x, r) \cap \Omega|^q \geq c_0 |B(x, r)|^p, \ 0 < r < 1, \tag{1.1}$$

where Φ is an additive set function associated with the extension operator and a constant $c_0 = c_0(p, q, n)$ depends on p, q and n only. In the case $p = q$ the measure density condition was introduced in [9] (see, also [24]) and the study of the case $q < p$ requires the use of set functions associated with the extension operators [23, 28].

It is well known [3, 21] that if $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, then there exists the bounded extension operator $E : W^1_p(\Omega) \to W^1_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. In [11], the notion of (ε, δ)-domains was introduced and it was proved that in every (ε, δ)-domain there exists the bounded extension operator $E : W^k_p(\Omega) \to W^k_p(\mathbb{R}^n)$, for all $k \geq 1$ and $p \geq 1$.

The complete description of extension operators of the homogeneous Sobolev space $L^1_2(\Omega)$, $\Omega \subset \mathbb{R}^2$, was obtained in [26] in terms of the quasi-hyperbolic (quasiconformal) geometry of domains. Namely, it was proved that a simply connected domain $\Omega \subset \mathbb{R}^2$ is the L^1_2-extension domain iff Ω is an Ahlfors domain (quasi-disc). In the case of spaces $L^k_p(\Omega)$, $2 < p < \infty$, defined in domains $\Omega \subset \mathbb{R}^2$, the necessary and sufficient conditions were obtained in [20] and formulated in terms of sub-hyperbolic metrics. Note that extension operators on Sobolev spaces $W^k_p(\Omega)$ were intensively studied in the last decade (see, for example, [4, 9, 12, 13, 19]), but the problem of the complete characterization of Sobolev extension domains in the general case remains still open.

In the case $p > n$, the necessary conditions on W^1_p-extension domains written in terms of an intrinsic metric and a measure density were obtained in [24]. Extension operators of Sobolev spaces defined in domains of Carnot groups $E : W^1_p(\Omega) \to W^1_p(\mathbb{G})$ were considered in [8] and extensions of Sobolev spaces on metric measure spaces can be found in [10].

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The results of [9, 24] state that the extension operator

$$E : W^1_p(\Omega) \to W^1_p(\mathbb{R}^n), \quad 1 \leq p < \infty,$$

does not exist in Hölder cusp domains $\Omega \subset \mathbb{R}^n$. In [7], the extension operators with decreasing integrability from domains with the Hölder cusps were constructed by using the method of reflections. Later, the more general theory of composition operators on Sobolev spaces with decreasing integrability was founded in [22, 27]. Using another technique, the extension operators in such type of domains were considered in [16, 17]. An extension operator from Hölder singular domains with decreasing smoothness was studied in [2]. The detailed study of extension operators on Sobolev spaces defined in non-Lipschitz domains is given in [18].

Extension operators with decreasing integrability are considered in [23], where for the first time was introduced a set function (measure) associated with extension operators and were obtained the necessary conditions in integral terms. In the present article, we give a sharp necessary condition of the existence of extension operators on Sobolev spaces with decreasing integrability in capacitary terms and prove the generalized (p, q)-measure density condition that refines the results of [23] and generalized [9] in the case $n < q < p < \infty$.

The necessary conditions written in terms of intrinsic metrics are considered also. On this base, the lower estimates of norms of extension operators are obtained. The norm estimates of extension operators have applications in the spectral theory of non-linear elliptic operators and give estimates of Neumann eigenvalues in terms of operator's norms [5].

2. Set Functions Associated with the Extension Operator

Let Ω be a domain in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, then the Sobolev space $W^1_p(\Omega)$, $1 \leq p \leq \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid W^1_p(\Omega)\| = \|f \mid L_p(\Omega)\| + \|\nabla f \mid L_p(\Omega)\|,$$

where ∇f is the weak gradient of the function f and $\|\nabla f \mid L_p(\Omega)\|$ its norm in the Lebesgue space $L_p(\Omega)$. The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leq p \leq \infty$, is considered with the seminorm

$$\|f \mid L^1_p(\Omega)\| = \|\nabla f \mid L_p(\Omega)\|.$$

We consider the Sobolev spaces as Banach spaces of equivalence classes of functions up to a set of p-capacity zero [15].

