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DYNAMICS OF $2D$ SOLITONS IN MEDIA WITH VARIABLE DISPERSION: SIMULATION AND APPLICATIONS

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Abstract. Dynamics of multidimensional solitons in media with variable dispersion is studied numerically. The application of the obtained results to the dynamics of FMS waves in a magnetized plasma, and the 2-dimensional surface waves on shallow water are discussed.

In this paper we consider the problem of dynamics the multidimensional solitons which are described by the Kadomtsev–Petviashvili (KP) equation

$$\partial_t u + \alpha u \partial_x u + \beta \partial_x^3 u = \varkappa \int\limits_{-\infty}^{x} \Delta_\perp u dx, \tag{1}$$

in complex media with the varying in time and/or space dispersive parameter $\beta = \beta(t, \mathbf{r})$. This problem is mainly interesting from the point of view of its evident applications in physics of real complex media with the dispersion. For example, such situation can have place in the problems of the propagation of the 2-dimensional (2D) gravity and gravity-capillary waves on the surface of "shallow" water [5, 7] when β is defined respectively as

$$\beta = c_0 H^2/6$$

and

$$\beta = (c_0/6)[H^2 - 3\sigma/\rho g]$$

where H is the depth, ρ is the density, and σ is the coefficient of surface tension of fluid. If $H = H(t, x, y)$, β also becomes the function of the coordinates and time. Similar situation may have place on studying of the evolution of the 3D fast magnetosonic (FMS) waves in magnetized plasma [1, 6] in case of the inhomogeneous and/or non-stationary plasma and magnetic field when β is a function of the Alfv'en velocity $v_A = f[B(t, \mathbf{r}), n(t, \mathbf{r})]$ and angle $\theta = (\mathbf{k}\,\hat{}\,\mathbf{B})$:

$$\beta = v_A (c^2/2\omega_{0i}^2)(\cot^2\theta - m/M)$$

where m and M are the masses of electron and ion, respectively. It is well known [8] that the 1D solutions of the Korteweg-de Vries (KdV) equation (equation (1) with $\varkappa = 0$) with $\beta = \text{const}$ in dependence on value of β are divided into two classes: at $|\beta| < u_0(0, x)l/12$ (l is the characteristic wave length) they have soliton character, in an opposite case they are the wave packets with asymptotes being proportional to the derivative of the Airy function [7,8]. In these cases, the KdV equation can be integrated by the inverse scattering transform (IST) method [5, 7]. But, if $\beta = \beta(x, t)$ it is impossible principally, and it is necessary to resort to a numerical simulation. Similar situation has place for the multidimensional KP equation: in case $\beta = \beta(t, \mathbf{r})$ the dispersive term becomes quasi-linear and the model being not exactly integrable [7].

Here, the problem of study of structure and evolution of the nonlinear waves described by the KP equation with $\beta = \beta(t, \mathbf{r})$ is considered distracting from a specific type of the propagation medium. The numerical experiments were conducted for several model types of function β when at $t < t_{cr}$

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$\beta = \beta_0 = \text{const}$, and at $t \geq t_{cr}$

$$1)\ \beta(x) = \begin{cases} \beta, & x \leq a; \\ \beta_0 + c, & x > a; \end{cases} \tag{2}$$

$$2)\ \beta(x,t) = \begin{cases} \beta_0, & x \leq a; \\ \beta_0 + nc, n = (t - t_{cr})/\tau = 1, 2 \ldots; & x > a \end{cases} \tag{3}$$

$$3)\ \beta(t) = \beta_0(1 + k_0\overline{\beta}\sin\omega t), \quad \overline{\beta} = (\beta_{\max} - \beta_{\min})/2, \tag{4}$$
$$0 < k_0 < l, \quad \pi/2\tau < \omega < 2\pi/\tau,$$

a and c are constants. In terms of the propagation of the waves on shallow water that means respectively, that after reaching of time t_{cr}: 1) sharp "break of bottom"; 2) gradual "change of a height" of a segment of bottom; and 3) the "oscillations of bottom" with time take place.

