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ON THE VECTOR FIBER SURFACE OF THE SPACE $Lm(Vn)$ OF TRIPLET CONNECTEDNESS

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Abstract. In this paper, we consider the theory of the surface of metric vector fibers for the space $Lm(Vn)$ with triplet connectedness. It is proved that in metric vector fibers there always exists an internal triplet connectedness. Analogues of Gauss–Weingarten derivation formulas and also analogues of generalized Gauss, Peterson–Codazzi–Mainardi equations are found.

Let us consider the vector fiber space $Lm(Vn)$, where the local coordinates of a point transform by the law [2]

$$\overline{x^i} = \overline{x^l}(x^k); \qquad \overline{y^\alpha} = A^\alpha_\beta(x) y^\beta;$$

$$\det\left\|\frac{\partial\overline{x^i}}{\partial\overline{x^k}}\right\| \neq 0; \quad \det\|A^\alpha_\beta\| \neq 0; \quad i,j,k = 1,\dots,n; \quad \alpha,\beta,\gamma = 1,\dots,m. \tag{1}$$

Assume that the tensor field $G_{AB}(A,B,C = 1,2,\dots,n+m)$ is given on the space $Lm(Vn)$, i.e.

$$\overline{G_{AB}} = \overset{*}{\mathfrak{X}}{}^C_A \overset{*}{\mathfrak{X}}{}^D_B G_{CD},$$

where

$$\mathfrak{X}^A_B = \left\|\frac{\partial\overline{\mathfrak{X}}^A}{\partial\overline{\mathfrak{X}}^B}\right\| = \begin{Vmatrix} \dfrac{\partial\overline{x}^i}{\partial x^j} & \dfrac{\partial\overline{x}^i}{\partial x^\alpha} \\[2ex] \dfrac{\partial\overline{x}^\alpha}{\partial x^i} & \dfrac{\partial\overline{x}^\alpha}{\partial x^\beta} \end{Vmatrix} = \begin{Vmatrix} x^i_j & 0 \\ A^\alpha_{\beta k} y^\beta & A^\alpha_\beta \end{Vmatrix}.$$

An inverse matrix of the matrix has the form

$$\overset{*}{\mathfrak{X}}{}^A_B = \left\|\frac{\partial\overline{\mathfrak{X}}^A}{\partial\overline{\mathfrak{X}}^B}\right\| = \begin{Vmatrix} \dfrac{\partial x^i}{\partial\overline{x}^k} & \dfrac{\partial x^i}{\partial\overline{x}^\alpha} \\[2ex] \dfrac{\partial x^\alpha}{\partial\overline{x}^i} & \dfrac{\partial x^\alpha}{\partial\overline{x}^\beta} \end{Vmatrix} = \begin{Vmatrix} \overset{*}{x}{}^i_j & 0 \\ \overset{*}{A}{}^\alpha_{\beta k} A^\beta_\gamma y^\gamma & \overset{*}{A}{}^\alpha_\beta \end{Vmatrix}.$$

Since $G^{\alpha\beta} = A^\alpha_\gamma A^\beta_\delta G^{\gamma\delta}$, where $G_{\beta\gamma}G^{\alpha\beta} = \delta^\alpha_\gamma$ and $G_{\beta i} = \overset{*}{A}{}^\gamma_\beta \overset{*}{x}{}^k_i G_{\gamma k} + \overset{*}{A}{}^\gamma_\beta \overset{*}{A}{}^\alpha_{\varepsilon i} A^\varepsilon_\rho y^\rho G_{\gamma\alpha}$, we can use them to construct the values Γ^α_i as follows: $\Gamma^\alpha_i = G^{\alpha\beta}G_{\beta i}$.

Furthermore,

$$G^{\alpha\beta}G_{\beta i} = A^\alpha_\gamma A^\beta_\delta G^{\gamma\delta} \overset{*}{A}{}^\rho_\beta \overset{*}{x}{}^k_i G_{\rho k} + A^\beta_\gamma G^{\gamma\delta} \overset{*}{A}{}^\omega_\beta \overset{*}{A}{}^\sigma_{\varepsilon i} A^\varepsilon_\rho y^\rho G_{\omega\sigma}.$$