2.1. **Set functions and capacity.**

Let $A \subset \mathbb{R}^n$ be an open bounded set such that $A \cap \Omega \neq \emptyset$. Denote by $W_0(A; \Omega)$ the class of continuous functions $f \in L^1_p(\Omega)$ such that $f\eta$ belongs to $L^1_p(A \cap \Omega) \cap C_0(A \cap \Omega)$ for all smooth functions $\eta \in C^\infty_0(\Omega)$. We define the set function

$$\Phi(A) = \sup_{f \in W_0(A;\Omega)} \left(\frac{\|E(f) \mid L^1_q(A)\|}{\|f \mid L^1_p(A \cap \Omega)\|} \right)^\kappa, \quad \frac{1}{\kappa} = \frac{1}{q} - \frac{1}{p}.$$

This set function was introduced in [23] in connection with the lower estimates of norms of extension operators on Sobolev spaces. For readers convenience we give the detailed proof of the following theorem announced in [23]:

Theorem 2.1. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad 1 \leq q < p < \infty.$$

Then the function $\Phi(A)$ is a bounded monotone countably additive set function defined on open bounded subsets $A \subset \mathbb{R}^n$.

Proof. Let $A_1 \subset A_2$ be open subsets of $\mathbb{R}^n$. Then extending functions of $C_0(A_1)$ by zero, we have $C_0(A_1) \subset C_0(A_2)$ and obtain

$$\Phi(A_1) = \sup_{f\in W_0(A_1;\Omega)} \left(\frac{\|E(f) \mid L^1_q(A_1)\|}{\|f \mid L^1_p(A_1\cap\Omega)\|}\right)^{\kappa} \le \sup_{f\in W_0(A_1;\Omega)} \left(\frac{\|E(f) \mid L^1_q(A_2)\|}{\|f \mid L^1_p(A_2\cap\Omega)\|}\right)^{\kappa}$$
$$\le \sup_{f\in W_0(A_2;\Omega)} \left(\frac{\|E(f) \mid L^1_q(A_2)\|}{\|f \mid L^1_p(A_2\cap\Omega)\|}\right)^{\kappa} = \Phi(A_2).$$

Hence Φ is the monotone set function.

Consider open bounded disjoint sets A_k, $k = 1, 2, \ldots$ such that $A_0 = \bigcup_{k=1}^{\infty} A_k$. We choose arbitrary functions $f_k \in W_0(A_k;\Omega)$ such that

$$\|E(f_k) \mid L^1_q(A_k)\| \ge \left(\Phi(A_k)\left(1-\frac{\varepsilon}{2^k}\right)\right)^{\frac{1}{\kappa}} \|f_k \mid L^1_p(A_k\cap\Omega)\|,$$
$$\|f_k \mid L^1_p(A_k\cap\Omega)\|^p = \Phi(A_k)\left(1-\frac{\varepsilon}{2^k}\right),$$

where $k = 1, 2, \ldots$ and $\varepsilon \in (0, 1)$ is a fixed number. Setting $g_N = \sum\limits_{k=1}^{N} f_k$, we find that

$$\left\| E(g_N) \middle| L^1_q\left(\bigcup_{k=1}^{N} A_k\right)\right\| \ge \left(\sum_{k=1}^{N}\left(\Phi(A_k)\left(1-\frac{\varepsilon}{2^k}\right)\right)^{\frac{q}{\kappa}} \|g_N \mid L^1_p(A_k\cap\Omega)\|^q\right)^{\frac{1}{q}}$$
$$= \left(\sum_{k=1}^{N}\Phi(A_k)\left(1-\frac{\varepsilon}{2^k}\right)\right)^{\frac{1}{\kappa}} \left\| g_N \middle| L^1_p\left(\left(\bigcup_{k=1}^{N} A_k\right)\cap\Omega\right)\right\|$$
$$\ge \left(\sum_{k=1}^{N}\Phi(A_k) - \varepsilon\Phi(A_0)\right)^{\frac{1}{\kappa}} \left\| g_N \middle| L^1_p\left(\left(\bigcup_{k=1}^{N} A_k\right)\cap\Omega\right)\right\|,$$

