Transactions of A. Razmadze
Mathematical Institute
Vol. **174** (2020), issue 3, 369–375

ON THE INFLUENCE OF BOUNDARY CONDITIONS OF RIGID FIXING ON EIGEN-OSCILLATIONS AND THERMOSTABILITY OF SHELLS OF REVOLUTION, CLOSE BY THEIR FORM TO CYLINDRICAL ONES, WITH AN ELASTIC FILLER, UNDER THE ACTION OF PRESSURE AND TEMPERATURE

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Abstract. The influence of boundary conditions of rigid fixing on eigen-oscillations and thermostability of shells of revolution which by their form are close to cylindrical ones, with an elastic filler, under the action of external pressure and temperature, is investigated. We consider closed shells of middle length whose form of midsurface generatrix is defined by a parabolic function. The shells of positive and negative Gaussian curvature are studied. Formulas and graphs of dependence of the least frequency and form of wave formation on the type of boundary conditions, external pressure, temperature, rigidity of an elastic filler, as well as on the amplitude of shell deviation from the cylinder, are presented. Comparison of the given parameters with the situation when the shell ends are freely supported, is carried out. The question of thermostability is considered and the formula for finding critical pressure is given.

In the present paper we investigate the influence of boundary conditions of rigid fixing, temperature, external pressure and rigidity of an elastic filler on eigen-oscillations and stability of closed shells of revolution, close by their form to cylindrical ones. We consider a light filler for which the influence of tangential stresses on the contact surface and inertia forces may be neglected. The shell is assumed to be thin and elastic. Temperature is uniformly distributed in the shell body. An elastic filler is modelled by Winkler's base; its extension upon heating comes out of account. We investigate the shells of middle length whose form of the midsurface generatrix is defined by a parabolic function. We consider the shells of positive, as well as of negative Gaussian curvature. Formulas and universal curves of dependence of the least frequency and critical load on the Gaussian curvature, type of boundary conditions, temperature, rigidity of an elasic filler, as well as on the amplitude of shell deviation from the cylinder, are obtained. The question of thermostability is also considered and the formula for determination of critical pressure is given.

1. We consider the shell whose middle surface is formed by the rotation of square parabola around the z-axis of the rectangular system of coordinates x, y, z with the origin at the bisecting point of a segment of the axis of revolution. It is assumed that the radius R of the midsurface cross-section is defined by the equality $R = r + \delta_0[1 - \xi^2(r/\ell)^2]$, where r is the end-wall cross-section, δ_0 is the maximal deviation from the cylindrical form (for $\delta_0 > 0$, the shell is convex and for $\delta_0 < 0$, it is concave), $L = 2\ell$ is the shell length, $\xi = z/r$. We consider the shells of middle length [6] and it is assumed that

$$\big(\delta_0/r\big)^2,\ \big(\delta_0/\ell\big)^2 \ll 1. \tag{1}$$

For the shells of middle length, the forms of oscillation corresponding to the lower frequencies are accompanied by a weakly-marked wave formation in longitudinal direction as compared with the circumferential one, therefore the relation

$$\partial^2 f/\partial\xi^2 \ll \partial^2 f/\partial\varphi^2 \quad (f = u, v, w), \tag{2}$$

is valid, where u, v, w are, respectively, meridional, circumferential and radial displacement components characterizing oscillation form. Hence, according to Novozhilov's statement [3], as the basic

2020 *Mathematics Subject Classification.* 35J60.

Key words and phrases. Shells; Oscillation; Stability; Boundary conditions; Filler; Temperature; Frequency; Critical load.

equations of oscillations we can take those corresponding to Vlasov's semimomentless theory [5]. As a result of simplification, the system of equations takes the form (due to the adopted assumption, temperature terms are equal to zero [4])