Since $A^\alpha_\gamma \overset{*}{A}{}^\alpha_\beta = \delta^\alpha_\beta$, $A_\gamma k^\alpha \overset{*}{A}{}^\gamma_\varepsilon + A^\alpha_\gamma \overset{*}{A}{}^\gamma_{\varepsilon i} x^i_k = 0$, $-\overset{*}{x}{}^k_i A^k_{\gamma k} A^\alpha_{\gamma k} \overset{*}{A}{}^\gamma_\varepsilon = A^\alpha_\gamma \overset{*}{A}{}^\gamma_{\varepsilon i}$, we observe that the values Γ^α_i form an object of linear connectedness with the following transformation law

$$\overset{*}{\Gamma}{}^\alpha_i = A^\alpha_\gamma \overset{*}{x}{}^k_i \Gamma^\gamma_k - \overset{*}{x}{}^k_i A^\alpha_{\gamma k} y^\gamma,$$

and the values

$$g_{ij} = G_{ij} - \Gamma^\alpha_i G_{\alpha j} - \Gamma^\beta_j G_{\beta i} + \Gamma^\alpha_i \Gamma^\beta_j G_{\alpha\beta}$$

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are a double covariant symmetric tensor so that we can construct an object of affine connectedness Γ^i_{jk} in the following manner

$$\Gamma^i_{jk} - \frac{1}{2} g^{ip}(\nabla_k g_{pj} + \nabla_j g_{kp} - \nabla_p g_{jk}).$$

Note that the linear connectedness Γ^α_i induces the vertical affine connectedness defined by the object $\nabla_\beta \Gamma^\alpha_i \equiv \Gamma^\alpha_{\beta i}$ with the following transformation law

$$\overline{\Gamma^\alpha_{\beta i}} = \overset{*}{x}{}^k_i \overset{*}{A}{}^\gamma_\beta A^\alpha_\delta \Gamma^\delta_{\gamma k} - A^\alpha_{\beta i}.$$

Structural equations of the space $Lm(Vn)$ with triplet connectedness have the form [3, 4]:

$$\begin{cases} D\omega^i = \omega^k \wedge \widetilde{\omega}^i_k, \\ D\widetilde{\theta}^\alpha = \widetilde{\theta}^\beta \wedge \widetilde{\omega}^\alpha_\beta + R^\alpha_{ik}\omega^i \wedge \omega^k, \\ D\widetilde{\omega}^\alpha_\beta = \widetilde{\omega}^\gamma_\beta \wedge \widetilde{\omega}^\alpha_\gamma + R^\alpha_{\beta ik}\omega^i \wedge \omega^k + R^\alpha_{\beta i\gamma}\omega^i \wedge \widetilde{\theta}^\gamma, \\ D\widetilde{\omega}^i_j = \widetilde{\omega}^k_j \wedge \widetilde{\omega}^i_k + R^i_{jpq}\omega^p \wedge \omega^q + R^i_{jp\gamma}\omega^p \wedge \widetilde{\theta}^\gamma. \end{cases} \tag{2}$$

Assume that a hypersurface $\mathfrak{N}$ is given on the space $Lm(Vn)$

$$\omega^i = M^i_a \psi^a \tag{3}$$

and the 1-forms ψ^a re such that

$$\begin{cases} D\psi^a = \psi^b \wedge \psi^a_b, \\ D\psi^a_b = \psi^c_b \wedge \psi^a_c + \psi^c \wedge \psi^a_{bc}. \end{cases}$$

Note that

$$D\widetilde{\theta}^\alpha = \widetilde{\theta}^\beta \wedge \widetilde{\omega}^\alpha_\beta + R^\alpha_{ik} \wedge \omega^k = \widetilde{\theta}^\beta \wedge \widetilde{\omega}^\alpha_\beta + R^\alpha_{ik} M^i_a \psi^a \wedge M^k_a \psi^a = R^\alpha_{ab}\psi^a \wedge \psi^b,$$

where $R^\alpha_{ab} = R^\alpha_{ik} M^i_a M^k_b$.

The extension of system (3) is given by

$$\begin{cases} \nabla M^i_a = M^i_{ab}\psi^b, \quad \nabla M^i_{ab} + M^i_c \psi^c_{ab} = M^i_{abc}\psi^c, \\ \nabla M^i_{abc} + 2M^i_{(a|d|}\psi^d_{b)c} - M^i_d \psi^d_{bc} = M^i_{abd}\psi^d, \end{cases}$$

where

$$\begin{cases} M^i_{[ab]} = 0, \quad M^i_{a[bc]} = -R^i_{jpq} M^p_a M^q_b M^j_c, \\ M^i_{ab[cd]} = -R^i_{pqj} M^p_{ab} M^q_c M^j_d. \end{cases}$$

The values M^i_a, M^i_{ab} and M^i_{abc} form a fundamental third-order difference-geometric object of the surface $\mathfrak{N}$.