since the sets, where $\nabla E(f_k)$ do not vanish, are disjoint. By the last inequality, we have

$$\Phi(A_0)^{\frac{1}{\kappa}} \ge \sup \frac{\left\| E(g_N) \middle| L^1_q\left(\bigcup_{k=1}^{N} A_k\right)\right\|}{\left\| g_N \middle| L^1_p\left(\left(\bigcup_{k=1}^{N} A_k\right)\cap\Omega\right)\right\|} \ge \left(\sum_{k=1}^{N}\Phi(A_k) - \varepsilon\Phi(A_0)\right)^{\frac{1}{\kappa}},$$

where the upper bound is taken over all the above functions $g_N \in W_0\left(\left(\bigcup_{k=1}^{N} A_k\right);\Omega\right)$. Since both N and ε are arbitrary, we have

$$\sum_{k=1}^{\infty}\Phi(A_k) \le \Phi\left(\bigcup_{k=1}^{\infty} A_k\right).$$

The inverse inequality can be proved directly. $\square$

Corollary 2.2. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad 1 \le q < p < \infty.$$

Then

$$\|E(f) \mid L^1_q(A)\| \le \Phi(A)^{\frac{1}{\kappa}} \|f \mid L^1_p(A\cap\Omega)\|, \quad \frac{1}{\kappa} = \frac{1}{q} - \frac{1}{p}, \tag{2.1}$$

for any function $f \in W^1_\infty(A) \cap C_0(A)$.

Recall the notion of a variational p-capacity [6]. The condenser in the domain $\Omega \subset \mathbb{R}^n$ is the pair (F_0, F_1) of connected, closed relatively to Ω, sets $F_0, F_1 \subset \Omega$. A continuous function $f \in L^1_p(\Omega)$ is called an admissible function for the condenser (F_0, F_1), if the set $F_i \cap \Omega$ is contained in some connected component of the set $\operatorname{Int}\{x \in \Omega : f(x) = i\}$, $i = 0, 1$. We call as the p-capacity of the condenser (F_0, F_1) relatively to the domain Ω the following quantity:

$$\operatorname{cap}_p(F_0, F_1; \Omega) = \inf \|f|L^1_p(\Omega)\|^p. \tag{2.2}$$

Here the greatest lower bond is taken over all functions, admissible for the condenser $(F_0, F_1) \subset \Omega$. If the condenser has no admissible functions, we put the capacity equal to infinity.

Let $F_1 = E$ be a subset of open set $U \subset \Omega$ and $F_0 = \Omega \setminus U$, then the condenser $R = (E, U) = (\Omega \setminus U, E)$ is called a ring condenser, or a ring. Note that the infimum in (2.2) can be taken on over functions $f \in C^\infty_0(\Omega)$ such that $f = 1$ on E and $f = 0$ on $\Omega \setminus U$.

Theorem 2.3. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad 1 \leq q < p < \infty.$$

Then for any compact set $E \subset (U \cap \Omega)$ the inequality

$$\operatorname{cap}_q^{\frac{1}{q}}(E, U) \leq \Phi(U)^{\frac{1}{\kappa}} \operatorname{cap}_p^{\frac{1}{p}}(E, (U \cap \Omega)), \quad \frac{1}{\kappa} = \frac{1}{q} - \frac{1}{p}, \tag{2.3}$$

holds for any open set $U \subset \mathbb{R}^n$.