In the first series of numerical experiments we investigated the evolution of initial pulse in case when at t_{cr} the spasmodic change of $\beta = \beta(t, x, y)$ has a place behind soliton ["negative" step when $c < 0$ in (2), (3)]. At this, the dependence of spatial structure of solution on parameter a was studied. The obtained results (see Figure 1) showed that in all cases the evolution leads to the formation of waving tail which is not connected with soliton going away and caused only by local influence of sudden change of the "relief" $\beta(t, x, y)$. Consequently, the formation of oscillatory structure is connected not so much with decreasing of a role of the dispersion effects behind soliton as with the spasmodic changing of β in space.

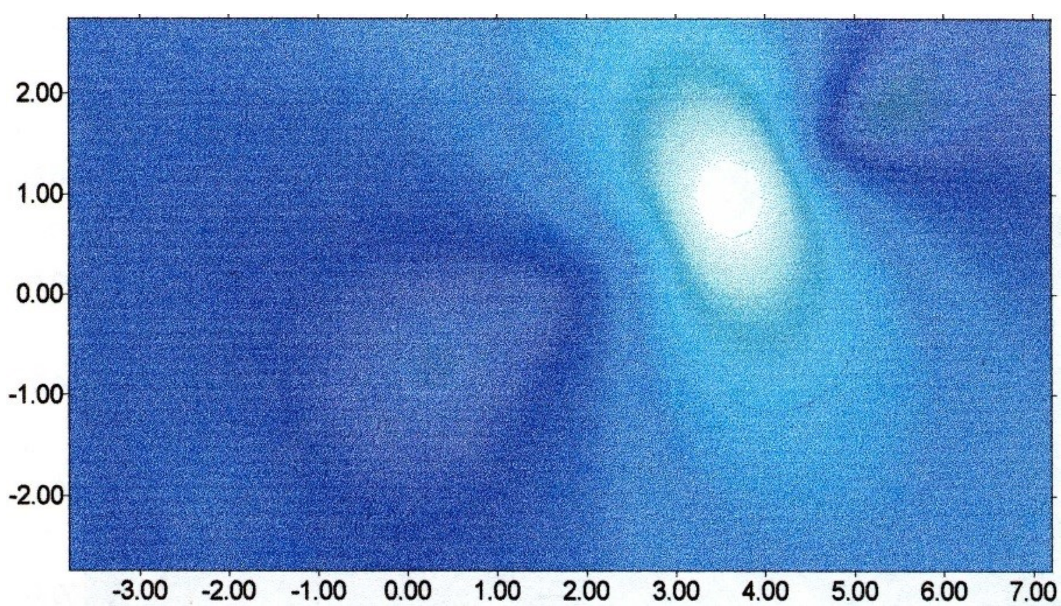

Figure 1. Evolution of a 2D soliton of equation (1) for the dispersion change law (3) at $a = 5.0$, $c = -0.0038$ for $t = 0.6$.

In the next series of simulations we considered a case when the sudden change of β takes place directly under or in front of an initial pulse ("negative" step). An example of the results is shown in Figure 2. One can see that for such character of the "relief" the disturbance caused by sudden change of β has also local character, i.e. it doesn't propagate together with the going away soliton.