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial \varphi^2} = -\big[1+2(1+\nu)\delta\big]\frac{\partial w}{\partial \xi}, \quad \frac{\partial^2 v}{\partial \varphi^2} = (1+2\nu\delta)\frac{\partial w}{\partial \varphi}\,,\\
&\varepsilon\,\frac{\partial^8 w}{\partial \varphi^8} + \frac{\partial^4 w}{\partial \xi^4} + 4\delta\frac{\partial^4 w}{\partial \xi^2\,\partial \varphi^2} + 4\delta^2\frac{\partial^4 w}{\partial \varphi^4} - t_1^0\frac{\partial^6 w}{\partial \xi^2\,\partial \varphi^4} - t_2^0\frac{\partial^6 w}{\partial \varphi^6}\\
&\qquad - 2s^0\frac{\partial^6 w}{\partial \xi\,\partial \varphi^5} + \gamma\frac{\partial^4 w}{\partial \varphi^4} + \frac{\rho r^2}{E}\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial \varphi^4}\Big) = 0,\\
&\qquad \varepsilon = h^2/12r^2(1-\nu^2), \quad \delta = \delta_0 r/\ell^2,\\
&\qquad t_i = T_i^0/Eh \quad (i=1,2), \quad s^0 = S^0/Eh, \quad \gamma = \beta r^2/Eh,
\end{aligned}
\tag{3}
$$

where E,ν is an elastic module and the Poison coefficient; T_1^0 and T_2^0 are, respectively, meridional and circumferential stresses of the initial state, S^0 is a shearing stress of the initial state; ρ is density of the shell material; β is the "bed" coefficient of the elastic filler (characterizing elastic rigidity of the filler); φ is angular coordinate, t is time.

The initial state is assumed to be momentless. With rigid fixing of the shell ends there are no meridional displacements at the ends. On the basis of a corresponding solution, taking into account the filler reaction, temperature and also equations (1), we obtain the following approximate expressions

$$
\begin{aligned}
&T_1^0 = -qr\Big\{\nu + \frac{\delta_0}{r}\Big[\frac{1+\nu}{3} + 2(1-2\nu^2)\big(r/\ell\big)^2 - (1-\nu^2)\xi^2\big(r/\ell\big)^2\Big]\Big\} - \frac{\alpha TEh}{1-\nu},\\
&T_2^0 = -qr\Big[1 - 2\nu\frac{\delta_0}{r}\Big(\frac{r}{\ell}\Big)^2\Big] + w_0\beta_0 r, \quad S^0 = 0,
\end{aligned}
\tag{4}
$$

where w_0 and β_0 are, respectively, deflection and "bed" coefficient of the filler in the initial state; α is the coefficient of linear extension; T is temperature; q is external pressure ($q>0$).

Taking into account relations (1) and (2), we find that

$$
\frac{\delta_0}{r}\Big[\frac{1+\nu}{3} + 2(1-2\nu^2)\big(r/\ell\big)^2 - (1-\nu^2)\xi^2\big(r/\ell\big)^2\Big]\frac{\partial^2 w}{\partial \xi^2} \ll \frac{\partial^2 w}{\partial \varphi^2}\,, \quad \nu\frac{\partial^2 w}{\partial \xi^2} \ll \frac{\partial^2 w}{\partial \varphi^2}\,.
$$

Therefore expressions (4) after substitution into (3) can be simplified and they take the form

$$
T_1^0 = -\frac{\alpha TEh}{1-\nu}\,, \quad T_2^0 = -qr\Big[1 - 2\nu\frac{\delta_0}{r}\Big(\frac{r}{\ell}\Big)^2\Big] + w_0\beta_0 r, \quad T_i^0 = \sigma_i^0 h \quad (i=1,2). \tag{4$'$}
$$

Bering in mind that in the initial state the shell deformation in a circumferential direction is defined by the equalities

$$
\varepsilon_\varphi^0 = \frac{\sigma_2^0 - \nu\sigma_1^0}{E} + \alpha T, \quad \varepsilon_\varphi^0 = -\frac{w_0}{r}
$$

we get

$$
w_0 = \big(-\sigma_2^0 + \nu\sigma_1^0\big)\frac{r}{E} - \alpha Tr. \tag{5}
$$

Substituting (5) into (4$'$), we obtain

$$
\begin{aligned}
&\frac{T_1^0}{Eh} = \frac{\sigma_1^0}{E} = -\frac{\alpha T}{1-\nu}\,,\\
&\frac{T_2^0}{Eh} = \frac{\sigma_2^0}{E} = -\frac{qr}{Eh}\Big[1 - 2\nu\frac{\delta_0}{r}\Big(\frac{r}{\ell}\Big)^2\Big] + \frac{\beta_0 r}{Eh}\Big[\big(-\sigma_2^0 + \nu\sigma_1^0\big)\frac{r}{E} - \alpha Tr\Big].
\end{aligned}
\tag{6}
$$