Proof. Let a smooth function $u \in L^1_p(\Omega)$ be an admissible function for the condenser $(E, (U \cap \Omega)) \subset \Omega$. Then, extending u by zero on the set $U \setminus \Omega$ we obtain the function $E(u) \in L^1_q(\mathbb{R}^n)$ which is an admissible function for the condenser $(E, U) \subset \mathbb{R}^n$. Hence, by inequality (2.1), we have

$$\operatorname{cap}_q^{\frac{1}{q}}(E, U) \leq \Phi(U)^{\frac{1}{\kappa}} \|u \mid L^1_p(\Omega)\|.$$

Since u is an arbitrary admissible function for the condenser $(E, (U \cap \Omega)) \subset \Omega$, therefore

$$\operatorname{cap}_q^{\frac{1}{q}}(E, U) \leq \Phi(U)^{\frac{1}{\kappa}} \operatorname{cap}_p^{\frac{1}{p}}(E, (U \cap \Omega)). \qquad \square$$

2.2. **Generalized (p, q)-measure density conditions.** Consider measure density conditions in domains allowing the extension of operators with decreasing integrability.

Theorem 2.4. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad n < q < p < \infty.$$

Then the domain Ω satisfies the generalized (p, q)-measure density condition

$$\Phi(B(x, r))^{p-q} |B(x, r) \cap \Omega|^q \geq c_0 |B(x, r)|^p, \quad 0 < r < 1,$$

where $x \in \overline{\Omega}$ and a constant $c_0 = c_0(p, q, n)$ depends on p, q and n only.

Proof. Fix a smooth test function $\eta : \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{supp}(\eta) \subset B(0, 1)$ such that η is equal to 1 in the neighborhood of $0 \in \mathbb{R}^n$ and $0 \leq \eta(x) \leq 1$ for all $x \in \mathbb{R}^n$. Consider the points $x \in \overline{\Omega}$, $y \in \Omega$, and denote by $r := |x - y|$. Then the function

$$f(z) = \eta\left(\frac{x - z}{r}\right)$$

is a smooth function such that $f = 1$ in the neighborhood of $x \in \overline{\Omega}$, $f(y) = 0$ and

$$|\nabla f(z)| \leq \frac{\widetilde{C}}{r} \quad \text{for all} \quad z \in \mathbb{R}^n.$$

Substituting this test function f into inequality (2.1), we obtain

$$\|f \mid L^1_q(B(x, r)\| \leq \Phi(B(x, r))^{\frac{p-q}{pq}} \|f \mid L^1_p(B(x, r) \cap \Omega)\|$$
$$\leq \Phi(B(x, r))^{\frac{p-q}{pq}} \frac{\widetilde{C}}{r} |B(x, r) \cap \Omega|^{\frac{1}{p}}.$$

Because $q > n$, applying the embedding theorem of the space of compact supported Sobolev functions to the space of Hölder continuous functions (see, for e.g., [15])

$$L^1_q(B(x, r)) \hookrightarrow H^\gamma(B(x, r)), \quad \gamma = 1 - n/q,$$

we have

$$\frac{1}{|x-y|^{1-\frac{n}{q}}} = \frac{|f(x)-f(y)|}{|x-y|^{1-\frac{n}{q}}} \leq \|f \mid H^{\gamma}B(x,r)\| \leq C\|f \mid L^1_q(B(x,r)\|.$$

So, using these inequalities, we obtain

$$\frac{(r^n)^{\frac{1}{q}}}{r} = \frac{1}{|x-y|^{1-\frac{n}{q}}} \leq \Phi(B(x,r))^{\frac{p-q}{pq}} C\frac{\widetilde{C}}{r}|B(x,r)\cap\Omega|^{\frac{1}{p}}.$$

Hence

$$(r^n)^{\frac{1}{q}} \leq \Phi(B(x,r))^{\frac{p-q}{pq}} C\widetilde{C}|B(x,r)\cap\Omega|^{\frac{1}{p}},$$

and the required inequality is proved. □

To prove the sharpness of condition (1.1), we consider as an example the Hölder singular domain Ω_α, $\alpha > 1$, [7, 14, 18]:

$$\Omega_\alpha = \left\{(x_1,x_2)\in\mathbb{R}^2 : 0 < x_1 \leq 1, |x_2| < x_1^\alpha\right\} \cup B((2,0),\sqrt{2}).$$