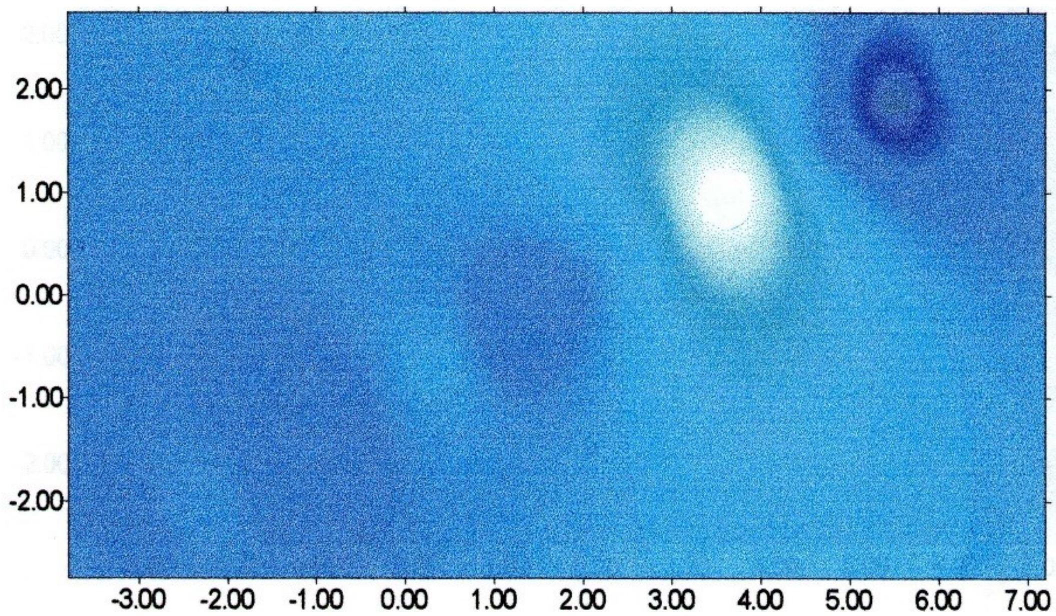

FIGURE 2. Evolution of a 2D soliton of equation 1 for the dispersion change law (2) at $a = 4.0$, $c = -0.0038$ for $t = 0.6$.

But, unlike the cases of the first series, the asymptotes of leaving soliton become oscillating, besides, against a background of the long-wave oscillations of the waving tail we can also see the appearance of the wave fluctuations. The effects noted can be interpreted as a result of those that for the areas of the wave surface with different values of local wave number k_x the value of the dispersive effects is different. As a result, the dispersive confusion of the Fourier-harmonics phases takes place in the (x, y)-region not equally intensity everywhere and, consequently, it counteracts with different extent of activity to the generation due to nonlinearity of the harmonics with big k_x.

In the next series of simulations with β changing with the laws (2) and (3) we considered the cases of "positive" step ($c > 0$) being both in front of and behind of initial pulse for the wide range of values of a. The examples of the most interesting results are shown in Figure 3.

One can see that when "positive" step is far in front of maximum of function $u(0, x, y)$ the soliton evolution on the initial stage does not differ qualitatively from that for $\beta = \text{const}$ (Figure 3a), but in the future their character is defined by presence of the step, namely the processes, caused by the same causes which have been noted for the results of the second series, begin to be developed (Figure 3b).

As we can see, the appreciable change of the soliton structure which can lead to wave falling is observed owing to intensive generation of the harmonics with big k_x in the soliton front region, even for rather small height of the step. Thus, the disturbance of the propagating 2D soliton has also local character.

As to equation (4), the simulation for different $k_0 = \text{const}$ and variable frequency ω showed that for some values of ω the stationary (locally) standing waves can be formed, in another cases the formation of the stationary periodical wave structures is possible, and in the intermediate cases a chaotic regime is usually realized.

In conclusion, we studied propagation of 2D solitons in complex media with variable dispersion, considering as a concrete example evolution of 2D solitary waves on shallow water. Let us note that such approach can be useful and effective in the problems of nonlinear dynamics of the FMS waves and wave beams in a magnetized plasma [1, 5–7], and also in problems of investigation of evolution and transformation of the internal gravity waves (IGW) and travelling ionospheric disturbances at