Introduce the notation

$$
\overline{q} = \frac{qr}{Eh}\,, \quad \delta = \frac{\delta_0}{r}\Big(\frac{r}{\ell}\Big)^2, \quad \gamma_0 = \frac{\beta_0 r^2}{Eh}\,, \quad g = 1+\gamma_0.
$$

Then (6) takes the form

$$\frac{\sigma_1^0}{E} = -\frac{\alpha T}{1-\nu}, \quad \frac{\sigma_2^0}{E} = -\overline{q}(1-2\nu\delta) + \left[\Big(-\frac{\sigma_2^0}{E} + \nu\,\frac{\sigma_1^0}{E}\Big)\gamma_0 - \alpha T\gamma_0\right], \tag{7}$$

whence we arrive at

$$\frac{\sigma_2^0}{E}(1+\gamma_0) = -\overline{q}(1-2\nu\delta) + \nu\gamma_0\,\frac{\sigma_1^0}{E} - \alpha T\gamma_0. \tag{8}$$

Substituting into (8) the first expression of (7), we obtain

$$\frac{\sigma_2^0}{E} = -\left[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\right]g^{-1}.$$

Consequently,

$$-\frac{\sigma_1^0}{E} = \frac{\alpha T}{1-\nu}, \quad -\frac{\sigma_2^0}{E} = \left[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\right]g^{-1}. \tag{9}$$

In view of the fact that R is close to r, in the expressions for stresses (9) we adopted $R \approx r$.

As a result, the third equation of system (3) takes the form

$$\varepsilon\,\frac{\partial^8 w}{\partial\varphi^8} + \frac{\partial^4 w}{\partial\xi^4} + 4\delta\,\frac{\partial^4 w}{\partial\xi^2\,\partial\varphi^2} + 4\delta^2\,\frac{\partial^4 w}{\partial\varphi^4} + \left[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\right]g^{-1}\frac{\partial^6 w}{\partial\varphi^6}$$
$$+ \frac{\alpha T}{1-\nu}\frac{\partial^6 w}{\partial\xi^2\,\partial\varphi^4} + \gamma\,\frac{\partial^4 w}{\partial\varphi^4} + \frac{\rho r^4}{E}\,\frac{\partial^2}{\partial\varphi^2}\Big(\frac{\partial^4 w}{\partial\varphi^4}\Big) = 0. \tag{10}$$

A solution of system (3) for harmonic oscillations of closed shells will be sought in the form

$$u = U(\xi)\sin n\varphi\,\cos\omega t,$$
$$v = V(\xi)\cos n\varphi\,\cos\omega t,$$
$$w = W(\xi)\sin n\varphi\,\cos\omega t.$$

From the first two equations of system (3) we obtain

$$n^2 U = \big[1 - 2(1-\nu)\delta\big]W', \tag{11}$$

$$nV = \big(1 + 2\nu\delta\big)W. \tag{12}$$

Note certain simplifications of boundary conditions of rigid fixing for the shells (for $\xi = \text{const}$) having the form

$$u = v = w = w'_\xi = 0. \tag{13}$$

On the basis of equality (12) we find that the fulfilment of the condition $w = 0$ leads to that of the condition $v = 0$, while in view of (11), the fulfilment of the condition $w'_\xi = 0$ leads to that of the condition $u = 0$.

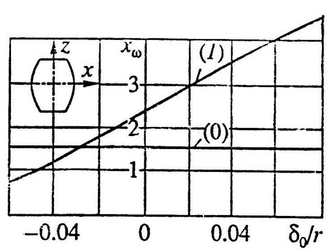

FIGURE 1

Thus if conditions $w = w'_\xi = 0$ ($\xi = \text{const}$) are fulfilled, then all conditions (13) are likewise fulfilled.

Let the shell edges be rigidly fixed. In addition, the solution should satisfy the condition of periodicity with respect to φ and also the following boundary conditions with respect to the coordinate ξ,

$$w = 0 \quad (\xi = \pm\,\ell/r), \quad w'_\xi = 0 \quad (\xi = \pm\,\ell/r). \tag{14}$$