Then $|B(0,r)\cap\Omega_\alpha| = cr^{\alpha+1}$ and, substituting it into inequality (1.1), we obtain

$$\Phi(B(0,r))^{p-q} r^{(\alpha+1)q} \geq Cr^{2p}, \quad 0<r<1.$$

Hence $1 \leq q < 2p/(\alpha+1)$ that coincide with the sufficient condition of the existence of (p,q)-extension operators [7, 14, 18]. So, the necessary condition of Theorem 2.4 is sharp.

Recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is called α-integral regular domain [23] if the function

$$K(x) = \limsup_{r\to 0} \frac{|B(x,r)|}{|B(x,r)\cap\Omega|}$$

belongs to the Lebesgue spaces $L_\alpha(\overline{\Omega})$. From Theorem 2.4 follows the assertion which was originally formulated in [23]:

Theorem 2.5. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad n < q < p < \infty.$$

Then the domain Ω is α-integral regular for $\alpha = q/(p-q)$ and

$$\|E\| \geq c_1 \|K \mid L_\alpha(\overline{\Omega})\|^{\frac{1}{p}},$$

where a constant $c_1 = c_1(p,q,n)$ depends on p, q and n only.

Proof. Rewrite inequality (1.1) in the form

$$\left(\frac{|B(x,r)|}{|B(x,r)\cap\Omega|}\right)^{\frac{q}{p-q}} \leq \frac{1}{c_1^\kappa}\frac{\Phi(B(x,r))}{|B(x,r)|}.$$

Putting $r \to 0$ and using the Lebesgue type differentiability theorem, we have

$$K(x)^\alpha \leq \frac{1}{c_1^\kappa}\Phi'(x), \text{ for almost all } x \in \overline{\Omega}.$$

Integrating the last inequality on the closed domain $\overline{\Omega}$, we find that for any bounded open set $\overline{\Omega} \subset U \subset \mathbb{R}^n$,

$$\int\limits_{\overline{\Omega}} K(x)^\alpha\ dx \leq \frac{1}{c_1^\kappa}\int\limits_{\overline{\Omega}} \Phi'(x)\ dx \leq \frac{1}{c_1^\kappa}\int\limits_{U} \Phi'(x)\ dx = \frac{1}{c_1^\kappa}\Phi(U) \leq \frac{1}{c_1^\kappa}\|E\|^\kappa.$$ □

2.3. **Intrinsic metrics in extension domains.** Let $\gamma : [a,b] \to \Omega$ be a rectifiable curve, then the length $l(\gamma)$ can be calculated by the formula

$$l(\gamma) = \int\limits_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle^{\frac{1}{2}}\ dt.$$

In the domain $\Omega \subset \mathbb{R}^n$, we define an intrinsic metric in Alexandrov's sense [1]:

$$d_\Omega(x,y) = \inf l(\gamma(x,y)), \quad x, y \in \Omega,$$

where infimum is taken over all rectifiable curves $\gamma \subset \Omega$, joint points $x, y \in \Omega$.

We use the following lemma [8, 25].

Lemma 2.6. *For any points $x, y \in \Omega$ there exists a function $f \in W^1_\infty(\Omega)$ such that:*
(1) $0 \leq f(t) \leq 1$ *for any* $t \in \Omega$, $f(x) = 1$ *and* $f(y) = 0$,
(2) $|f(t) - f(s)| \leq d_\Omega(t,s)/d_\Omega(x,y)$,
(3) $\operatorname{supp}(f) \subset B(x, d_\Omega(x,y)) := B(x,R)$,
(4) $|\nabla f| \leq 1/d_\Omega(x,y)$ *a.e. in* Ω.