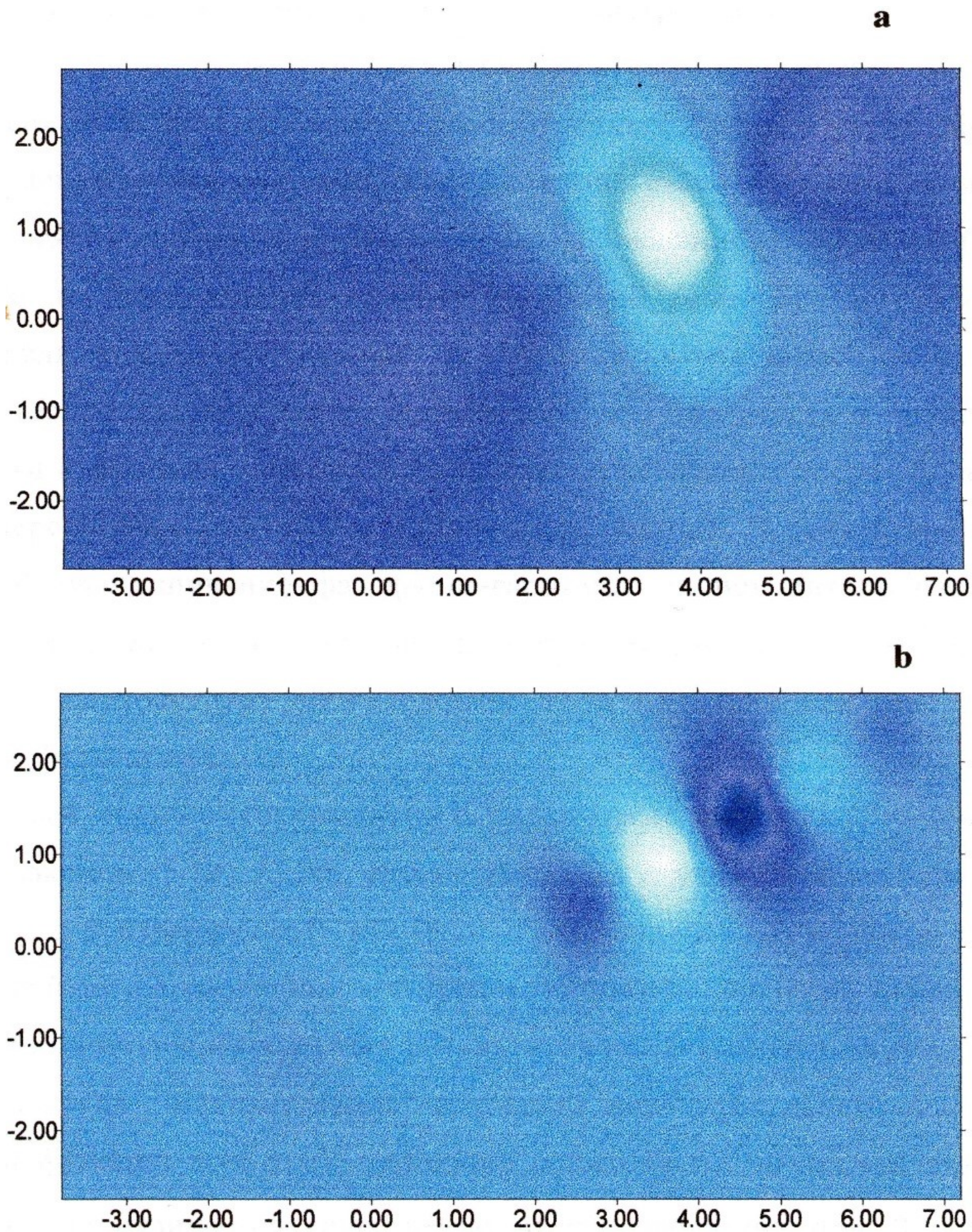

Figure 3. Evolution of a 2D soliton of equation (1) for the dispersion change law (3) at $a = 5.0$, $c = 0.0038$: (a) $t = 0.6$, (b) $t = 0.8$.

heights of the ionosphere F-region on fronts of the solar terminator and the solar eclipse spot [2, 4] and in regions where basic ionosphere characteristics are changed in time and space [3].

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