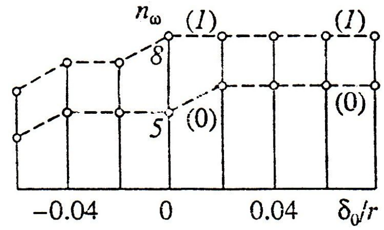

FIGURE 2

Solution w of equation (10), as is mentioned above, for harmonic oscillations is sought in the form

$$w = W \sin n\varphi \cos \omega t. \tag{15}$$

From (10) and (15) follows

$$W^{(4)} - \Big(4\delta n^2 - \frac{\alpha T}{1-\nu}\,n^4\Big)W^{(2)} - n^4\Big\{\frac{\rho r^2}{E}\,\omega^2 + \Big[\overline{q}(1-2\nu\delta) + \frac{\alpha T \gamma_0}{1-\nu}\Big]g^{-1}n^2 - \varepsilon n^4 - 4\overline{\delta}^{\,2}\Big\}W = 0, \quad \overline{\delta}^{\,2} = \delta^2 + \gamma/4. \tag{16}$$

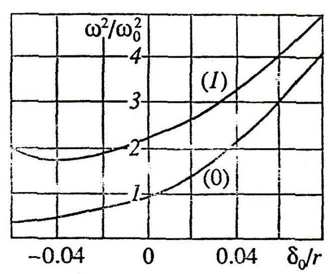

FIGURE 3

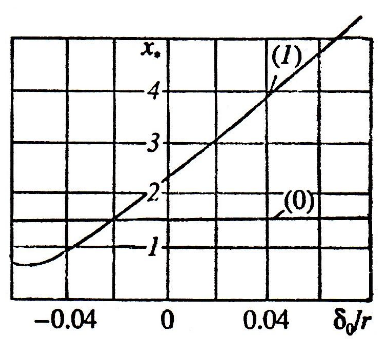

FIGURE 4

Assuming $W = Ce^{\alpha\xi}$, we obtain the following characteristic equation

$$\alpha^4 - \Big(4\delta n^2 - \frac{\alpha T}{1-\nu}\,n^4\Big)\alpha^2 - n^4\Big\{\frac{\rho r^2}{E}\,\omega^2 + \Big[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big]g^{-1}n^2 - \varepsilon n^4 - 4\overline{\delta}^{\,2}\Big\} = 0$$

which can be written as

$$p^2 - ap - b = 0, \quad \Omega = \rho r^2/E, \tag{17}$$

$$\begin{aligned} p &= \alpha^2, \quad a = 4\delta n^2 + \frac{\alpha T}{1-\nu}\, n^4, \\ b &= n^4\bigg\{\Omega\omega^2 + \bigg[\overline{q}(1-2\delta\nu) + \frac{\alpha T\gamma_2}{1-\nu}\bigg] g^{-1} n^2 - \varepsilon n^4 - 4\overline{\delta}^{\,2}\bigg\}. \end{aligned} \tag{18}$$

Proceeding from the condition $b > 0$, from (17) and (18) we have

$$\alpha_{1,2} = \pm\sqrt{p_1}, \quad \alpha_{3,4} = \pm i\sqrt{-p_2},$$
$$p_1 = \frac{a}{2} + \sqrt{\frac{a^2}{4} + b} > 0, \quad p_2 = \frac{a}{2} - \sqrt{\frac{a^2}{4} + b} < 0. \tag{19}$$

General solution of equation (16) takes the form

$$W = A\operatorname{ch} k_1\xi + B\operatorname{sh} k_1\xi + C\cos k_2\xi + D\sin k_2\xi,$$
$$k_1 = \sqrt{p_1}, \quad k_2 = \sqrt{-p_2}.$$

Satisfying boundary conditions (14), we obtain the system of four homogeneous equations. Since the determinant of that system is equal to zero, we get

$$\operatorname{th} k_1\overline{\ell} = \frac{k_1}{k_2}\,\operatorname{tg} k_2\overline{\ell} = -\frac{k_2}{k_1}\,\operatorname{tg} k_2\overline{\ell}, \quad \overline{\ell} = \ell/r. \tag{20}$$

Consequently, this system falls into two independent systems and hence a solution falls into odd and even functions. To the even function there correspond symmetric with respect to ξ forms of oscillations, while to the odd function there correspond skew-symmetric ones. Thus we obtain

$$W = D\bigg(\sin k_2\xi - \frac{\sin k_2\overline{\ell}}{\operatorname{sh} k_1\overline{\ell}}\,\operatorname{sh} k_1\xi\bigg),$$
$$W = C\bigg(\cos k_2\xi - \frac{\cos k_2\overline{\ell}}{\operatorname{ch} k_1\overline{\ell}}\,\operatorname{ch} k_1\xi\bigg).$$