Note that the proof of this lemma is based on the test function

$$f(t) = \frac{d_\Omega(t, \Omega_x)}{d_\Omega(x,y)}, \quad t \in \Omega,$$

for fixed $x, y \in \Omega$, which was introduced in [25] (see, also [8]). The sets Ω_x and $d_\Omega(t, \Omega_x)$ are defined for the fixed points $x, y \in \Omega$ by the formulas

$$\Omega_x = \{s \in \Omega : d_\Omega(x,s) \geq d_\Omega(x,y)\}$$

and

$$d_\Omega(t, \Omega_x) = \inf\{d_\Omega(t,s) : s \in \Omega_x\}.$$

In the following theorem we give the relation between the intrinsic metric and the Euclidean metric in (p,q)-extension domains that can be considered as a generalized Ahlfors type metric condition.

Theorem 2.7. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad n < q \leq p < \infty.$$

Then in the domain Ω, the intrinsic metric is (p,q)-equivalent to the Euclidean metric

$$d_\Omega(x,y)^{1-\frac{n}{p}} \leq C_0 \Phi(B(x,R))^{\frac{1}{\kappa}} |x-y|^{1-\frac{n}{q}}, \quad R = d_\Omega(x,y), \tag{2.4}$$

for all $|x-y| < 1$, where a constant $C_0 = C_0(p,q,n)$ depends on p, q and n only.

Proof. Substituting the test function f from Lemma 2.6 into inequality (2.1), we obtain

$$\|E(f) \mid L^1_q(B(x,R)\| \leq \Phi(B(x,R))^{\frac{1}{\kappa}} \cdot \frac{1}{d_\Omega(x,y)^{1-\frac{n}{p}}}, \tag{2.5}$$

because of

$$\|f \mid L^1_p(\Omega)\| \leq \Bigg(\int\limits_{B(x,R)} |\nabla f(z)|^p\ dz\Bigg)^{\frac{1}{p}}$$

$$\leq \Bigg(\int\limits_{B(x,R)} \left(\frac{1}{R}\right)^p\ dz\Bigg)^{\frac{1}{p}} = \frac{1}{R^{1-\frac{n}{p}}} = \frac{1}{d_\Omega(x,y)^{1-\frac{n}{p}}}.$$

In the left-hand side of inequality (2.5) we apply the embedding theorem of the space of compact supported Sobolev functions to the space of Hölder continuous functions $L^1_q(B) \hookrightarrow H^\gamma(B)$, $\gamma = 1 - n/q$. So, we obtain

$$\frac{1}{|x-y|^{1-\frac{n}{q}}} = \frac{|f(x) - f(y)|}{|x-y|^{1-\frac{n}{q}}} \leq \|f \mid H^\gamma(B(x,R))\| \leq C_0 \|f \mid L^1_q(B(x,R)\|.$$

Hence

$$\frac{1}{|x-y|^{1-\frac{n}{q}}} \leq C_0 \Phi(B(x,R))^{\frac{1}{\kappa}} \cdot \frac{1}{d_\Omega(x,y)^{1-\frac{n}{p}}}, \ |x-y|<1.$$

The theorem is proved. □

Let $x \in \Omega$, we define the value [23]

$$M(x) = \limsup_{r\to 0} M(x,r) := \limsup_{r \to 0} \left\{ \inf_{|x-y|\leq r} \left\{ m : d_{\Omega(x,y)} \leq m|x-y| \right\} \right\}.$$

Inequality (2.4) leads to the following lower estimate of the extension operator formulated in [23]:

Theorem 2.8. *Let there exist a continuous linear extension operator*

$$E : L^1_p(\Omega) \to L^1_q(\mathbb{R}^n), \quad n < q \leq p < \infty.$$

Then

$$\|E\| \geq C_0 \|M \mid L_\alpha(\Omega)\|^{1-\frac{n}{q}}, \tag{2.6}$$

where $\alpha = (pq-pn)/(p-q)$ *and a constant* $C_0 = C_0(p,q,n)$ *depends on* p*,* q *and* n *only.*

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