The normal vector of the hypersurface $\mathfrak{N}$ at the point T satisfies the equations

$$g_{ij} n^i M^i_a = 0, \qquad g_{ij} n^i n^j = 1.$$

A metric tensor of the hypersurface $\mathfrak{N}$ is written in the form

$$g_{ab} = g_{ij} M^i_a M^j_b$$

and $\nabla g_{ab} = g_{abc}\psi^c$, where $g_{abc} = g_{ij} M^i_a M^j_{bc} + g_{ij} M^i_{ac} M^j_b$.

The vectors $M^i_{ab} e_i$ and $n^i_a e_i$ admit representations in the form of a linear combination of vectors of the reference point $\{T, M_a, n\}$:

$$M^i_{ab} e_i = \mathbb{Q}_{ab}{}^c M_c + \mathcal{L}_{ab} n, \tag{4}$$

$$n^i_a e_i = \mathcal{L}^b_a M_b + n_a n, \tag{5}$$

where

$$\mathbb{Q}^c_{ab} = g^{cd} g_{ik} M^i_{ab} M^k_d, \quad \mathcal{L}_{ab} = g_{ki} n^k M^i_{ab}, \quad \mathcal{L}^b_a = -g^{cb}\mathcal{L}_{ca}, \quad n_a = g_{ki} n^k n^i_a. \tag{6}$$

We call equations (4) and (5) the Gauss-Weingarten formulas of the hypersurface $\mathfrak{N}$. From (6) we obtain

$$\nabla \mathbb{Q}^c_{ab} + M^c_{ab} = \mathbb{Q}^c_{abd}\psi^d, \tag{7}$$

$$\nabla \mathcal{L}_{ab} = \mathcal{L}_{abc}\psi^c, \tag{8}$$

where

$$\begin{cases} \mathbb{Q}^c_{abd} = g^{ec}_{\dots,d} g_{ik} M^i_{ab} M^k_e + g^{ei} g_{ik} M^i_{abd} M^k_e + g^{ec} g_{ik} M^i_{ab} M^k_{ed}, \\ \mathcal{L}_{abc} = g_{ki} n^k_c M^i_{ab} + g_{ki} n^k M^i_{abc}, \\ g^{ec}_{\dots,d} = -g^{ea} g^{bc} g_{abd}. \end{cases}$$

From (7) and (8) it follows that $\mathbb{Q}^c_{ab}$ is the object of affine connectedness and $\mathcal{L}_{ab}$ is the tensor.

We call the object $\mathbb{Q}^c_{ab}$ the object of induced affine connectedness of the hypersurface $\mathfrak{N}$. It is easy to prove that the induced affine connectedness and the internal affine connectedness coincide. The 1-forms of this connectedness have the form

$$\widetilde{\psi}^a_b = \psi^a_b + \mathbb{Q}^a_{bc}\psi^c.$$

It is obvious that

$$D\psi^a = \psi^b \wedge \widetilde{\psi}^a_b, \quad D\widetilde{\psi}^a_b = \widetilde{\psi}^c_b \wedge \widetilde{\psi}^a_c + M^a_{bcd}\psi^c \wedge \psi^d,$$

where

$$M^a_{bcd} = \mathbb{Q}^a_{b[cd]} - \mathbb{Q}^a_{e[c}\mathbb{Q}^e_{|b|d]}.$$

The values M^a_{bcd} form the tensor which we call the curvature tensor of the hypersurface $\mathfrak{N}$. By extending equation (4) we obtain

$$R^i_{jpq} M^k_a M^p_b M^q_c = \big(M^d_{abc} - \mathcal{L}_{a[b}\mathcal{L}^d_{c]}\big) M^i_d - \big(\overset{k}{\nabla}_{[c}\mathcal{L}_{|a|b]} - M^d_{bc}\mathcal{L}_{ad} + \mathcal{L}_{a[b}n_{c]}\big) n^i,$$

where $\overset{k}{\nabla}_c$ is the symbol of nonholomorphic covariant differentiation.

From the above equalities we obtain the generalized Gauss equations

$$R_{ipq} M^j_a M^p_b M^q_c M^i_e = M^i_{abce} + \mathcal{L}_{a[b}\mathcal{L}_{c]d} \tag{9}$$

and the generalized Peterson–Codazzi–Mainardi equations

$$R_{kpqr} M^k_a M^q_b M^r_c n^p = M^d_{bc}\mathcal{L}_{ad} - \overset{k}{\nabla}_{[c}\mathcal{L}_{|a|b]} - \mathcal{L}_{a[b}n_{c]}, \tag{10}$$

where

$$R_{ipqr} = q_{ij} R^j_{pqr}, \quad M_{abce} = g_{de} M^d_{abc}.$$

Equations (9) and (10) establish the connection between the curvature tensor of the space $Lm(Vn)$ and the curvature tensor of the hypersurface $\mathfrak{N}$ in $Lm(Vn)$ [1,5].

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