First, let us consider the case $\delta = 0$, $\overline{q} = \gamma = T = 0$ where $p_1 = -p_2 = \sqrt{b}$, $k_1 = k_2 = \sqrt[4]{b} = k$. Equation (20) corresponding to skew-symmetric forms of oscillations takes the form

$$\operatorname{th} k\overline{\ell} = \operatorname{tg} k\overline{\ell}.$$

To the lower root of that equation there corresponds the value

$$k = 3{,}927\, r/\ell,$$

where as equation (20) corresponding to the symmetric forms of oscillation take for $\delta = 0$, $q = \gamma = T = 0$ the form

$$\operatorname{th} k\overline{\ell} = -\operatorname{tg} k\overline{\ell}.$$

To the lower root of that equation there corresponds the value

$$k = 2{,}365\, r/\ell = 0{,}75\pi\, r/\ell, \tag{21}$$

i.e., the lower value k corresponds to the symmetric form of oscillation. Therefore in the sequel we will consider oscillations with symmetric form of deflection with respect to ξ. Taking into account that

$$-p_1p_2 = b, \quad b = n^4(\Omega\omega^2 - \varepsilon n^4),$$

for $\delta = 0$, $q = \gamma = T = 0$, we get

$$k^4 = n^4(\Omega\omega^2 - \varepsilon n^4).$$

This implies that to the lower root (21) for fixed n there corresponds the least value of eigen-frequency defined by the expression

$$\Omega\omega^2 = \varepsilon n^4 + (d_1\lambda_1)^4 n^{-4}, \quad d_1 = 1{,}55, \quad \lambda_1 = \pi r/2\ell.$$

The least frequency value depending on n is realized for

$$n_0^2 = d_1\lambda_1\varepsilon^{-1/4}. \tag{22}$$

For $n = n_0$, from (22), for the least frequency of cylindrical shell of middle length with rigidly fixing ends we obtain the known formula [1]

$$\Omega\omega_{01}^2 = 2\,d_1^2\lambda_1^2\varepsilon^{1/2}.$$

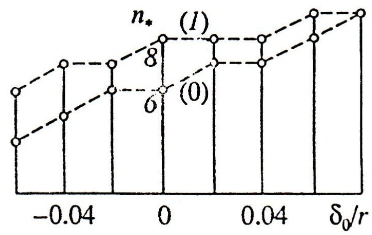

FIGURE 5

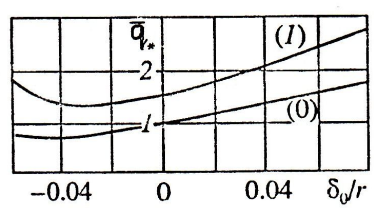

FIGURE 6

For freely supported ends, the least frequency of cylindrical shell is, as is known, defined by the formula

$$\Omega\omega_0^2 = 2\lambda_1^2\varepsilon^{1/2}. \tag{23}$$

Let us turn now to the general case and investigate axially symmetric forms of oscillations corresponding to lower frequencies. Relying on (19), we have

$$-p_2 = p_1 - a, \quad a = \Big(4\delta - \frac{\alpha T}{1-\nu}\,n^2\Big)n^2.$$

from which, putting $x = \bar{\ell}\sqrt{p_1}$, we obtain

$$-p_2\bar{\ell}^{\,2} = x^2 - \beta, \quad \beta = 4\,n^2\,\frac{\delta_0}{r} - \frac{\alpha T}{1-\nu}\Big(\frac{\ell}{r}\Big)^2 n^4. \tag{24}$$

Then equation (20) corresponding to symmetric forms of oscillations can be represented as

$$x\ \mathrm{th}\,x = -\sqrt{x^2-\beta}\ \mathrm{tg}\sqrt{x^2-\beta}. \tag{25}$$

On the basis of the first equality of (19), we have $p_1(p_1 - a) = b$ from which we find that

$$\Omega\omega^2 = \varepsilon n^4 + x^2(x^2-\beta)\Big(\frac{r}{\ell}\Big)^4 n^{-4} + 4\bar{\delta}^{\,2} - \Big[\bar{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big]g^{-1}n^2. \tag{26}$$

Consequently, in a general case, the eigen-frequencies ω for the shells under consideration are defined by formula (26), where x is any root of equation (25). The least frequency ω is obtained by minimizing the right-hand side of (26) with respect to n, when as x we take the least root of equation (25) which we denote by x_ω. On the basis of (24) and (25), it is not difficult to see that x_ω depends both on δ_0/r, T and on n. Such a minimization is realized by sorting out natural values n in the neighbourhood n_0 defined by equality (22). Below we present results of our calculations for the shells with geometric dimensions $\ell = r$, $h/r = 10^{-2}$, $\nu = 0,3$ for different values δ_0/r (for $\overline{q} = \gamma = T = 0$). In Figure 1 we can see dependence of x_ω on δ_0/r (curve (1) corresponds to the rigid shell ends fixing; straight line (0) corresponds to the freely supported ends). Figure 2 presents dependence of n_ω on δ_0/r ((1) corresponds to rigidly fixing ends and (0) to freely supported ends). In Figure 3 we can see the curves of dependence of the least frequencies ω^2/ω_0^2 on δ_0/r ((1) is the case of rigidly fixing ends and (0) for fxreely supported ends [2]), ω_0^2 is defined by expression (23).

For $\omega = 0$, from (26) we obtain

$$\overline{q}(1-2v\delta) = \left[\varepsilon n^2 + x^2(x^2-\beta)n^{-6}\left(\frac{r}{\ell}\right)^4 + 4\overline{\delta}^{\,2}n^{-2}\right]g - \frac{\alpha T\gamma_0}{1-v}\,. \tag{27}$$

The least value $\overline{q}$ is obtained after minimization of the right-hand side of equality (28) depending on n, when as x we take the least positive root of equation (26) which is denoted by x_*. It is not difficult to see that on the basis of the value x_* depends on n_*. Corresponding values x_*, n_*, $\overline{q}_*/\overline{q}_{0*}$ are critical and presented depending on δ_0/r by the curves (1) in Figures 4, 5, 6 for $\gamma_0 = T = 0$. In Figure 6, over the Oy-axis is drawn the dimensionless critical pressure $\overline{q}_*/\overline{q}_{0*}$ ($\overline{q}_{0*}$ characterizes critical pressure for freely supported cylindrical shell and is defined by the equality $\overline{q}_{0*} = 0,855(1-\nu^2)^{-3/4}(h/r)^{3/2}r/L$ [6]). Comparing curves 1 in Figures 3, 6, it is not difficult to notice that their behaviour is qualitativly close: if for $\delta_0 > 0$ the values of the least frequency and of critical pressure increase, then for $\delta_0 < 0$ they first decrease up to $\delta_0/r \approx -(0,03 \div 0,04)$ and then increase. According to (27), the formula for finding critical pressure $\overline{q}_*$ has the form

$$\overline{q}_* = \frac{1+\gamma_0}{1-2\nu\delta}\left[\varepsilon n_*^2 + x_*^2(x_*^2-\beta)n^{-6}\big(r/\ell\big)^4 + 4\overline{\delta}^{\,2}n_*^{-2}\right] - \frac{\alpha T\gamma_0}{(1-\nu)(1-2\nu\delta)}\,.$$

Thus we have obtained formulas for determination of lower frequencies for the shells of revolution which by their form are close to cylindrical ones, depending on the boundary conditions of rigid fixing, amplitude of cylinder deviation, rigidity of an elastic filler, external pressure and temperature. The formula for determination of critical pressure depending on the above-mentioned factors, is also given.

References

1. A. L. Goldenveĭzer, V. B. Lidskiĭ, P. E. Tovstik, *Free Oscillations of Thin Elastic Shells.* (Russian) Nauka, Moscow, 1979.
2. S. N. Kukudzhanov, On the influence of normal pressure on frequencies of eigen oscillations of shells of revolution, which are close to cylindrical ones. *Izvestiya RAN MTT* **6** (1996), 121–126.
3. B. V. Novojilov, *Theory of Thin Shells.* (Russian) Sudpromgis, Leningrad, 1962.
4. P. M. Ogibalov, V. P. Gribanov, *Thermal Stability of Plates and Shells.* (Russian) Moscow, 1968.
5. V. Z. Vlasov, *General Theory of Shells and Its Applications in Technology.* (Russian) Gos. Izdat. Tekh. Teor. Lit., Moscow-Leningrad, 1949.
6. A. S. Volmir, *Stability of Deformable Systems.* (Russian) Nauka, Moscow (1967).

(Received 29.01.2020)

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